Name:

MWF 9 Oliver Knill
MWF 10 Hansheng Diao
MWF 10 Joe Rabinoff
MWF 11 John Hall
MWF 11 Meredith Hegg
MWF 12 Charmaine Sia
TTH 10 Bence Béky
TTH 10 Gijs Heuts
TTH 11:30 Francesco Cavazzani
TTH 11:30 Andrew Cotton-Clay

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- All unspecified functions which appear are nice and differentiable.
- **Show your work**. Except for problems 1-3, we need to see details of your computation.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

Problem 1) True/False questions (20 points). No justifications are needed.

1) T F For any two vectors $\vec{u}$ and $\vec{v}$ we have $|\vec{u}| \leq |\vec{v}| + |\vec{v} - \vec{u}|$.

Solution:
This is the triangle inequality.

2) T F The grid curves of a parametric surface are always perpendicular to each other at any point on the surface.

Solution:
Parametrize a plane with two nonperpendicular vectors.

3) T F The triple scalar product $\vec{u} \cdot (\vec{v} \times \vec{w})$ of $\vec{u} = \langle 1, 0, 0 \rangle, \vec{v} = \langle 0, 1, 0 \rangle$ and $\vec{w} = \langle 2, 1, 1 \rangle$ is equal to 1.

Solution:
Compute it

4) T F For $f(x, y) = x^4 + y^4$ and $\vec{r}(t) = \langle t, t^2 \rangle$, we have $\frac{d}{dt} f(\vec{r}(t)) = \langle 4t^3, 4t^6 \rangle \cdot \langle 1, 2t \rangle$.

Solution:
This is the chain rule.

5) T F The flux of $\vec{F} = \langle x, 0, 0 \rangle$ through the outwardly-oriented boundary S of a parallelepiped spanned by edges $\langle 1, 0, 0 \rangle, \langle 0, 2, 0 \rangle, \langle 1, 1, 3 \rangle$ is equal to 6.

Solution:
It is by the divergence theorem equal to the volume.

6) T F The differential equation $u_x = u_t$ for a function $u(x, t)$ is called the heat equation.

Solution:
It is called the transport equation

7) | T | **F** | If a function $f(x,y)$ has a critical point at $(0,0)$ then div(grad(f))$(0,0)$ is zero.

Solution:
Take $f(x,y) = x^2 + y^2$. This is a counter example.

8) | T | **F** | A function of two variables always has an odd number of critical points.

Solution:
There can be any number of critical points. Also zero.

9) | T | **F** | If $f(x,y)$ is a function of two variables and $(0,0)$ is a maximum of $g(x) = f(x,0)$ and as well as a maximum of $h(y) = f(0,y)$ then $(0,0)$ is a maximum of f.

Solution:
$f_{xx} >$ and $f_{yy} > 0$ does not imply $D = f_{xx}f_{yy} - f_{xy}^2$ is positive.

10) | T | **F** | The divergence of a vector field $\vec{F}$ is always equal to the divergence of the curl of $\vec{F}$.

Solution:
Take $\vec{f}(x,y,z) = \langle x,y,z\rangle$. The curl is zero but the divergence is 3.

11) | **T** | F | The flux of a vector field $\vec{F}$ of length $|\vec{F}| = 1$ through a triangular surface S can not be larger than the surface area of the triangle.

Solution:
Use $|\vec{F}(r(u,v) \cdot \vec{r}_u \times \vec{r}_v| \leq |\vec{F}(\vec{r}(u,v))||\vec{r}_u \times \vec{r}_v| = |\vec{r}_u \times \vec{r}_v|$.

12) | **T** | F | The arc length of the boundary of a surface is independent of the parametrization of the surface.

Solution:
The arc length is independent of parametrization in general.

13) | T | **F** | The vector field $\text{curl}(\vec{F})$ is at every point (x, y, z) perpendicular to $\vec{F}(x, y, z)$.

Solution:
A counter example is $\langle 0, x, 1 \rangle$ which has curl $\langle 0, 0, 1 \rangle$.

14) | T | **F** | The scalar function $f = \text{div}(\vec{F})$ has the property that $\text{grad}(f(x, y, z))$ is perpendicular to $\vec{F}(x, y, z)$.

