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- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work.** Except for problems 1-3, we need to see details of your computation.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 180 minutes time to complete your work.

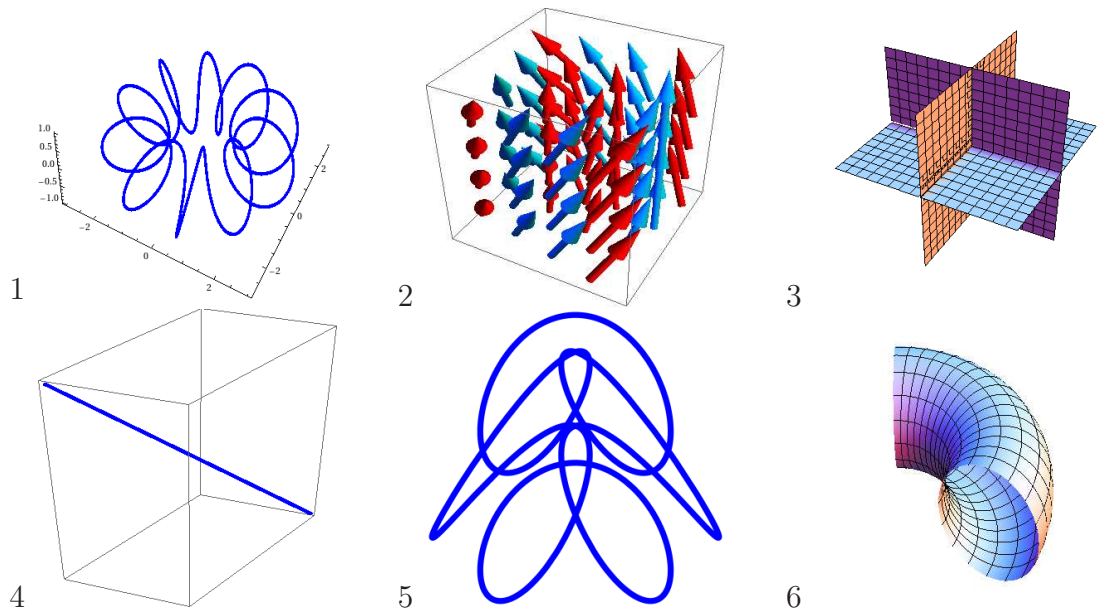
1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
14		10
Total:		150

Problem 1) True/False questions (20 points). No justifications are needed.

