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- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- All unspecified functions which appear are nice and differentiable.
- **Show your work**. Except for problems 1-3, we need to see details of your computation.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
14		10
Total:		150

Problem 1) True/False questions (20 points). No justifications are needed.

1) T F The tangent plane of the surface $z = f(x, y)$ at a local maximum of f is parallel to the xy-plane

Solution:
The tangent plane to the function $g(x, y, z) = z - f(x, y)$ can be written $\langle -f_x, -f_y, 1\rangle$ which is $\langle 0, 0, 1\rangle$ at critical points.

2) T F For any smooth functions $f(x, y), x(t), y(t)$, we have $\frac{d}{dt}f(x(t), y(t)) = f_x(x(t), y(t))x'(t) + f_y(x(t), y(t))y'(t)$.

Solution:
This is the chain rule.

3) T F At a local maximum of a function $f(x, y)$ we always have $f_{xx} \leq 0$ and $f_{yy} \leq 0$.

Solution:
Indeed if one of the conditions is not satisfied, then the function increases in some direction.

4) T F If $(0, 0)$ is a critical point for a function $f(x, y)$ as well as for a function $g(x, y)$ then $(0, 0)$ is a critical point of the function $f(x, y) + g(x, y)$.

Solution:
The gradient of the sum is also zero at $(0, 0)$.

5) T F The curves $\vec{r}(t) = \langle t, 2t\rangle$ and $\vec{s}(t) = \langle 2t, -t\rangle$ intersect at a right angle at $(0, 0)$.

Solution:
The velocity vectors are perpendicular at $(0, 0)$.

6) T F The quadric $x - y^2 + z^2 = 5$ is a hyperbolic paraboloid.

Solution:
It is, shifted and turned a bit

7) | T | F |

If $\vec{u}, \vec{v}, \vec{w}$ are unit vectors, then the length of the vector projection of $\vec{u} \times \vec{v}$ onto $\vec{w}$ is the same as the length of the vector projection of $\vec{v} \times \vec{w}$ onto $\vec{u}$.

Solution:
In both cases, we have the same triple scalar product

8) | T | F |

The partial differential equation $u_{tt} = u_{xx}$ is called the Clairaut equation.

Solution:
This is the wave equation

9) | T | F |

$\iint_R \sqrt{1 - x^2 - y^2}\ dx\ dy = \frac{2\pi}{3}$, where R is the region $\{(x, y) \mid x^2 + y^2 \leq 1\ \}$ in the xy-plane.

Solution:
The integral is the volume of half of the unit sphere.

10) | T | F |

There exists a vector field $\vec{F}(x, y, z)$ in space such that $\mathrm{curl}(\vec{F}) = \langle 5x, -11y, 7z \rangle$.

Solution:
The divergence would have to be zero.

11) | T | F |

Let S is the upper hemisphere $x^2 + y^2 + z^2 = 1, z \geq 0$ with normal pointing away from the center. Then the flux integral is $\iint_S \langle 0, 0, 1 \rangle \cdot\ d\vec{S} = 2\pi$.

Solution:
The flux integral is the same than the flux integral through the disc by the divergence theorem which is π, not 2π.

12) | T | F |

The points that satisfy $\theta = \pi/4$ and $\phi = \pi/4$ form a surface which is part of a cone.

Solution:
This is a curve, not a surface

13) | T | F | (T highlighted)

The curvature of the curve $\vec{r}(t) = \langle t, t, t^2 \rangle$ at $t = 0$ is equal to the curvature of the curve $\vec{s}(t) = \langle t^3, t^3, t^6 \rangle$ at $t = 0$.

Solution:
Different parametrization of the same curve

14) | T | F | (F highlighted)

If $f(x, y, z)$ is a function and $\vec{F} = \nabla f$ then $\text{div}(\vec{F}) = 0$ everywhere (i.e. $\vec{F}$ is incompressible).

Solution:
No, the divergence of the gradient is the Laplacian and not encessarily zero.

15) | T | F | (T highlighted)

For any function $f(x, y, z)$ we have $\text{curl}(\text{curl}(\text{grad}(f))) = \vec{0}$.

Solution:
Already the inner part $\text{curl}(\text{grad}(f)) = 0$ everywhere.

