Name:

MWF 9 Oliver Knill
MWF 10 Hansheng Diao
MWF 10 Joe Rabinoff
MWF 11 John Hall
MWF 11 Meredith Hegg
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TTH 10 Bence Béky
TTH 10 Gijs Heuts
TTH 11:30 Francesco Cavazzani
TTH 11:30 Andrew Cotton-Clay

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- All unspecified functions which appear are nice and differentiable.
- **Show your work**. Except for problems 1-3, we need to see details of your computation.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

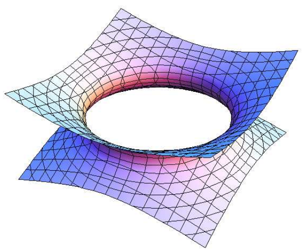
Problem 1) True/False questions (20 points). No justifications are needed.

1)	T	F	For any two vectors $\vec{u}$ and $\vec{v}$ we have $|\vec{u}| \leq |\vec{v}| + |\vec{v} - \vec{u}|$.
2)	T	F	The grid curves of a parametric surface are always perpendicular to each other at any point on the surface.
3)	T	F	The triple scalar product $\vec{u} \cdot (\vec{v} \times \vec{w})$ of $\vec{u} = \langle 1,0,0 \rangle, \vec{v} = \langle 0,1,0 \rangle$ and $\vec{w} = \langle 2,1,1 \rangle$ is equal to 1.
4)	T	F	For $f(x,y) = x^4 + y^4$ and $\vec{r}(t) = \langle t, t^2 \rangle$, we have $\frac{d}{dt} f(\vec{r}(t)) = \langle 4t^3, 4t^6 \rangle \cdot \langle 1, 2t \rangle$.
5)	T	F	The flux of $\vec{F} = \langle x, 0, 0 \rangle$ through the outwardly-oriented boundary S of a parallelepiped spanned by edges $\langle 1,0,0 \rangle$, $\langle 0,2,0 \rangle$, $\langle 1,1,3 \rangle$ is equal to 6.
6)	T	F	The differential equation $u_x = u_t$ for a function $u(x,t)$ is called the heat equation.
7)	T	F	If a function $f(x,y)$ has a critical point at $(0,0)$ then $\mathrm{div}(\mathrm{grad}(f))(0,0)$ is zero.
8)	T	F	A function of two variables always has an odd number of critical points.
9)	T	F	If $f(x,y)$ is a function of two variables and $(0,0)$ is a maximum of $g(x) = f(x,0)$ and as well as a maximum of $h(y) = f(0,y)$ then $(0,0)$ is a maximum of f.
10)	T	F	The divergence of a vector field $\vec{F}$ is always equal to the divergence of the curl of $\vec{F}$.
11)	T	F	The flux of a vector field $\vec{F}$ of length $|\vec{F}| = 1$ through a triangular surface S can not be larger than the surface area of the triangle.
12)	T	F	The arc length of the boundary of a surface is independent of the parametrization of the surface.
13)	T	F	The vector field $\mathrm{curl}(\vec{F})$ is at every point (x,y,z) perpendicular to $\vec{F}(x,y,z)$.
14)	T	F	The scalar function $f = \mathrm{div}(\vec{F})$ has the property that $\mathrm{grad}(f(x,y,z))$ is perpendicular to $\vec{F}(x,y,z)$.
15)	T	F	The Lagrange equations $\nabla f(x,y) = \lambda \nabla g(x,y), g(x,y) = x^2 + y^2 = 1$ have infinitely many solutions if $f = g$.
16)	T	F	If a vector is perpendicular to itself, then it is the zero vector.
17)	T	F	The gradient of the divergence of the curl of a vector field $\vec{F}$ is the vector field which assigns the zero vector to each point.
18)	T	F	The identity $\mathrm{Proj}_{\vec{v}}(\vec{w}) = \mathrm{Proj}_{\vec{w}}(\vec{v})$ holds for all vectors $\vec{v}, \vec{w}$.
19)	T	F	The function $f(x,y) = \sin(xy)$ is a solution to the Laplace equation $f_{xx} + f_{yy} = 0$.
20)	T	F	The formula $\vec{r}(u,v) = \langle 2u, (9+u^2)\cos(v), (9+u^2)\sin(v) \rangle$ gives a parametrization of a one-sheeted hyperboloid.