- 1) T F The functions $e^{x^2+y^3-y}$ and $x^2 + y^3 - y$ have the same critical points.
- 2) T F The line $\vec{r}(t) = \langle t^2, t^2, t^2 \rangle$ hits the plane $x + y + z = 100$ at a right angle.
- 3) T F The quadric $x^2 - 2y^2 + z^2 = 5$ is a one sheeted hyperboloid.
- 4) T F The relation $|\vec{u} \times \vec{v}| = |\vec{u} \cdot \vec{v}|$ is only possible if at least one of the vectors \vec{u} and \vec{v} is the zero vector.
- 5) T F The partial differential equation $u_x = u_{tt}$ is called the **Heat equation**.
- 6) T F The curvature of the curve $\vec{r}(t) = \langle \sin(2t), \cos(2t)/\sqrt{2}, \cos(2t)/\sqrt{2} \rangle$ at $t = 0$ is equal to the curvature of the curve $\vec{s}(t) = \langle 0, \cos(3t), \sin(3t) \rangle$ at $t = 0$.
- 7) T F The space curve $\vec{r}(t) = \langle \sin(t), t^2, \cos(t) \rangle$ for $t \in [0, 10\pi]$ is located on a cylinder.
- 8) T F If a smooth function $f(x, y)$ has a global maximum, then it has a global minimum.
- 9) T F If $L(x, y)$ is the linearization of $f(x, y)$ at (x_0, y_0) and $\vec{s}(t)$ is the line tangent to the curve $\vec{r}(t)$ on $f = c$ at the point $\vec{r}(t_0) = \vec{s}(t_0) = (x_0, y_0)$ so that $|\vec{r}'(t_0)| = |\vec{s}'(t_0)| = 1$, then $|d/dtL(\vec{s}(t))| = |d/dtf(\vec{r}(t))|$ at the time $t = t_0$.
- 10) T F If \vec{F} is a gradient field and $\vec{r}(t)$ is a flow line defined by $\vec{r}'(t) = \vec{F}(\vec{r}(t))$, then the line integral $\int_0^1 \vec{F} \cdot d\vec{r}$ is either positive or zero.
- 11) T F The flux of the vector field $\vec{F} = \nabla f$ through the surface $f(x, y, z) = x^4 + y^4 + z^4 = 1$ is positive if the surface is oriented so that $\vec{r}_u \times \vec{r}_v$ points in the direction of the gradient of f .
- 12) T F If we extremize the function $f(x, y)$ under the constraint $g(x, y) = 1$, and the functions are the same $f = g$, all points on the constraint curve are extrema for f .
- 13) T F If a point (x_0, y_0) is a minimum of $f(x, y)$ under the constraint $g(x, y) = 1$, then it is also a local minimum of the function $f(x, y)$ without constraints.
- 14) T F If a vector field $\vec{F}(x, y)$ is a gradient field, then any line integral along any closed ellipse is zero.
- 15) T F The flux of an irrotational vector field is zero through any surface S in space.
- 16) T F The divergence of a gradient field $\vec{F}(x, y, z) = \nabla f(x, y, z)$ is everywhere zero.
- 17) T F The line integral of the vector field $\vec{F}(x, y, z) = \langle x, y, z \rangle$ along a circle in the xy - plane is zero.
- 18) T F For any solid E , the moment of inertia $\iiint_E x^2 + y^2 dx dy dz$ is always larger than the volume $\iiint_E 1 dx dy dz$ of E .
- 19) T F The curvature of a parametrized curve satisfying $|\vec{r}'(t)| = 1$ is bounded above by the length $|\vec{r}''|$ of the acceleration.
- 20) T F Given a vector field $\vec{F} = \langle P, Q, R \rangle$, the directional derivative of $\text{div}(\vec{F}(x, y, z))$ in the direction $\vec{v} = \langle 1, 0, 0 \rangle$ is $P_{xx} + Q_{xy} + R_{xz}$.

Problem 2) (6 points)

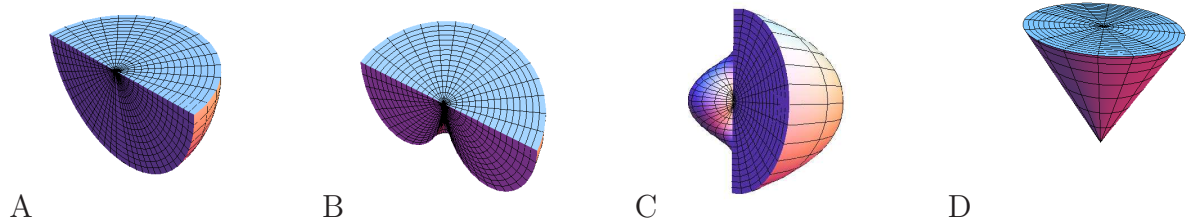
a) (6 points) Match the objects with their definitions



Enter 1-6	Object definition
	$\vec{r}(t) = \langle (2 + \cos(10t)) \cos(t), (2 + \cos(10t)) \sin(t), \sin(10t) \rangle$
	$\vec{F}(x, y, z) = \langle -y, x, 2 \rangle$
	$\vec{r}(t, s) = \langle (2 + \cos(s)) \cos(t), (2 + \cos(s)) \sin(t), \sin(s) \rangle$
	$x^2 y^2 z^2 = 0$
	$(x - 1)/5 = (y - 2)/10 = (z - 1)/3$
	$\vec{r}(t) = \langle \sin(t) + \cos(5t), \cos(t) + \cos(6t) \rangle$