Solution:
Take $\vec{F}(x, y, z) = \langle x^2, 0, 0 \rangle$ which has the divergence $2x$, the gradient of which is $\langle 1, 0, 0 \rangle$.

15) | **T** | F | The Lagrange equations $\nabla f(x, y) = \lambda \nabla g(x, y), g(x, y) = x^2 + y^2 = 1$ have infinitely many solutions if $f = g$.

Solution:
All the points are critical points.

16) | **T** | F | If a vector is perpendicular to itself, then it is the zero vector.

Solution:
Yes, $\vec{v} \cdot \vec{v} = 0$ implies $|\vec{v}| = 0$.

17) | **T** | F | The gradient of the divergence of the curl of a vector field $\vec{F}$ is the vector field which assigns the zero vector to each point.

Solution:
Take $\vec{F}(x, y, z) = \langle x^2, y^2, z^2 \rangle$. It has divergence $2x + 2y + 2z$ and the gradient $\langle 2, 2, 2 \rangle$.

18) | T | **F** | The identity $\text{Proj}_{\vec{v}}(\vec{w}) = \text{Proj}_{\vec{w}}(\vec{v})$ holds for all vectors $\vec{v}, \vec{w}$.

Solution:
If the two vectors are not parallel, then the projection vectors are not even parallel.

19) T F (F highlighted) The function $f(x,y) = \sin(xy)$ is a solution to the Laplace equation $f_{xx} + f_{yy} = 0$.

Solution:
Differentiate and see it is not the same.

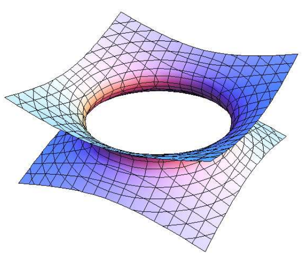
20) T F (T highlighted) The formula $\vec{r}(u,v) = \langle 2u, (9+u^2)\cos(v), (9+u^2)\sin(v)\rangle$ gives a parametrization of a one-sheeted hyperboloid.

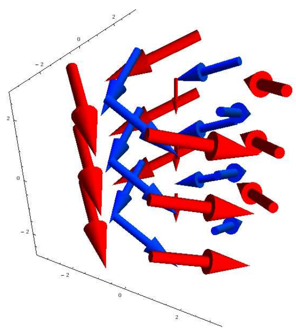
Solution:
$y^2 + z^2 = 9 + u^2 = 9 + (x/2)^2$.

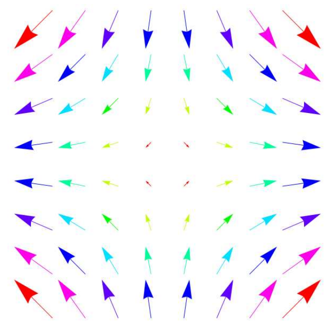
Problem 2) (10 points)

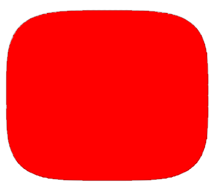
a) (6 points) Match the following objects.

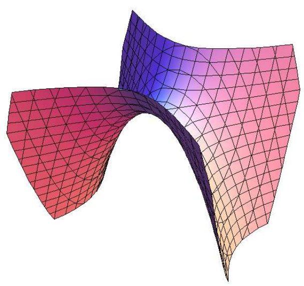
Formula	Enter 1-6
$\rho(\phi,\theta) \leq e^{\phi}$	
$\vec{F}(x,y,z) = \langle -y, x, -2 \rangle$	
$x^2 + y^2 - 5z^2 = 1$	
$z + x^2 - y^2 = 2$	
$\vec{F}(x,y) = \langle x, -y \rangle$	
$x^4 + 2y^4 \leq 3$	