16) | T | F | (F highlighted)

For any vector field $\vec{F}$ and any curve $\vec{r}$ parametrized on $[a, b]$ we have $\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt = \vec{F}(\vec{r}(b)) - \vec{F}(\vec{r}(a))$.

Solution:
If $\nabla f = \vec{F}$ and we would replace $\vec{F}$ on the right hand side with f, we would get the fundamental theorem of line integrals. As it is, it already does not make sense because the left hand side is a scalar and the right hand side is a vector field.

17) | T | F | (F highlighted)

There exist vector fields $\vec{F}$ and $\vec{G}$ in space such that $\text{curl}(\vec{F}) = \text{grad}(\vec{G})$.

Solution:
The right expression $\text{grad}(\vec{G})$ is not even defined.

18) | T | F |

If $\vec{F}$ is a smooth vector field in space and S is a closed oriented surface, then $\iint_S \text{curl}(\vec{F}) \cdot d\vec{S} = 0$.

Solution:
This follows from the divergence theorem or from Stokes theorem.

19) | T | F |

The solid enclosed by the surfaces $z = 2 - \sqrt{x^2 + y^2}$ and $z = \sqrt{x^2 + y^2}$ has the volume $\int_0^{2\pi} \int_0^1 \int_r^{2-r} r \; dzdrd\theta$.

Solution:
This is the volume in cylindrical coordinates.

20) | T | F |

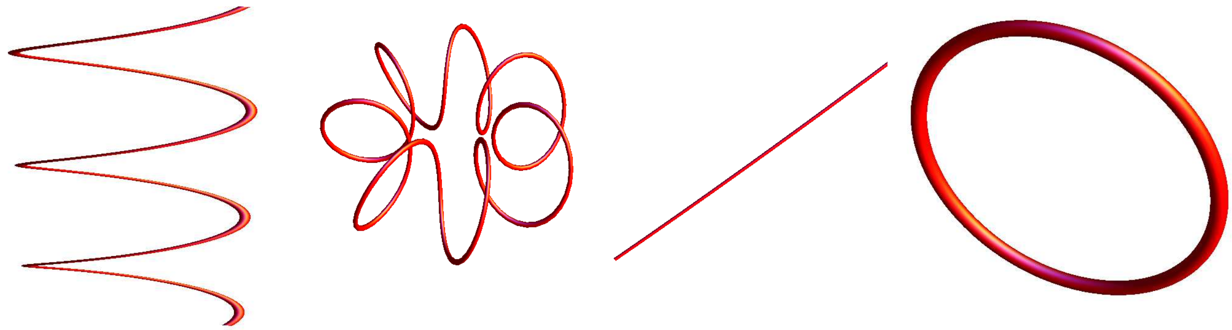
If $\vec{r}\,''(t) = \langle 0, 0, \sin(t) \rangle$, $\vec{r}(0) = \langle 0, 1, 0 \rangle$, $\vec{r}\,'(0) = \langle 1, 0, 0 \rangle$, then $\vec{r}(t) = \langle t, 1 + t, t - \sin(t) \rangle$.

Solution:
It is almost right. But already $\vec{r}\,'(0)$ does not fit.

Problem 2) (6 points)

a) (4 points) Match the curves. There is an exact match.

