Name:

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MWF 10 Hansheng Diao
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TTH 11:30 Andrew Cotton-Clay

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3, we need to see details of your computation.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 180 minutes time to complete your work.

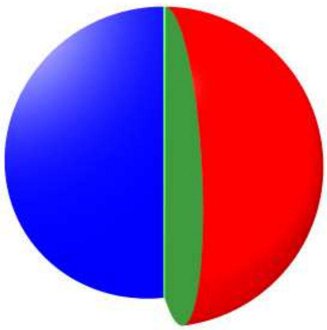
1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
14		10
Total:		150

Problem 1) True/False questions (20 points). No justifications are needed.

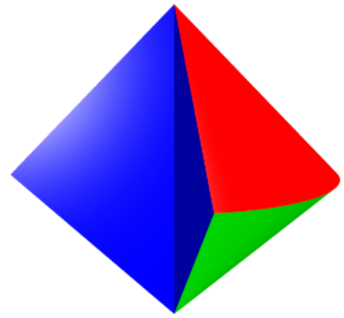
1)	T	F	There are two unit vectors $\vec{v}$, $\vec{w}$ for which the sum $\vec{v}+\vec{w}$ has length 1/3.
2)	T	F	For any three vectors, we have $\|(\vec{u}\times\vec{v})\times\vec{w}\| = \|(\vec{v}\times\vec{w})\times\vec{u}\|$.
3)	T	F	Denote by $d(P,L)$ the distance from a point P to a line L in space. For any point P and any two lines L,K in space, we have $d(P,L)+d(P,K) \geq d(L,K)$.
4)	T	F	For any three vectors $\vec{u},\vec{v},\vec{w}$, the relation $\|\vec{u}\times(\vec{v}\times\vec{w})\| \leq \|\vec{u}\|\|\vec{v}\|\|\vec{w}\|$ holds.
5)	T	F	If $\vec{r}(t)$ has speed 1 and curvature 1 everywhere, then $\vec{r}(2t)$ has constant speed 2 and constant curvature 1/2 everywhere.
6)	T	F	If the curvature of a space curve is constant 1 and the speed $\|\vec{r}\,'(t)\| = 1$ everywhere, then the acceleration satisfies $\|\vec{r}\,''(t)\| = 1$ everywhere.
7)	T	F	If a vector field $\vec{F} = \langle P,Q\rangle$ has $\text{curl}(\vec{F}) = Q_x - P_y = 0$ everywhere and divergence $\text{div}(\vec{F}) = P_x + Q_y = 0$ everywhere, then $\vec{F}$ must be constant.
8)	T	F	If the level curve $f(x,y)=1$ contains both the lines $x=y$ and $x=-y$, then $(0,0)$ must be a critical point for which $D<0$.
9)	T	F	The surface $\vec{r}(u,v) = \langle u^3\cos(v), u^3\sin(v), u^3\rangle$ with $v\in[0,2\pi)$ and $-\infty \leq u \leq \infty$ is a double cone.
10)	T	F	There is a non-constant function $f(x,y,z)$ of three variables such that $\text{div}(\text{grad}(f)) = f$.
11)	T	F	If $\text{curl}(\vec{F}) = \vec{F}$, then the vector field $\vec{F}$ satisfies $\text{div}(\vec{F}) = 0$ everywhere.
12)	T	F	The equation $\phi = \pi/4$ in spherical coordinates defines a half plane.
13)	T	F	The tangent plane of $x^3+y^2+z^4=9$ at $(0,3,0)$ is $y=3$.
14)	T	F	Assume (x_0,y_0) is not a critical point of $f(x,y)$. It is possible that f increases at (x_0,y_0) most rapidly in the direction $\langle 1,0\rangle$ and decreases most rapidly in the direction $\langle 4/5,-3/5\rangle$.
15)	T	F	Assume $\vec{F}(x,y,z)$ is defined everywhere except on the z-axis and satisfies $\text{curl}(\vec{F}) = \vec{0}$ everywhere except on the z-axis, then $\int_C \vec{F}\cdot d\vec{r} = 0$ for all curves C.
16)	T	F	A point (x_0,y_0) is an extremum of $f(x,y)$ under the constraint $g(x,y)=0$. If $D = f_{xx}f_{yy} - f_{xy}^2 > 0$, then (x_0,y_0) can not be a local maximum on the constraint curve.
17)	T	F	The vector field $\vec{F}(x,y,z) = \langle x^2,y^2,z^2\rangle$ can be the curl of another vector field $\vec{G}$.
18)	T	F	If $f(x,y)$ and $g(x,y)$ are two functions and $(2,3,3)$ is a critical point of the function $F(x,y,\lambda) = f(x,y) - \lambda g(x,y)$, then $(2,3)$ is a solution of the Lagrange equations for extremizing $f(x,y)$ under the constraint $g(x,y)=0$.
19)	T	F	Assume $(0,0)$ is a global maximum of $f(x,y)$ on the disc $D = \{x^2+y^2 \leq 1\}$, then $\int\int_D f(x,y)\,dxdy \leq \pi f(0,0)$.
20)	T	F	Let C be a curve parametrized by $\vec{r}(t), 0\leq t\leq 1$ for which the acceleration is constant 1. Then $\int_C \nabla f\cdot d\vec{r}$ is equal to $\int_0^1 D_{\vec{r}\,''(t)}(f(\vec{r}(t)))\,dt$.