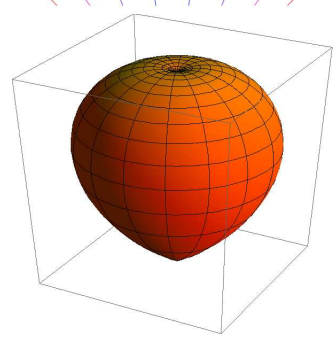

1


2


3

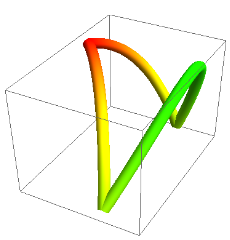

4

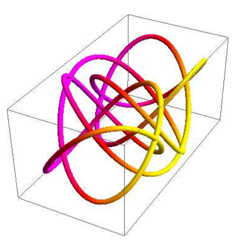

5

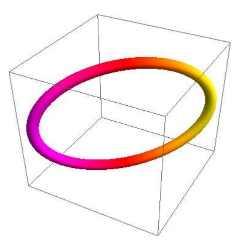

6

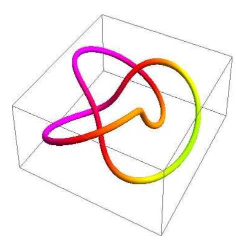
b) (4 points) A knot is a closed curve in space. Match the following knots

Formula	Enter A,B,C,D
$\vec{r}(t) = \langle \cos(5t), \cos(t) + \sin(5t), \cos(7t) \rangle$	
$\vec{r}(t) = \langle |\cos(t)|, |\sin(t) + \cos(t)|, |\cos(2t)| \rangle$	
$\vec{r}(t) = \langle (2 + \cos(\frac{3t}{2})) \cos(t), (2 + \cos(\frac{3t}{2})) \sin(t), \sin(\frac{3t}{2}) \rangle$	
$\vec{r}(t) = \langle \cos(t), \cos(t), \sin(t) \rangle$	


A


B


C


D

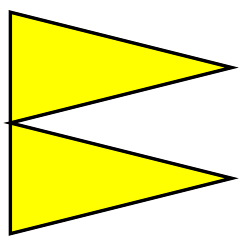
Solution:
a) 6,2,1,5,3,4
b) B,A,D,C

Problem 3) (10 points)

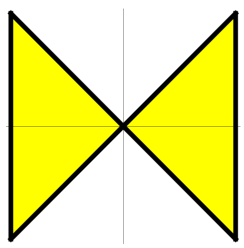
a) (4 points) It is Hobbit time. The following regions resemble ancient runes of the Anglo Saxons studied by JRR Tolkien. (A is "b", B is "d", C is "st", and D is "oe").

Integral	Enter A,B,C,D
$\int_{-1}^{1} \int_{-\lvert x\rvert}^{\lvert x\rvert} f(x,y)\ dydx$	
$\int_{-1}^{1} \int_{-1-\lvert x\rvert/2}^{1+\lvert x\rvert/2} f(x,y)\ dydx$	
$\int_{-1}^{1} \int_{0}^{\lvert\lvert y+\frac{1}{2}\rvert-\lvert y-\frac{1}{2}\rvert\rvert-\lvert y\rvert} f(x,y)\ dxdy$	
$\int_{-1}^{1} \int_{\lvert y\rvert-1}^{1-\lvert y\rvert} f(x,y)\ dxdy$	

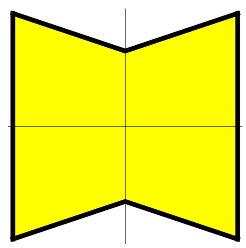
[1]



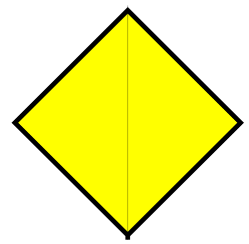
A



B



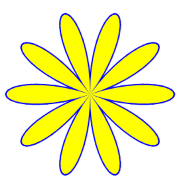
C



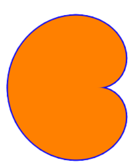
D

b) (4 points) Matching polar regions

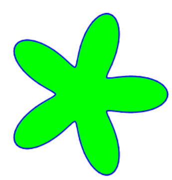
Formula	Enter E,F,G,H
$r \leq \theta(2\pi - \theta)$	
$r \leq \lvert\cos(5\theta)\rvert$	
$r \leq 2 + \cos(5\theta)$	
$r \leq 2\pi - \theta$	



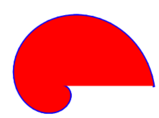
E



F



G



H

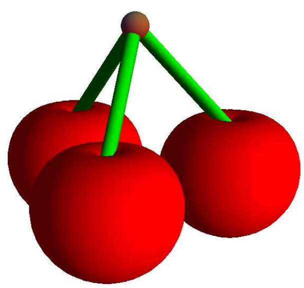
c) (2 points) Which derivatives and integrals do appear in the statements of the following theorems? Check each box which applies. Multiple entries are allowed in each row or column.