1 2 3 4

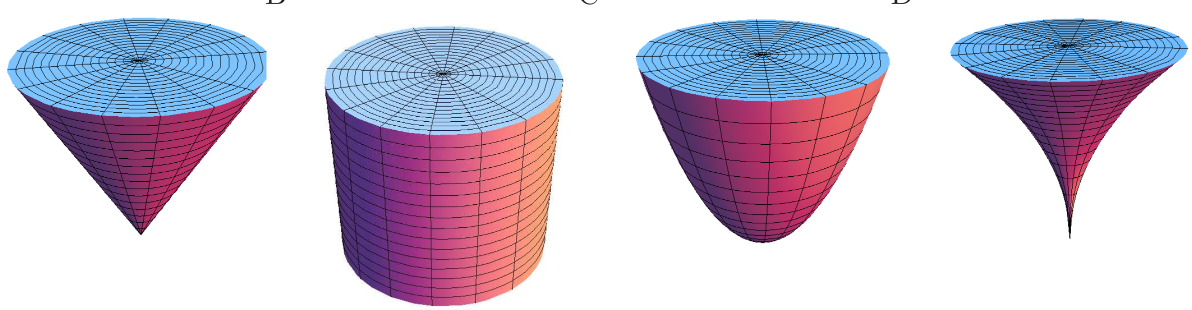


Enter 1-4	Object definition
	$\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$
	$\vec{r}(t) = \langle \cos(t), 0, \sin(t) \rangle$
	$\vec{r}(t) = \langle (2+\cos(7t))\cos(t), (2+\cos(7t))\sin(t), \sin(7t) \rangle$
	$\vec{r}(t) = \langle t, t, t \rangle$

b) (4 points) Match the solids with the triple integrals. Also here, there is an exact match:

Enter A-D	3D integral computing volume
	$\int_0^{2\pi} \int_0^1 \int_r^1 r \; dzdrd\theta$
	$\int_0^{2\pi} \int_0^1 \int_{r^2}^1 r \; dzdrd\theta$
	$\int_0^{2\pi} \int_0^1 \int_{\sqrt{r}}^1 r \; dzdrd\theta$
	$\int_0^{2\pi} \int_0^1 \int_0^1 r \; dzdrd\theta$

A B C D



c) (2 points) What was the name again?

Enter one word	PDE
	$g_x = g_y$
	$g_{xx} = -g_{yy}$

Solution:
a) 1,4,2,3
b) A,C,D,B
c) transport and Laplace

Problem 3) (10 points)

a) (5 points) For the following quantities, decide whether they are vector fields or scalar fields (functions) or nonsense. Here $\vec{F} = \langle P, Q, R \rangle$ is a vector field in space, $f(x, y, z)$ is a scalar function and $\nabla = \langle \partial_x, \partial_y, \partial_z \rangle$. Recall that $\nabla \times \vec{F} = \text{curl}(\vec{F}), \nabla \cdot \vec{F} = \text{div}(\vec{F})$ and $\nabla f = \text{grad}(f)$.

object	scalar	vector	not defined
$\nabla \vec{F}$			
$\nabla \cdot \vec{F}$			
$\nabla \times \vec{F}$			
$\nabla(\nabla \cdot \vec{F})$			
$\nabla \times (\nabla \times \vec{F})$			
$\nabla \times (\nabla \cdot \vec{F})$			
∇f			
$\nabla f \times \vec{F}$			
$\nabla f \cdot \vec{F}$			
$\nabla \times (\nabla f)$			

Solution:
a)

object	scalar	vector	not defined
$\nabla \vec{F}$			*
$\nabla \cdot \vec{F}$	*		
$\nabla \times \vec{F}$		*	
$\nabla(\nabla \cdot \vec{F})$		*	
$\nabla \times (\nabla \times \vec{F})$		*	
$\nabla \times (\nabla \cdot \vec{F})$			*
∇f		*	
$\nabla f \times \vec{F}$		*	
$\nabla f \cdot \vec{F}$	*		
$\nabla \times (\nabla f)$		*	

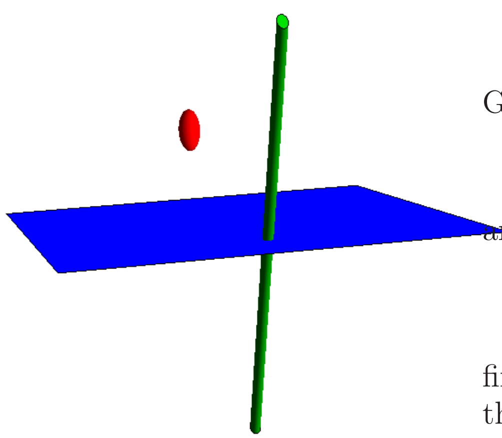
b) (5 points) Match the formulas for the position vector $\vec{r}(t)$ of a curve in space:

label	formula
A	$\vec{r}\,''(t)$
B	$\int_a^b \lvert\vec{r}\,'(t)\rvert \, dt$
C	$\vec{r}\,'(t)/\lvert\vec{r}\,'(t)\rvert$
D	$\vec{T}\,'(t)/\lvert\vec{T}\,'(t)\rvert$
E	$\lvert\vec{T}\,'(t)\rvert/\lvert\vec{r}\,'(t)\rvert$

expression	enter A-E
curvature	
unit normal vector	
unit tangent vector	
arc length	
acceleration	