Problem 2) (10 points)

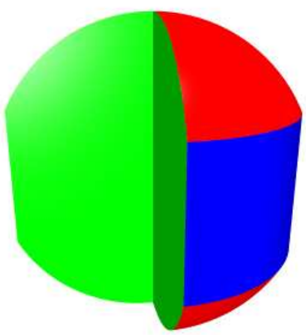
a) (4 points) Match the following triple integrals with the regions.



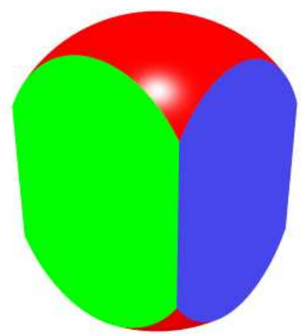
I



II



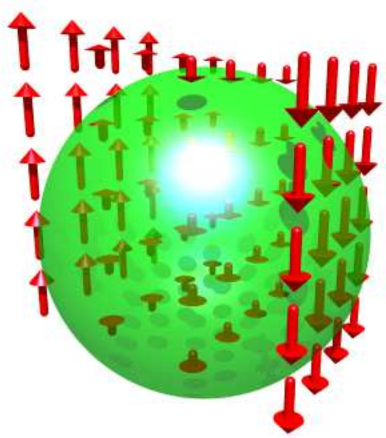
III



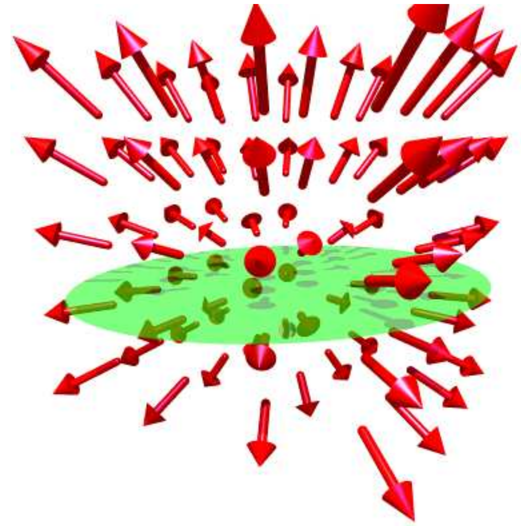
IV

Enter I,II,III,IV here	Equation
	$\int_0^{3\pi/2} \int_0^1 \int_{-\sqrt{2-r^2}}^{\sqrt{2-r^2}} f(r\cos(\theta), r\sin(\theta), z) r \; dzdrd\theta$
	$\int_0^{3\pi/2} \int_0^1 \int_{r-1}^{1-r} f(r\cos(\theta), r\sin(\theta), z) r \; dzdrd\theta$
	$\int_0^{3\pi/2} \int_0^1 \int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} f(r\cos(\theta), r\sin(\theta), z) r \; dzdrd\theta$
	$\int_{-1}^1 \int_{-1}^1 \int_{-\sqrt{2-x^2-y^2}}^{\sqrt{2-x^2-y^2}} f(x,y,z) \; dzdydx$