Problem 2) (10 points)

a) (6 points) Match the following objects.

Formula	Enter 1-6
$\rho(\phi,\theta) \leq e^{\phi}$	
$\vec{F}(x,y,z) = \langle -y, x, -2 \rangle$	
$x^2 + y^2 - 5z^2 = 1$	
$z + x^2 - y^2 = 2$	
$\vec{F}(x,y) = \langle x, -y \rangle$	
$x^4 + 2y^4 \leq 3$	



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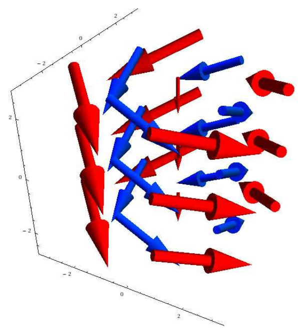

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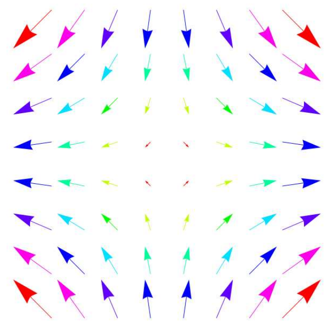

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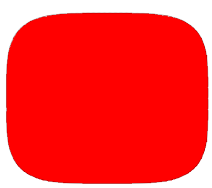

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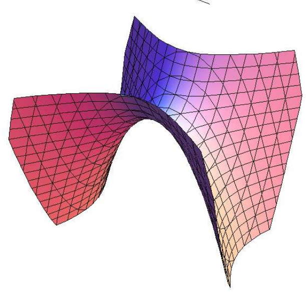

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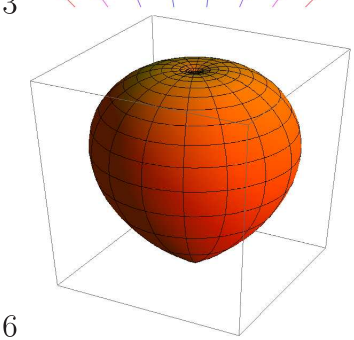
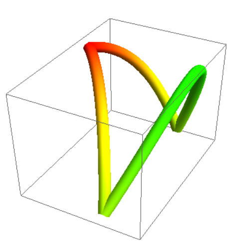


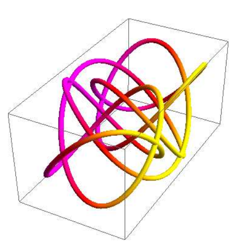
6

b) (4 points) A knot is a closed curve in space. Match the following knots

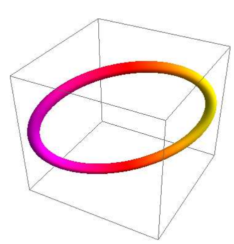
Formula	Enter A,B,C,D
$\vec{r}(t) = \langle \cos(5t), \cos(t) + \sin(5t), \cos(7t) \rangle$	
$\vec{r}(t) = \langle \lvert\cos(t)\rvert, \lvert\sin(t) + \cos(t)\rvert, \lvert\cos(2t)\rvert \rangle$	
$\vec{r}(t) = \langle (2 + \cos(\frac{3t}{2}))\cos(t), (2 + \cos(\frac{3t}{2}))\sin(t), \sin(\frac{3t}{2}) \rangle$	
$\vec{r}(t) = \langle \cos(t), \cos(t), \sin(t) \rangle$	



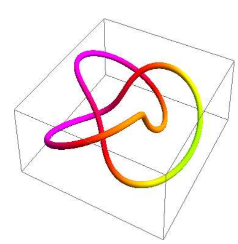
A



B



C



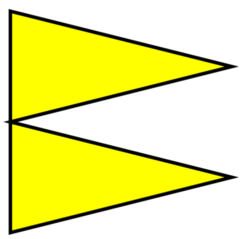
D

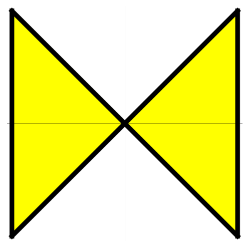
Problem 3) (10 points)

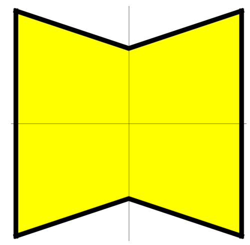
a) (4 points) It is Hobbit time. The following regions resemble ancient runes of the Anglo Saxons studied by JRR Tolkien. (A is "b", B is "d", C is "st", and D is "oe").