Solution:
b) E,D,C,B,A

Problem 4) (10 points)



Given a point $P = (4, 3, 1)$, a plane

$$\Sigma : 3x + 4y - 12z = 0$$

and a line

$$L : \frac{x-1}{3} = \frac{y-2}{4} = \frac{z-1}{12} \ ,$$

find the sum $d(P, L) + d(P, \Sigma)$ of the distances of P to the line and plane.

Solution:
A point on the plane is $(0,0,0)$. The distance of the point to the plane is

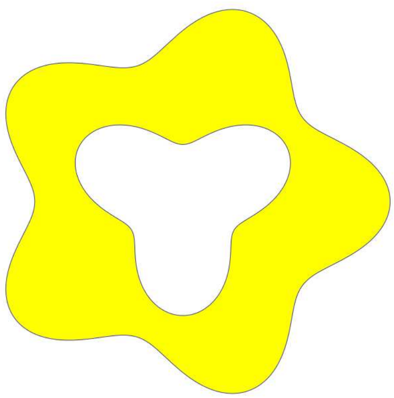
$$d(P,\Sigma) = \frac{|\langle 4,3,1\rangle \cdot \langle 3,4,12\rangle|}{|\langle 3,4,12\rangle|} = 12/13 \; .$$

A point on the line is $(1,2,1)$. The distance of the point to the line is

$$d(P,L) = \frac{|\langle 3,1,0\rangle \times \langle 3,4,12\rangle|}{|\langle 3,4,12\rangle|} = \sqrt{1521}/13 = 39/13 \; .$$

The sum is $\boxed{51/13}$.

Problem 5) (10 points)



a) (5 points) Find the double integral

$$\int_0^3 \int_y^3 \frac{\sin(2x)}{x} \; dx \; dy \; .$$

b) (5 points) What is the area of the polar region

$$3 + \sin(3\theta) \leq r \leq 6 + \cos(5\theta) \; ?$$

Solution:
a) This is a typical switching of the order of integration problem. Writing it as a type I integral gets rid of the x

$$\int_0^3 \int_0^x \frac{\sin(2x)}{x} \; dydx = \frac{1-\cos(6)}{2} \; .$$

b) The integral in polar coordinates is

$$\int_0^{2\pi} \int_{3+\sin(3\theta)}^{6+\cos(5\theta)} r \; drd\theta = \int_0^{2\pi} \frac{(6+\cos(5\theta))^2}{2} - \frac{(3+\sin(3\theta))^2}{2} \; d\theta = 27\pi \; .$$

The result is $\boxed{27\pi}$.

Problem 6) (10 points)

a) (8 points) Locate and classify all the local maxima, minima and saddle points of the function

$$f(x,y) = x^4 + y^4 - 8x^2 - 8y^2 \ .$$

b) (2 points) Is there a global maximum or a global minimum of f? Explain.

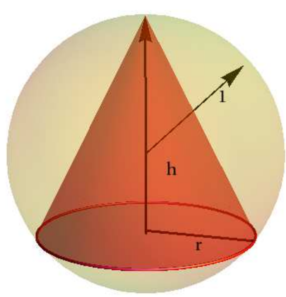
Solution:

a)

x	y	D	f_{xx}	Type	f
-2	-2	1024	32	minimum	-32
-2	0	-512	32	saddle	-16
-2	2	1024	32	minimum	-32
0	-2	-512	-16	saddle	-16
0	0	256	-16	maximum	0
0	2	-512	-16	saddle	-16
2	-2	1024	32	minimum	-32
2	0	-512	32	saddle	-16
2	2	1024	32	minimum	-32

b) There is **no global maximum** since for $y = 0$ already the function blows up for $x \to \infty$. There **is a global minimum** since the function goes to $+\infty$ at infinity in all directions. These global minima are critical points and are the points, where $f = -32$.