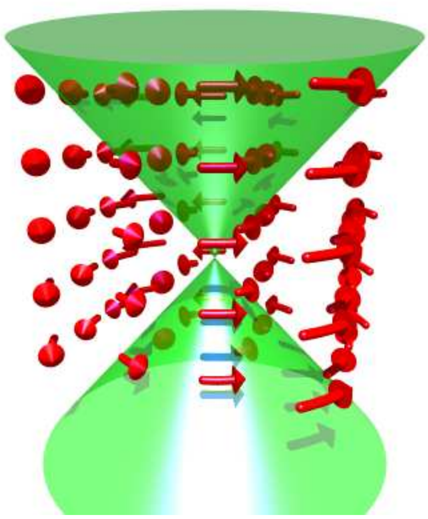
2b) (6 points) Match the following pictures with their vector fields and surfaces. Then check whether the flux integral is zero.



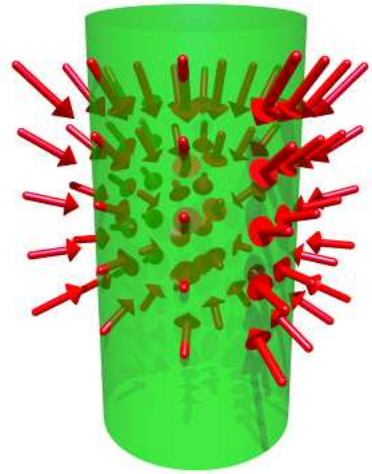
A



B



C

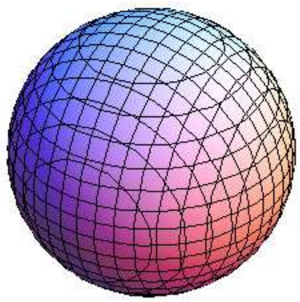


D

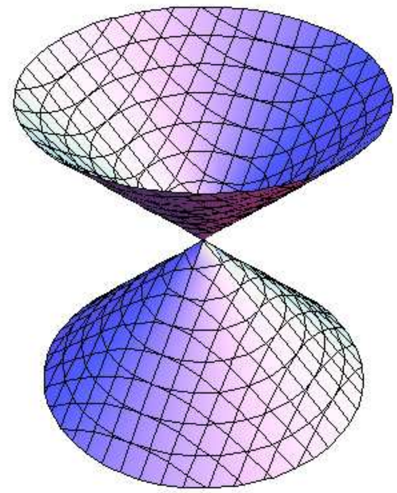
Enter A,B,C,D	Field	Surface	Flux zer
	$\vec{F}(x,y,z) = \langle x, y, z\rangle$	$\vec{r}(u,v) = \langle u, v, 0\rangle$	
	$\vec{F}(x,y,z) = \langle 0, 0, y\rangle$	$\vec{r}(\theta,\phi) = \langle \sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi)\rangle$	
	$\vec{F}(x,y,z) = \langle -x, -y, -z\rangle$	$\vec{r}(\theta,z) = \langle \cos(\theta), \sin(\theta), z\rangle$	
	$\vec{F}(x,y,z) = \langle -y, x, 0\rangle$	$\vec{r}(\theta,z) = \langle z\cos(\theta), z\sin(\theta), z\rangle$	

Problem 3) (10 points)

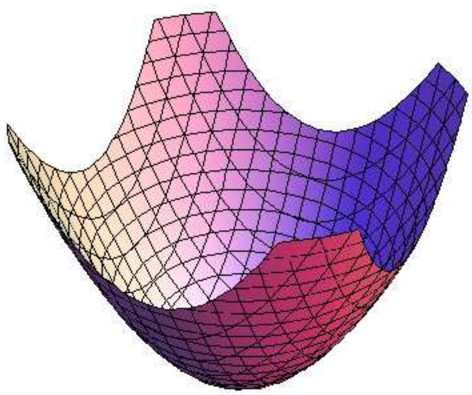
a) (6 points) Match the following level surfaces with functions $f(x,y,z)$ and also match the parametrization of part of the surface $f(x,y,z)=0$.