Integral	Enter A,B,C,D
$\int_{-1}^{1} \int_{-\vert x\vert}^{\vert x\vert} f(x,y)\ dydx$	
$\int_{-1}^{1} \int_{-1-\vert x\vert/2}^{1+\vert x\vert/2} f(x,y)\ dydx$	
$\int_{-1}^{1} \int_{0}^{\vert\vert y+\frac{1}{2}\vert-\vert y-\frac{1}{2}\vert\vert-\vert y\vert} f(x,y)\ dxdy$	
$\int_{-1}^{1} \int_{\vert y\vert-1}^{1-\vert y\vert} f(x,y)\ dxdy$	

[1]


A


B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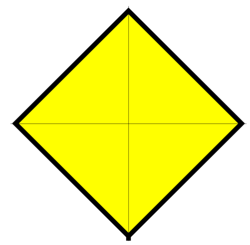
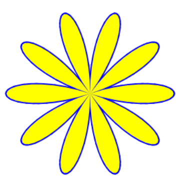
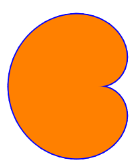
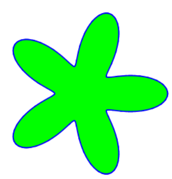

D

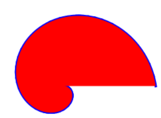
b) (4 points) Matching polar regions

Formula	Enter E,F,G,H
$r \leq \theta(2\pi - \theta)$	
$r \leq \vert\cos(5\theta)\vert$	
$r \leq 2 + \cos(5\theta)$	
$r \leq 2\pi - \theta$	


E


F


G

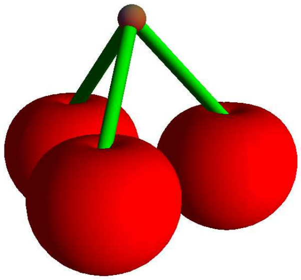

H

c) (2 points) Which derivatives and integrals do appear in the statements of the following theorems? Check each box which applies. Multiple entries are allowed in each row or column.

Integral theorem	Grad	Curl	Div	Line integral	Flux integral
Divergence theorem					
Stokes' theorem					

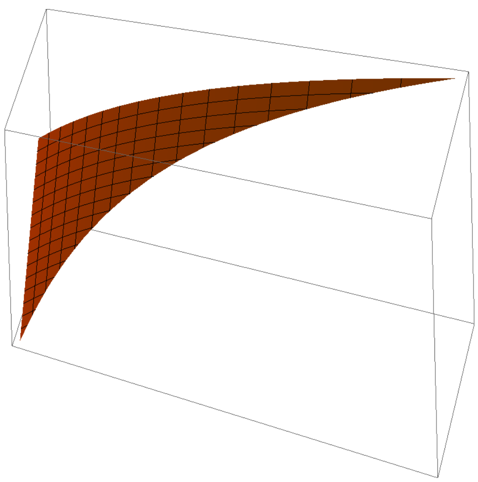
Problem 4) (10 points)

[1] More information on http://www.theshorterword.com/anglo-saxon-runes



Three cherries have centers at $A = (-1,-1,-1)$, $B = (1,0,-2)$ and $C = (0,1,-2)$ and are tied together at the origin $O = (0,0,0)$.[2]
Find the distance between O and the plane through A, B, C.