Integral theorem	Grad	Curl	Div	Line integral	Flux integral
Divergence theorem					
Stokes' theorem					

[1] More information on http://www.theshorterword.com/anglo-saxon-runes

Solution:
a) B,C,A,D
b) F,E,G,H
d) Divergence theorem: divergence and flux integral.
Stokes theorem: curl, line integral and flux integral.

Problem 4) (10 points)



Three cherries have centers at $A = (-1,-1,-1)$, $B = (1,0,-2)$ and $C = (0,1,-2)$ and are tied together at the origin $O = (0,0,0)$.[2]
Find the distance between O and the plane through A, B, C.

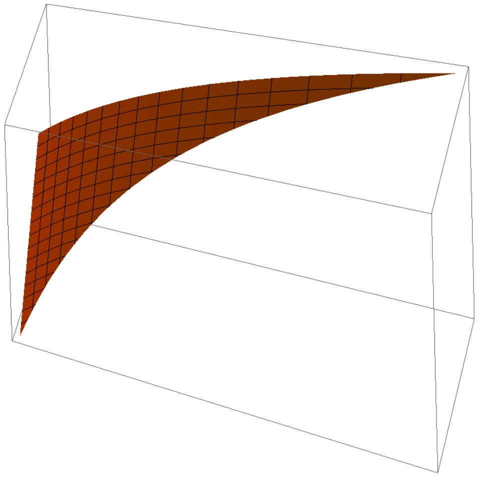
Solution:
The vector $\vec{n} = \vec{AB} \times \vec{AC} = \langle 1,1,3 \rangle$ is normal to the plane. The vector $\vec{AO} = \langle 1,0,-2 \rangle$ connects a O with a point on the plane. We have

$$d = \frac{|\vec{AO} \cdot \vec{n}|}{|\vec{n}|} = \langle 1,1,3 \rangle \cdot \langle 1,1,1 \rangle / \sqrt{11} = 5/\sqrt{11} \; .$$

The distance is $\boxed{5/\sqrt{11}}$.

Problem 5) (10 points)

[2]Picture from Mathematica project by Alissa Zhang



Find the surface area of the surface

$$\vec{r}(u,v) = \langle u^2 + v, u, v \rangle$$

for which $0 \leq v \leq 4$ and $\frac{v}{4} \leq u \leq 1$.

Solution:
We compute $\vec{r}_u = \langle 2u, 1, 0 \rangle, \vec{r}_v = \langle 1, 0, 1 \rangle$. Their cross product is $\langle 1, -2u, -1 \rangle$ which has length $\sqrt{2+4u^2}$. To integrate

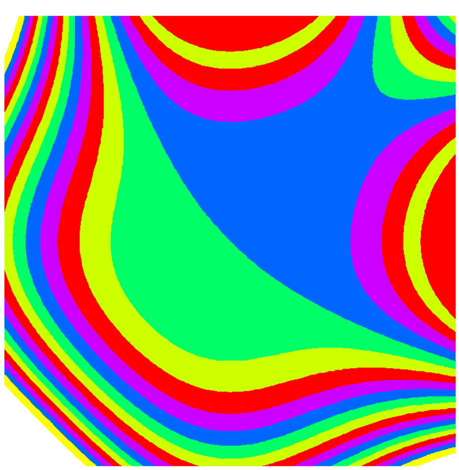
$$\int_0^4 \int_{v/4}^1 \sqrt{2+4u^2}\, dudv ,$$

we draw the triangle in the uv plane over which we integrate, then change the order of integration

$$\int_0^1 \int_0^{4u} \sqrt{2+4u^2}\, dvdu = \int_0^1 4u\sqrt{4u^2+2}\, du = (6^{3/2} - 2^{3/2})/3 .$$

It could also be written as $\boxed{2\sqrt{6} - \frac{2}{3}\sqrt{2}}$.