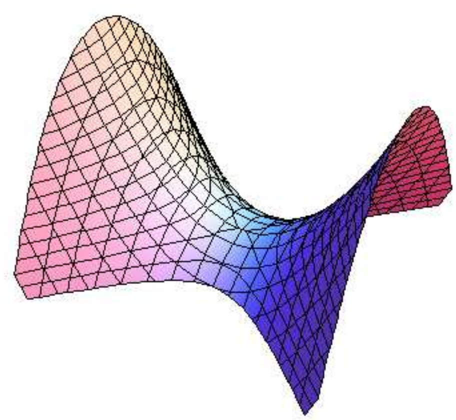
I



II



III



IV

Enter I,II,III,IV	$f(x,y,z)=0$
	$f(x,y,z)=-x^2+y^2+z$
	$f(x,y,z)=x^2+y^2+z^2-1$
	$f(x,y,z)=-x^2-y^2+z$
	$f(x,y,z)=-x^2-y^2+z^2$

Enter I,II,III,IV	parametrization
	$\langle u,v,u^2-v^2\rangle$
	$\langle u,v,u^2+v^2\rangle$
	$\langle u,v,\sqrt{1-u^2-v^2}\rangle$
	$\langle s\cos(t),s\sin(t),s\rangle$

b) (2 points) We know that $\vec{r}\,''(t) = \langle -\cos(t), -\sin(t), 0 \rangle$, $\vec{r}(0) = \langle 2, 3, 4 \rangle$ and $\vec{r}\,'(0) = \langle 0, 1, 1 \rangle$. The expression $\langle \cos(t)+1, \sin(t)+3, t+4 \rangle$ is equal to:

Check which applies	result
	the velocity $\vec{r}\,'(t)$
	the position $\vec{r}(t)$
	the curvature $\kappa(\vec{r}(t))$
	the unit tangent vector $\vec{T}(t)$

c) (2 points) What is the name of the partial differential equation $\text{div}(\text{grad}(f)) = 0$ for $f(x, y)$?

Check which applies	PDE
	Transport equation
	Wave equation
	Heat equation
	Laplace equation

Problem 4) (10 points)

Find the distance between the sphere $(x-4)^2 + y^2 + (z-6)^2 = 1$ and the cylinder of radius 2 around the line $x = y = z$.

Problem 5) (10 points)

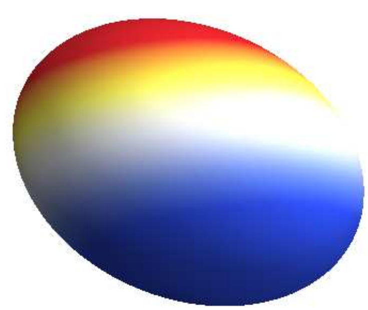
a) (3 points) Find the tangent plane to the surface $S : 4xy - z^2 = 0$ at $(1, 1, 2)$.

b) (4 points) Estimate $4 * 1.001 * 0.99 - 2.001^2$, where $*$ is the usual multiplication.

c) (3 points) Parametrize the line through $(1, 1, 2)$ which is perpendicular to the surface S at $(1, 1, 2)$.