Problem 5) (10 points)

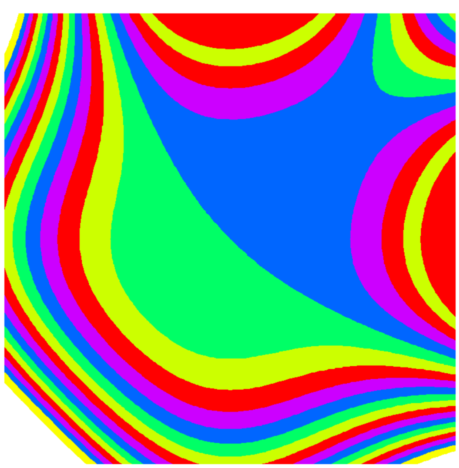


Find the surface area of the surface

$$\vec{r}(u,v) = \langle u^2 + v, u, v \rangle$$

for which $0 \leq v \leq 4$ and $\frac{v}{4} \leq u \leq 1$.

Problem 6) (10 points)

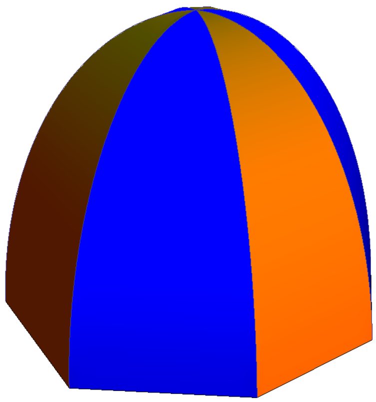


The function $f(x,y) = 2x^3 + 2y^3 - 3x^2y^2$ is called the "happy function" as you can see when you turn your head clockwise by $\pi/4$. Find and classify its extrema.

In one of the critical points, the discriminant D is zero. We want you nevertheless to decide whether this point is a "local maximum" a "local minimum" or "neither of them".

[2] Picture from Mathematica project by Alissa Zhang

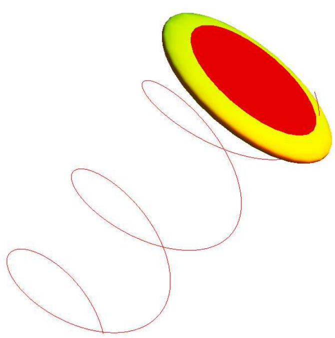
Problem 7) (10 points)



Archimedes computed volumes of solids which now bear his name. He showed that, as for the sphere, each "Archimedean globe" has volume equal to two thirds of the prism in which it is inscribed. Later it was discovered that also the surface area is two thirds of the surface area of a circumscribing prism. To find globes of minimal surface we are led to the problem:

Find values of r, h satisfying $g(r, h) = r^2 h = 3$ so that $f(r, h) = 3r^2 + 2rh$ is minimal.

Problem 8) (10 points)



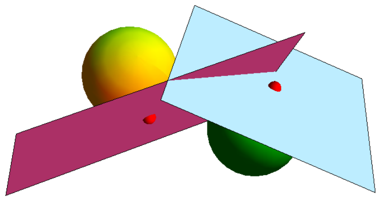
The frisbee was invented by Harvard students in 1845 when a student threw a cake plate to George Frisbie Hoar and shouted "Frisbie, catch!".
A point on the outer rim of a frisbee moves on a curve $\vec{r}(t)$ satisfying [3]

$$\vec{r}\,''(t) = \langle 0, -\cos(t), -\sin(t) \rangle \ .$$

We know that $\vec{r}(0) = \langle 0, 1, 0 \rangle$ and $\vec{r}\,'(0) = \langle 1, 0, 1 \rangle$. Find $\vec{r}(t)$ and the arc length of the curve $\vec{r}(t)$ from $t = 0$ to $t = 2\pi$.

Problem 9) (10 points)

[3] Some claim this incident happened at Yale, which is a fairy tale. **George Frisbie Hoar** graduated from Harvard in 1846 and became later a United States Senator. He fought against political corruption, and campaigned for the rights of African and Native Americans.