Problem 7) (10 points)



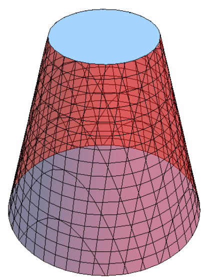
For which base radius r and height h does a cone inscribed into the unit sphere have maximal volume $f(r,h) = \pi r^2 h/3$? The constraint is given by Pythagoras as $g(r,h) = r^2 + (h-1)^2 = 1$. Use the Lagrange method.

Solution:
The gradient of f is $\nabla f(r,h) = \langle \pi 2rh/3, \pi r^2/3 \rangle$, the gradient of g is $\nabla g(r,h) = \langle 2r, 2h - 2 \rangle$. The Lagrange equations are

$$\begin{aligned} 2\pi hr/3 &= \lambda 2r \\ \pi r^2/3 &= \lambda(2h-2) \\ r^2 + (h-1)^2 &= 1 \; . \end{aligned}$$

Eliminating λ gives $2h/r = r/(h-1)$ or $r^2 = 2h(h-1)$. Plugging this into the third equation gives $\boxed{h = 4/3}$. Therefore $\boxed{r = 2\sqrt{2}/3}$.

Problem 8) (10 points)



A bird's feeding cage E is part of a cone $x^2 + y^2 = 4(3-z)^2$ with $1 < z < 2$. The cage is filled with different kind of seeds, the heavier have gone down and the density is $(3-z)$. We want to find the moment of inertia

$$\iiint_E (x^2+y^2)(3-z) \; dxdydz$$

so that we can know how much energy the feeding cage has if a squirrel spins it. You do not have to worry in this problem that squirrels are not birds.

Solution:
The integral in cylindrical coordinates is

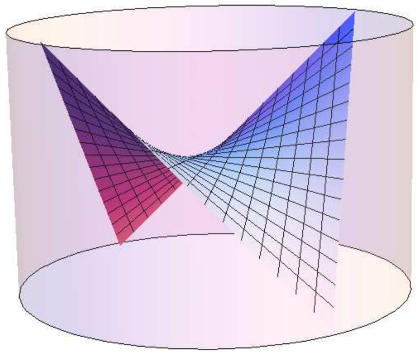
$$\int_0^{2\pi} \int_1^2 \int_0^{2(3-z)} r^3(3-z) \; drdzd\theta \; .$$

Start at the inner integral $\int_0^{2(3-z)} r^3(3-z)dr$ which is $2^4(3-z)^4(3-z)/4 = 4(3-z)^5$. Now integrate this (do not multiply out the polynomial!)

$$\int_1^2 4(3-z)^5 dz = -4(3-z)^6/6|_1^2 = 4(2^6 - 1^6)/6 = 42 \; .$$

Integrating over θ gives $\boxed{84\pi}$.

Problem 9) (10 points)



a) (4 points) Find the surface area of the surface

$$\vec{r}(s,t) = \langle s, -t, 2st \rangle$$

with $s^2 + t^2 \leq 9$.

b) (4 points) The coordinates of the surface satisfies $2xy + z = 0$. Find the tangent plane at $(1, 1, -2)$.

c) (2 points) What is the formula for the linearization of $f(x,y) = 2xy$ at the point $(1, 1)$.

Solution:

a) $r_s = \langle 1, 0, 2t \rangle$, $r_t = \langle 0, -1, 2s \rangle$ and $r_s \times r_t = \langle 2t, -2s, -1 \rangle$ with length $\sqrt{4t^2 + 4s^2 + 1}$. To integrate this over a disc of radius 3, we use polar coordinates:

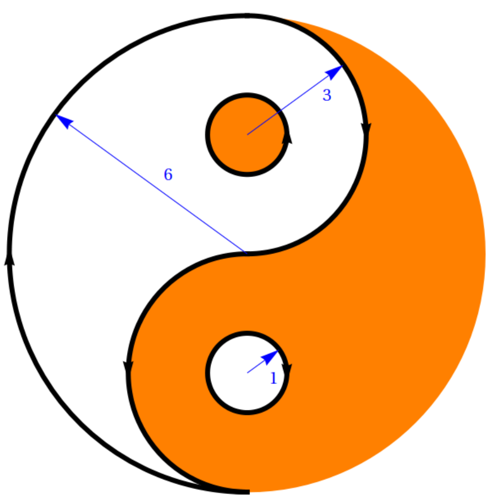
$$\int_0^3 \int_0^{2\pi} \sqrt{1 + 4r^2} r \, d\theta dr = 2\pi(4r^2 + 1)^{3/2}/12_0^3 .$$