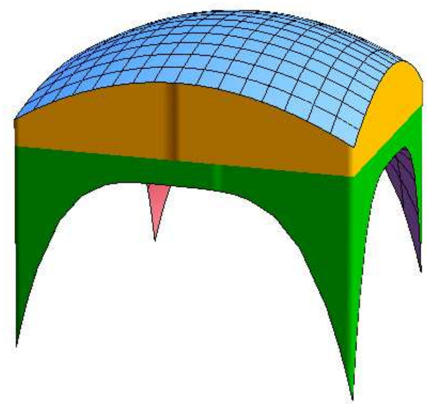
Problem 6) (10 points)

Find the place on the elliptical **asteroid** surface $g(x, y, z) = 5x^2 + y^2 + 3z^2 = 9$, where the temperature $f(x, y, z) = 750 + 5x - 2y + 9z$ is maximal.



Problem 7) (10 points)

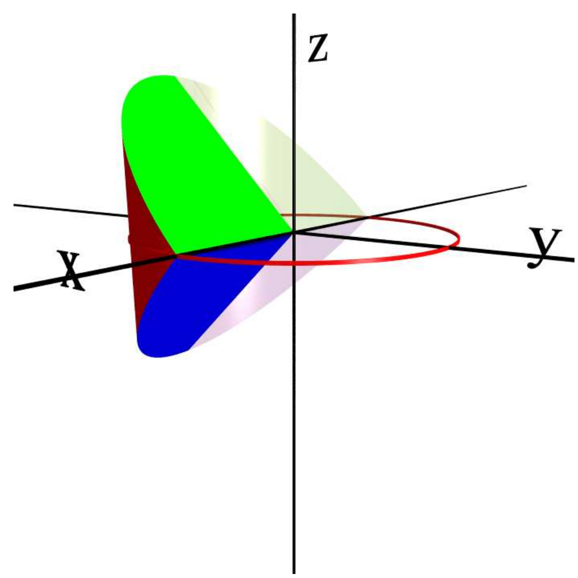
The thickness of the region enclosed by the two graphs $f_1(x,y) = 10 - 2x^2 - 2y^2$ and $f_2(x,y) = -x^4 - y^4 - 2$ is denoted by $f(x,y) = f_1(x,y) - f_2(x,y)$. Classify all critical points of f and find the global minimal thickness.



To the picture: over a square domain, the region looks like a chair. In the problem you consider the function over the entire plane.

Problem 8) (10 points)

Find the volume of the solid piece of **cheese** bound by the cylinder $x^2 + y^2 = 1$, the planes $y - z = 0$ (bottom boundary) and $y + z = 0$ (top boundary) which is on the quadrant $x \geq 0$ and $y \leq 0$.



Problem 9) (10 points)

Compute the surface area of the **Tsai** surface which is parametrized by

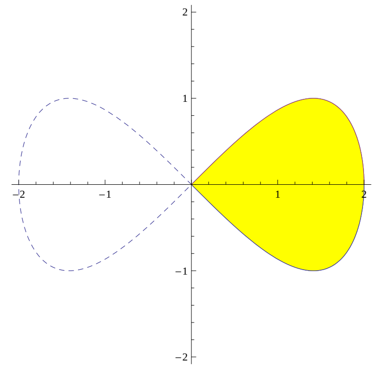
$$\vec{r}(u,v) = \langle 3u + 2v, 4u + v, \frac{2}{7}v^{\frac{7}{2}} \rangle \ ,$$

where $0 \leq u \leq 1$ and $u^{1/4} \leq v \leq 1$.

Problem 10) (10 points)

Find the area $\int\int_R 1 \; dxdy$ of the region R inside the right leaf of the **Gerono lemniscate** $x^4 = 4(x^2 - y^2)$ which has the parametrization

$$\vec{r}(t) = \langle 2\sin(t), 2\sin(t)\cos(t) \rangle \; .$$



Problem 11) (10 points)

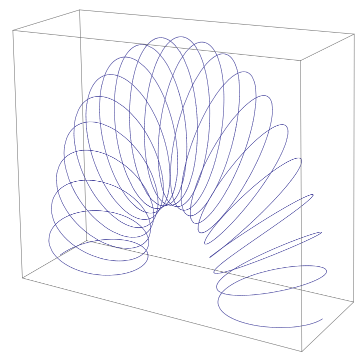
Find the line integral of the vector field

$\vec{F}(x,y,z) = \langle \cos(x+z), 2yze^{y^2z} + 7, \cos(x+z) + y^2e^{y^2z} \rangle$

along the **slinky** curve

$\vec{r}(t) = \langle \sin(40t), (2+\cos(40t))\cos(t), (2+\cos(40t))\sin(t) \rangle$

with $0 \leq t \leq \pi$.