Problem 6) (10 points)



The function $f(x,y) = 2x^3 + 2y^3 - 3x^2y^2$ is called the "happy function" as you can see when you turn your head clockwise by $\pi/4$. Find and classify its extrema.

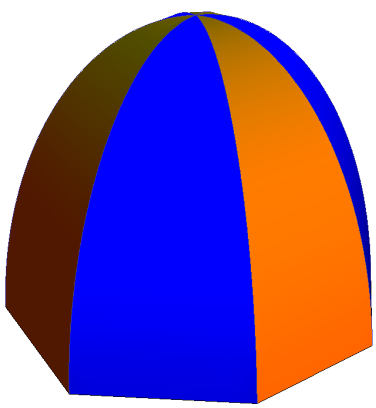
In one of the critical points, the discriminant D is zero. We want you nevertheless to decide whether this point is a "local maximum" a "local minimum" or "neither of them".

Solution:
The gradient is $\langle 5x^2 - 6xy^2, 6y^2 - 6x^2y \rangle$. If one of the x, y is zero, both are and $(0,0)$ is a critical point. If none are zero, then $x = y^2$ and $y = x^2$. Plugging in the second to the first gives $x = x^4$ and so $x = 1$. The discriminant D is

$$D = f_{xx}f_{yy} - f_{xy}^2 = (12x - 6y^2)(12y - 6x^2) - (-12xy)^2 .$$

At the first critical point $(0,0)$ this is zero. At the second critical point $(1,1)$, then $D = 36 - 144 < 0$ and the point is a saddle point.
To analyze the behavior at $(0,0)$, set $y = 0$ to see that $f(x,0) = 2x^3$. This function takes both positive and negative values arbitrarily close to 0. It is neither a local maximum, nor a local minimum. To summarize, $\boxed{\text{there is one saddle point } (1,1)}$. It is at the nose of the face. Furthermore, the point $(0,0)$ with $D = 0$ is **neither maximum,nor minimum.** This is the critical point on the lips.

Problem 7) (10 points)



Archimedes computed volumes of solids which now bear his name. He showed that, as for the sphere, each "Archimedean globe" has volume equal to two thirds of the prism in which it is inscribed. Later it was discovered that also the surface area is two thirds of the surface area of a circumscribing prism. To find globes of minimal surface we are led to the problem:

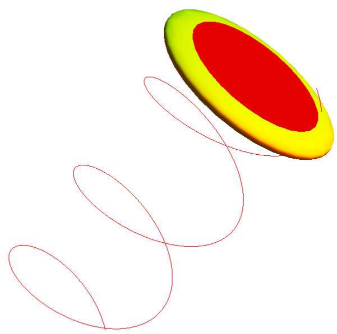
Find values of r, h satisfying $g(r,h) = r^2h = 3$ so that $f(r,h) = 3r^2 + 2rh$ is minimal.

Solution:
The Lagrange equations are

$$\begin{aligned} 6r + 2h &= \lambda 2rh \\ 2r &= \lambda r^2 \\ r^2h &= 3 . \end{aligned}$$

Eliminating λ gives $3r = h$. Plugging into the third equation gives $\boxed{r = 1 \text{ and } h = 3}$.

Problem 8) (10 points)



The frisbee was invented by Harvard students in 1845 when a student threw a cake plate to George Frisbie Hoar and shouted "Frisbie, catch!".[3]

A point on the outer rim of a frisbee moves on a curve $\vec{r}(t)$ satisfying

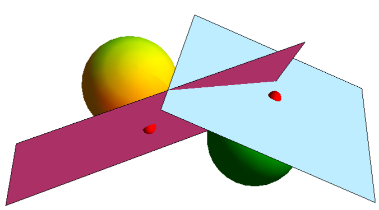
$$\vec{r}\,''(t) = \langle 0, -\cos(t), -\sin(t) \rangle \ .$$

We know that $\vec{r}(0) = \langle 0, 1, 0 \rangle$ and $\vec{r}\,'(0) = \langle 1, 0, 1 \rangle$. Find $\vec{r}(t)$ and the arc length of the curve $\vec{r}(t)$ from $t = 0$ to $t = 2\pi$.