This is $\boxed{\pi(37^{3/2} - 1)/6}$.

b) The gradient of $g(x,y,z) = z + 2xy = 0$ is $\langle 2y, 2x, 1 \rangle$ which is at the point $(1, 1, -2)$ equal to $\langle 2, 2, 1 \rangle$ so that the equation of the tangent plane is $2x + 2y + z = d$. The constant d is obtained by plugging in the point $(1, 1, -2)$ which gives $\boxed{2x + 2y + z = 2}$.

c) To get the linearization, compute the gradient of $f(x,y) = 2xy$ which is $2, 2$ if $f(1,1) + 2(x-1) + 2(y-1) = 2x + 2y - 2$. We have $\boxed{L(x,y) = 2x + 2y - 2}$.

Problem 10) (10 points)



Let C be the boundary curve of the white Yang part of the Ying-Yang symbol in the disc of radius 6. You can see in the image that the curve C has three parts, and that the orientation of each part is given. Find the line integral of the vector field

$$\vec{F}(x,y) = \langle -y + \sin(e^x), x \rangle$$

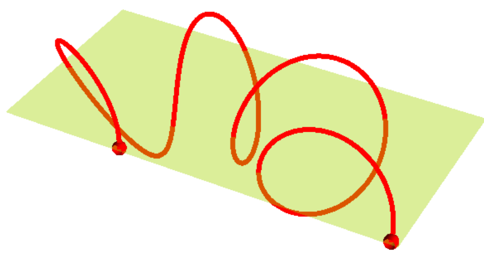
around C. Notice that the Ying and the Yang have the same area.

Solution:
We use Greens theorem noticing however that the orientation of the curve is negative:

$$\int_C F(r(t)) \cdot r'(t)\, dt = -\int\int \text{curl}(\vec{F})(x,y)\, dxdy$$

Because the curl of $\vec{F}$ is constant 2, the right hand side is $-2\text{Area(Yang)} = -2\pi 6^2/2$ which is $\boxed{-36\pi}$.

Problem 11) (10 points)



Let C be the curve

$\vec{r}(t) = \langle (2+\cos(7t))\cos(t), (2+\cos(7t))\sin(t), \sin(7t)\rangle$

parametrized by $0 \leq t \leq \pi$ starting at $t = 0$ and ending at $t = \pi$. Calculate the line integral

$$\int_0^\pi \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt\ ,$$

where $\vec{F}$ is the vector field

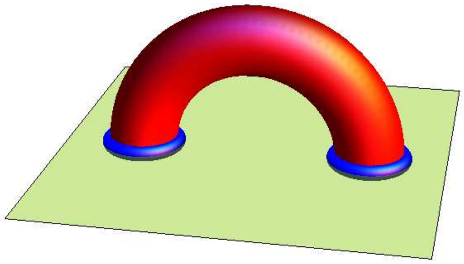
$\vec{F}(x,y,z) = \langle 4xe^{2x^2+3y^2+4z^2}, 6ye^{2x^2+3y^2+4z^2}, 8ze^{2x^2+3y^2+4z^2}\rangle$.

Solution:
We notice that

$$\vec{F} = \nabla f, f(x,y,z) = e^{2x^2+3y^2+4z^2}\ .$$

The fundamental theorem of line integrals assures that the line integral is the potential difference $f(\vec{r}(\pi)) - f(\vec{r}(0))$. We see that $\vec{r}(\pi) = \langle -1,0,0\rangle$ and $\vec{r}(0) = \langle 3,0,0\rangle$. Now $f(\vec{r}(\pi)) - f(\vec{r}(0)) = \boxed{e^2 - e^{18}}$.

Problem 12) (10 points)



Find the flux of the curl of $\vec{F}(x,y,z) = \langle -y, x^2, 0 \rangle$ through a half torus surface S given by $(\sqrt{x^2+z^2}-3)^2 + y^2 = 1, z \geq 0$ which intersects the xy-plane $z=0$ in two circles $C_1 : (x-3)^2 + y^2 = 1$ and $C_2 : (x+3)^2 + y^2 = 1$. The torus S is oriented outwards.