Problem 12) (10 points)

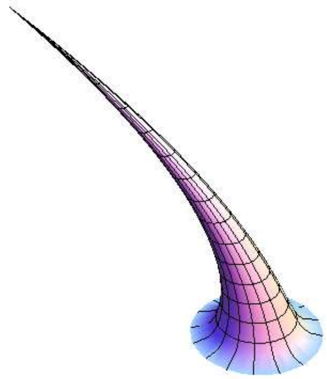
Find the flux integral $\int\int_S \mathrm{curl}(\vec{F}) \cdot d\vec{S}$, where

$$\vec{F}(x,y,z) = \langle 2\cos(\pi y)e^{2x} + z^2, x^2\cos(z\pi/2) - \pi\sin(\pi y)e^{2x}, 2xz \rangle$$

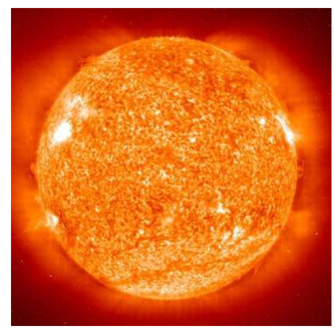
and S is the **thorn** surface parametrized by

$$\vec{r}(s,t) = \langle (1 - s^{1/3})\cos(t) - 4s^2, (1 - s^{1/3})\sin(t), 5s \rangle$$

with $0 \leq t \leq 2\pi, 0 \leq s \leq 1$ and oriented so that the normal vectors point to the outside of the thorn.



Problem 13) (10 points)

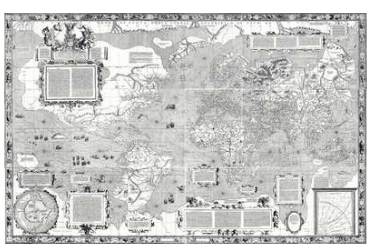


Assume the vector field

$\vec{F}(x,y,z) = \langle 5x^3 + 12xy^2, y^3 + e^y \sin(z), 5z^3 + e^y \cos(z) \rangle$

is the magnetic field of the **sun** whose surface is a sphere of radius 3 oriented with the outward orientation. Compute the magnetic flux $\int \int_S \vec{F} \cdot d\vec{S}$.

Problem 14) (10 points)



The **Mercator projection** is one of the most famous map projections. It was invented in 1569 and used for nautical voyages. The inverse of the projection is the parametrization of the sphere as

$$\vec{r}(u,v) = \langle \cos(u)\cos(\arctan(\sinh(v))), \sin(u)\cos(\arctan(\sinh(v))), \sin(\arctan(\sinh(v))) \rangle \ .$$

a) (3 points) Show that $|\vec{r}(u,v)| = 1$ verifying so that $\vec{r}(u,v)$ parametrizes the unit sphere, if $0 \leq u < 2\pi, -\infty < v < \infty$.
b) (3 points) Show that $|\vec{r}_u(u,v)| = |\vec{r}_v(u,v)| = 1/\cosh(v)$ and that $\vec{r}_u(u,v) \cdot \vec{r}_v(u,v) = 0$.
c) (2 points) Use *b*) to show that $|\vec{r}_u \times \vec{r}_v| = 1/\cosh(x)^2$.
d) (2 points) Use $\int 1/\cosh^2(x)\ dx = 2\arctan(\tanh(x/2)) + C$ to see that the surface area of the unit sphere is 4π.

Hint for b): you can use the identity $\cos(\arctan(\sinh(v)) = 1/\cosh(v)$.