Solution:

$\vec{r}\,''(t) = \langle 0, -\cos(t), -\sin(t) \rangle$
$\vec{r}\,'(t) = \langle 1, -\sin(t), \cos(t) \rangle$
$\vec{r}(t) = \langle t, \cos(t), \sin(t) \rangle$ The speed is $\sqrt{2}$. The arc length is $\sqrt{2}(2\pi)$ which is $\boxed{\sqrt{8}\pi}$.

Problem 9) (10 points)



Find a parametrization of the line of intersection of the tangent plane at the point $(1, -1, 0)$ of the sphere

$$x^2 + y^2 + z^2 = 2$$

and the tangent plane to the point $(5, 1, 1)$ of the sphere

$$(x - 5)^2 + y^2 + z^2 = 2 \ .$$

[3]Some claim this incident happened at Yale, which is a fairy tale. **George Frisbie Hoar** graduated from Harvard in 1846 and became later a United States Senator. He fought against political corruption, and campaigned for the rights of African and Native Americans.

Solution:
The tangent plane to the first is obtained by computing the gradient at $(1, -1, 0)$ and fixing the constant

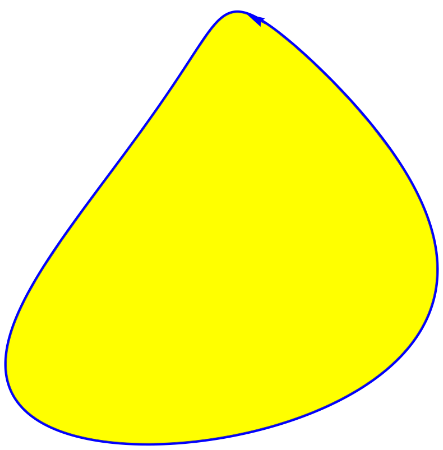
$$2x - 2y = 4$$

The tangent plane to the second is

$$2y + 2z = 4 \ .$$

To find the intersecting line, we find two poins like $(2, 0, 2)$ or $(0, -2, 4)$ on the line and parametrize $\boxed{\vec{r}(t) = \langle 2, 0, 2\rangle + t\langle -2, -2, 2\rangle}$. Note that there are many solutions. An other convenient solution is to take the cross product of $\langle 2, -2, 0\rangle$ and $\langle 0, 2, 2\rangle$ to get a vector $\langle -4, -4, 4\rangle$ in the intersection and to find one point in the intersection.

Problem 10) (10 points)



Find the area of the region enclosed by the curve

$$\vec{r}(t) = \langle 3\cos(t) - \sin(2t), 4\sin(t) + \cos(t)\rangle \ ,$$

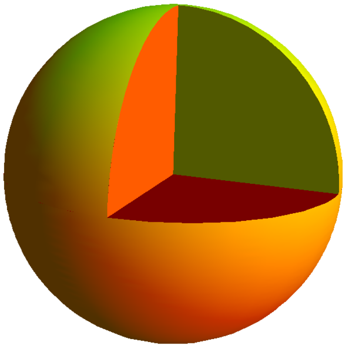
where $0 \leq t \leq 2\pi$.

Solution:
We use Greens theorem using $\vec{F} = \langle 0, x\rangle$. We get

$$\begin{aligned} \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) \ dt &= \int_0^{2\pi} \langle 0, 3\cos(t) - \sin(2t)\rangle \cdot \langle -3\sin(t) + 2\cos(2t), 4\cos(t) - \sin(t)\rangle \\ &= \int_0^{2} \pi 12\cos^2(t) \ dt = 12\pi \ . \end{aligned}$$

The answer is $\boxed{12\pi}$.

Problem 11) (10 points)



Find the flux of the vector field

$$\vec{F}(x,y,z) = \langle x^3 + x^2, y^3 - xy, z^3 - xz \rangle$$

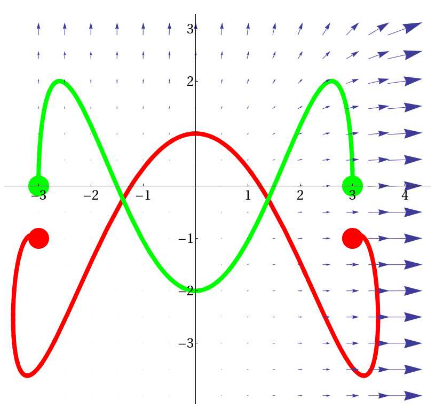
through the boundary surface of E (oriented outwards), where the solid E is a unit sphere from which the first octant has been removed.