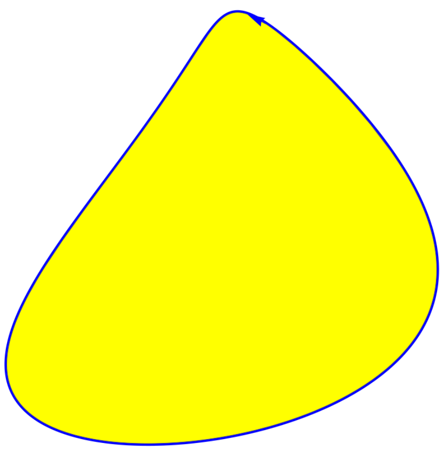
Find a parametrization of the line of intersection of the tangent plane at the point $(1, -1, 0)$ of the sphere

$$x^2 + y^2 + z^2 = 2$$

and the tangent plane to the point $(5, 1, 1)$ of the sphere

$$(x-5)^2 + y^2 + z^2 = 2 \; .$$

Problem 10) (10 points)

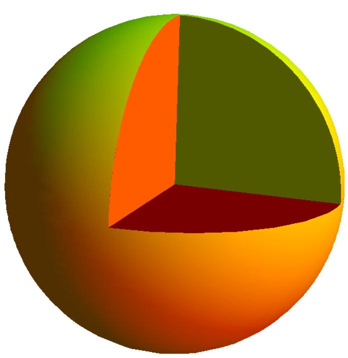


Find the area of the region enclosed by the curve

$$\vec{r}(t) = \langle 3\cos(t) - \sin(2t), 4\sin(t) + \cos(t) \rangle \; ,$$

where $0 \leq t \leq 2\pi$.

Problem 11) (10 points)

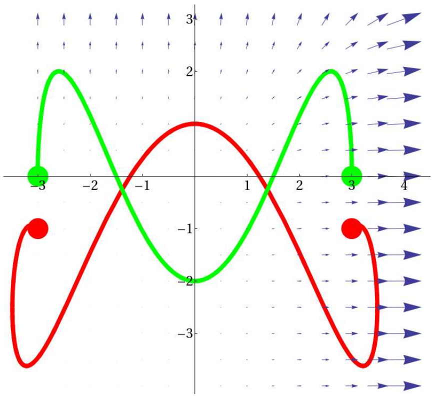


Find the flux of the vector field

$$\vec{F}(x, y, z) = \langle x^3 + x^2, y^3 - xy, z^3 - xz \rangle$$

through the boundary surface of E (oriented outwards), where the solid E is a unit sphere from which the first octant has been removed.

Problem 12) (10 points)



Assume the wind velocity on the Charles is

$$\vec{F}(x,y) = \langle e^x, e^y \rangle \ .$$

A sail boat takes the path

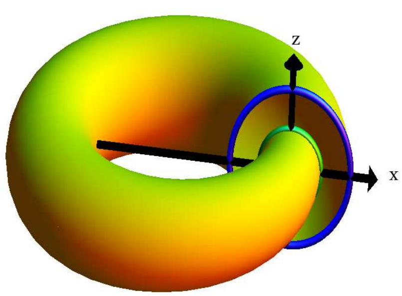
$$C_1 : \vec{r}(t) = \langle -3\cos(t) - \sin(2t), 2\sin(t) + 2\cos(4t) - 3 \rangle$$

from $(-3,-1)$ to $(3,-1)$. An other boat follows the path

$$C_2 : \vec{r}(t) = \langle -3\cos(t), 2\sin(3t) \rangle$$

from $(-3,0)$ to $(3,0)$. To find out which path needs more energy, compute both line integrals $\int_{C_1} \vec{F} \cdot \vec{dr}$ and $\int_{C_2} \vec{F} \cdot \vec{dr}$.

Problem 13) (10 points)



The "foot in the mouth" surface S seen in the picture is parametrized by

$$\vec{r}(u,v) = \langle \quad (4 + g(u,v)\cos(v))\cos(u) - 4, \\ (4 + g(u,v)\cos(v))\sin(u), \\ g(u,v)\sin(v) \rangle$$

with $g(u,v) = (2 - \frac{u}{2\pi})$ and $0 \leq u, v \leq 2\pi$. It is oriented outwards. Its boundary consists of two circles in the xz-plane centered at the origin, with radius 1 and 2. Find the flux of the curl of

$$\vec{F}(x,y,z) = \langle z + yz, x, \sin(x^3 y) + y^2 + z^4 \rangle$$

through S.