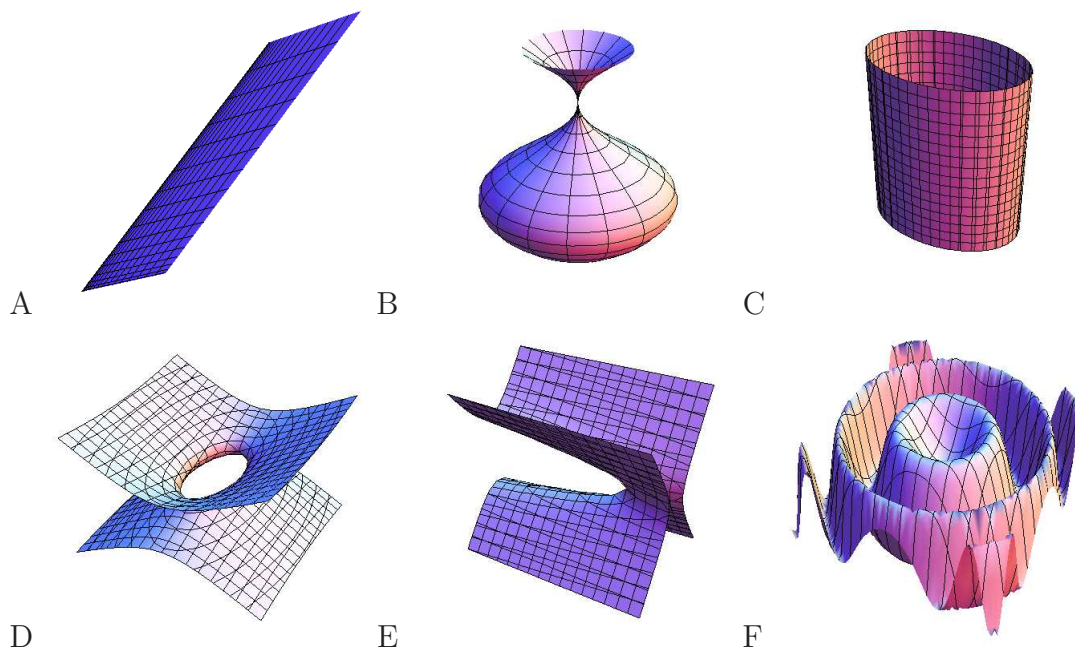
b) (4 points) Match the solids with the triple integrals:

Enter A-D	3D integral computing volume
	$\int_0^{2\pi} \int_0^{\pi/4} \int_0^{1/\cos(\phi)} \rho^2 \sin(\phi) \, d\rho d\phi d\theta$
	$\int_0^{\pi} \int_{\pi/2}^{\pi} \int_0^{\sin(\phi)} \rho^2 \sin(\phi) \, d\rho d\phi d\theta$
	$\int_0^{\pi} \int_{\pi/2}^{\pi} \int_0^1 \rho^2 \sin(\phi) \, d\rho d\phi d\theta$
	$\int_0^{2\pi} \int_0^{\pi} \int_0^{2\pi-\theta} \rho^2 \sin(\phi) \, d\rho d\phi d\theta$



Problem 3) (10 points)

a) (6 points) The surfaces are given either as a parametrization or implicitly. Match them. Each surface matches one definition.

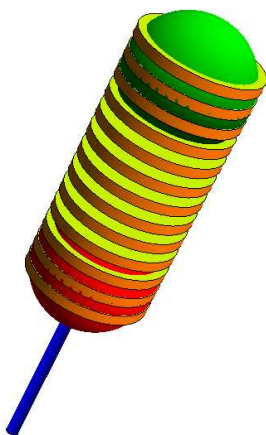


Enter A-F here	Function or parametrization
	$\vec{r}(u, v) = \langle u^2, v^2, u^2 + v^2 \rangle$
	$\vec{r}(u, v) = \langle (1 + \sin(u)) \cos(v), (1 + \sin(u)) \sin(v), u \rangle$
	$4x^2 + y^2 - 9z^2 = 1$
	$x - 9y^2 + 4z^2 = 1$
	$\vec{r}(u, v) = \langle u, v, \sin(u^2 + v^2) \rangle$
	$4x^2 + 9y^2 = 1$

b) (4 points) If the blank box is replaced by $\nabla f(5, 6)$ the statement becomes true or false. Determine which case we have. The function $f(x, y)$ is an arbitrary nice function like for example $f(x, y) = x - yx + y^2$. The curve $\vec{r}(t)$, wherever it appears, parametrizes the level curve $f(x, y) = f(5, 6)$ and has the property that $\vec{r}'(0) = \langle 5, 6 \rangle$.