Solution:
The divergence is $3x^2 + 3y^2 + 3z^2 = 3\rho^2$. By symmetry, we can compute $\int\int\int_E 3\rho^2 \, dV$ over the entire sphere and multiply with 7/8.

$$(7/8) \int_0^{2\pi} \int_0^{\pi} \int_0^1 3\rho^2 \; \rho^2 \sin(\phi) \; d\rho d\phi d\theta = (7/8) 2\pi 23/5 = 21\pi/10 \; .$$

Thw answer is $\boxed{\frac{21\pi}{10}}$

Problem 12) (10 points)



Assume the wind velocity on the Charles is

$$\vec{F}(x,y) = \langle e^x, e^y \rangle \; .$$

A sail boat takes the path

$C_1 : \vec{r}(t) = \langle -3\cos(t) - \sin(2t), 2\sin(t) + 2\cos(4t) - 3 \rangle$

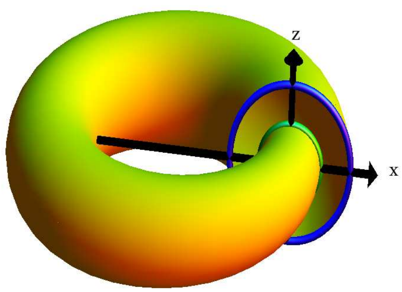
from $(-3,-1)$ to $(3,-1)$. An other boat follows the path

$$C_2 : \vec{r}(t) = \langle -3\cos(t), 2\sin(3t) \rangle$$

from $(-3,0)$ to $(3,0)$. To find out which path needs more energy, compute both line integrals $\int_{C_1} \vec{F} \cdot \vec{dr}$ and $\int_{C_2} \vec{F} \cdot \vec{dr}$.

Solution:
The vector field is a gradient field with potential $f(x,y) = e^x + e^y$. By the fundamental theorem of line integrals, we can just compute the potential at the beginning and end points. The first integral is $f(3,-1) - f(-3,-1) = e^3 - e^{-3}$. The second is $f(-3,0) - f(3,0) = e^3 - e^{-3}$. The two line integrals are the same. The answer in both cases is $\boxed{e^3 - e^{-3}}$.

Problem 13) (10 points)



The "foot in the mouth" surface S seen in the picture is parametrized by

$$\vec{r}(u,v) = \langle \begin{array}{l} (4 + g(u,v)\cos(v))\cos(u) - 4, \\ (4 + g(u,v)\cos(v))\sin(u), \\ g(u,v)\sin(v)\rangle \end{array}$$

with $g(u,v) = (2 - \frac{u}{2\pi})$ and $0 \leq u, v \leq 2\pi$. It is oriented outwards. Its boundary consists of two circles in the xz-plane centered at the origin, with radius 1 and 2. Find the flux of the curl of

$$\vec{F}(x,y,z) = \langle z + yz, x, \sin(x^3 y) + y^2 + z^4 \rangle$$

through S.

Solution:
We use Stokes theorem. The boundary consists of two curves. The outer one is oriented clockwise the inner one counter clockwise.

$$\vec{r}_1(t) = \langle 2\cos(t), 0, -2\sin(t) \rangle$$

$$\vec{r}_2(t) = \langle \cos(t), 0, \sin(t) \rangle$$

In the xz-plane, we have $y = 0$ and $\vec{F} = \langle z, x, z^4 \rangle$. Write down the two line integrals

$$\int_{C_1} \vec{F} \cdot \, \vec{dr} = \int_0^{2\pi} \langle -2\sin(t), 2\cos(t), 4\sin(t)^2 \rangle \cdot \langle -2\sin(t), 0, 2\cos(t) \rangle \; dt = 4\pi$$

$$\int_{C_2} \vec{F} \cdot \, \vec{dr} = \int_0^{2\pi} \langle -\sin(t), \cos(t), \sin(t)^2 \rangle \cdot \langle \sin(t), 0, -\cos(t) \rangle \; dt = -\pi$$

The answer is $\boxed{3\pi}$.