True/False	Topic	Statement
	Linearization	$L(x, y) = f(5, 6) + \boxed{} \cdot \langle x - 5, y - 6 \rangle$
	Chain rule	$\frac{d}{dt} f(\vec{r}(t)) _{t=0} = \boxed{} \cdot \vec{r}'(0)$
	Steepest descent	f decreases at $(5, 6)$ most in the direction of $\boxed{}$
	Estimation	$f(5 + 0.1, 5.99) \sim f(5, 6) + \boxed{} \cdot \langle 0.1, -0.01 \rangle$
	Directional derivative	$D_{\vec{v}} f(5, 6) = \boxed{} \cdot \vec{v}, \vec{v} = 1$
	Level curve	of f through $(5, 6)$ has the form $\boxed{} \cdot \langle x - 5, y - 6 \rangle = 0$
	Vector projection	of $\nabla f(5, 6)$ onto \vec{v} is $\vec{v}(\vec{v} \cdot \boxed{}) / \vec{v} ^2$
	Tangent line	of $\vec{r}(t)$ at $(5, 6)$ is parametrized by $\vec{R}(s) = \langle 5, 6 \rangle + s \boxed{}$

Problem 4) (10 points)



Two ice cream scoops given by spheres

$$x^2 + y^2 + (z + 1)^2 = 1$$

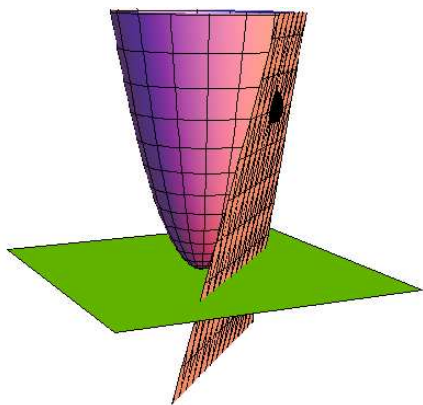
and

$$(x - 1)^2 + (y - 1)^2 + (z - 2)^2 = 1$$

are enclosed by a cylinder which is tangent to both spheres. Find the equation of the cylinder.

Hint: consider the distance of a general point (x, y, z) to the line passing through the centers of the spheres.

Problem 5) (10 points)



Find a parametrization

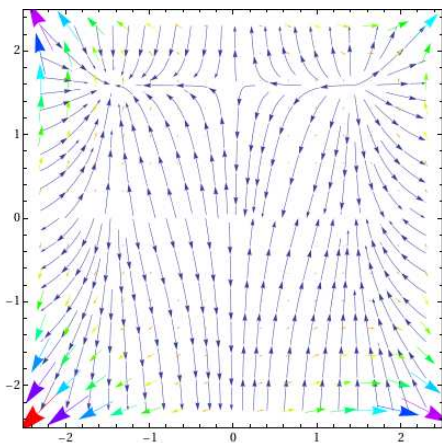
$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$$

for the line obtained by intersecting the tangent plane Σ to the surface

$$x^2 + y^2 - z = 0$$

at $(-1, -1, 2)$ with the xy -plane.

Problem 6) (10 points)



The vector field

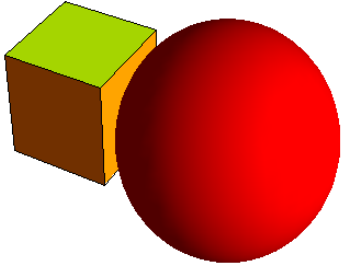
$$\vec{F}(x, y) = \langle P, Q \rangle = \langle y(x^4 - 2x^2), x(y^4 - 4y) \rangle$$

has the curl

$$f(x, y) = \text{curl}(\vec{F})(x, y) = Q_x(x, y) - P_y(x, y) .$$

Find and classify all critical points of f by deciding whether they are local maxima, local minima or saddle points. Is there a global maximum or global minimum of f ?

Problem 7) (10 points)



We want to minimize the volume of the union of a **sphere** of radius x and a **cube** of side length y under the constraint that the sum of the two surface areas is equal to 4. Find the minimal value using the Lagrange method.

Remark: You do not have to show any derivations of the volume and surface area of the sphere.

Problem 8) (10 points)

A solid E in space is determined by the inequalities

$$0 \leq z \leq 9 ,$$

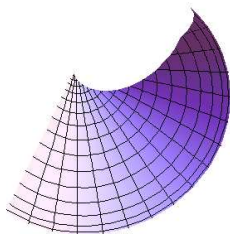
$$z^2 - x^2 - y^2 \geq 4$$

and

$$x^2 + y^2 \leq 1 .$$

Find the volume of E .

Problem 9) (10 points)



A surface S is parametrized by

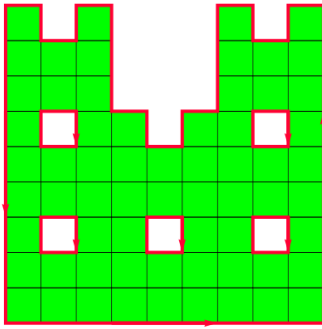
$$\vec{r}(u, v) = e^{-u^2} \langle 1, \sin(v), \cos(v) \rangle$$

where

$$0 \leq u \leq \sqrt{\pi}, u^2 \leq v \leq \pi .$$

Find its surface area.

Problem 10) (10 points)

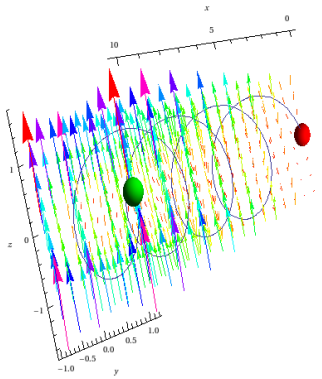


What is the line integral $\int_C \vec{F} \cdot d\vec{r}$ of the vector field

$$\vec{F}(x, y) = \langle 1 + y + 2xy, y^2 + x^2 \rangle$$

along the boundary C of the planar “castle region” shown in the picture? Each of the 5 windows is a unit square and the base of the castle has length 9. The boundary consists of 6 curves which are all oriented so that the region is to the left.

Problem 11) (10 points)



Compute the line integral of the vector field

$$\vec{F}(x, y, z) = \langle \cos(x), 2 + \cos(y), e^z + x(y^2 + z^2) \rangle$$

along the curve $\vec{r}(t) = \langle t, \cos(t), \sin(t) \rangle$ with $0 \leq t \leq 3\pi$.

Hint: you might want to find a split $\vec{F} = \vec{G} + \vec{H}$ and compute line integrals of \vec{G} and \vec{H} separately.

Problem 12) (10 points)



A biker in the Harvard Hemenway gym pedals. Assume that the force of a foot is

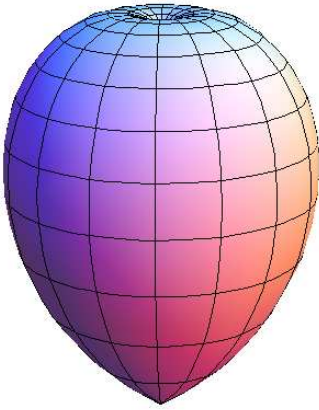
$$\vec{F} = \langle 0, 0, x^3 - x^2 + \sqrt{2 + \sin(z)} \rangle$$

and that one of the feet moves on a path $C : \vec{r}(t) = \langle 2 \cos(t), 0, 2 \sin(t) \rangle$. How much work

$$\int_C \vec{F} \cdot d\vec{r}$$

is done by this foot, when pedaling 10 times which means $0 \leq t \leq 20\pi$?

Problem 13) (10 points)



X-Rays have intensity and direction and are given by a vector field

$$\vec{F}(x, y, z) = \langle z^7, \sin(z) + y + z^{77}, z + \cos(xy) + \sin(y) \rangle .$$

A **tonsil** is given in spherical coordinates as $\rho \leq \phi$. Find the flux of the X-Ray field \vec{F} through the surface $\rho = \phi$ of the tonsil. The surface is oriented with normal vectors pointing outside. **Remark:** The flux is the amount of **ionizing radiation** absorbed by the tissue. This X-ray exposure is measured in the unit **Gray** which corresponds to the radiation amount to deposit 1 **joule** of energy in 1 **kilogram** of matter and corresponds to about 100 **Rem**. A typical dental X-ray is reported to lead to about one tenth to one half of a Rem.