Solution:
By Stokes theorem, it is the sum of the line integrals along the two circles. Parametrize them in the right orientation as

$$\vec{r}_1(t) = \langle 3+\cos(t), \sin(t), 0 \rangle, \vec{r}_2(t) = \langle -3+\cos(t), \sin(t), 0 \rangle \; .$$

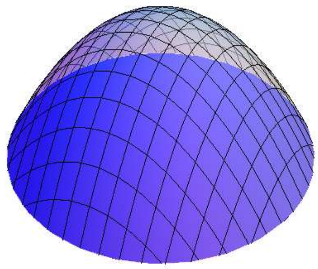
Now compute the line integrals

$$\int_0^{2\pi} \langle -\sin(t), (3+\cos(t))^2, 0 \rangle \cdot \langle -\sin(t), \cos(t), 0 \rangle \; dt = 7\pi$$

$$\int_0^{2\pi} \langle -\sin(t), (-3+\cos(t))^2, 0 \rangle \cdot \langle -\sin(t), \cos(t), 0 \rangle \; dt = -5\pi$$

The sum is $\boxed{2\pi}$. **Remark.**It was also possible to compute these line integrals by applying Stokes again and using the flux of the curl through the discs. Alternatively, one could have seen with the divergence theorem that the flux of the curl through the torus is the sum of the fluxes through the bottom closure discs. Computing the flux through the disc needs a parametrization of the discs and a computation of the curl. It could also be done by applying only then Stokes. A wide variety of solutions applied here and the class has explored essentially all possible cases.

Problem 13) (10 points)



Find the flux of the vector field

$$\vec{F}(x,y,z) = \langle x^3 z, y^3 z, 1 + e^{x^2+y^2} \rangle$$

through the paraboloid part S of the boundary of the solid

$$G : z + x^2 + y^2 \leq 1,\ z \geq 0 \ .$$

The paraboloid surface S is oriented upwards.

Solution:
Use the divergence theorem. The divergence of $\vec{F}$ is $(3x^2 + 3y^2)z$. Denote by G the solid bound by the xy-plane and the paraboloid. We have

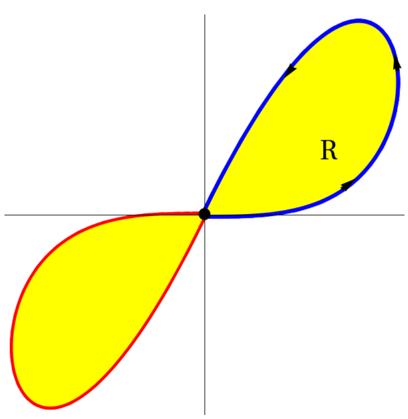
$$\int\int\int_G \text{div}(\vec{F})(x,y,z)\ dxdydz = \int_0^{2\pi}\int_0^1\int_0^{1-r^2} r(3r^2)z\ dzdrd\theta = -\pi/8\ .$$

Denote by D the floor surface. It is parametrized as $\vec{r}(s,t) = \langle s,t,0\rangle$ with $s^2 + t^2 \leq 1$. The vector field on the floor is $\vec{F}(x,y,z) = \langle 0,0,1+e^{x^2+y^2}\rangle$. The flux of this through the floor D parametrized by $r(u,v) = \langle u,v,0\rangle$ is

$$\int\int_D 1 + \exp(u^2+v^2)\ dudv = \int_0^{2\pi}\int_0^1 (1+\exp(r^2)r\ drd\theta = e\pi\ .$$

The total result is $\boxed{\pi/8 + e\pi}$.

Problem 14) (10 points)



Find the area of the **propeller** shaped region enclosed by the figure 8 curve

$$\vec{r}(t) = \langle t - t^3, 2t^3 - 2t^5\rangle\ ,$$

parametrized by $-1 \leq t \leq 1$. To find the total area compute the area of the region R enclosed by the right loop $0 \leq t \leq 1$ and multiply by 2.

Solution:
This is a typical Green problem. We compute the line integral of the vector field $\vec{F}(x,y) = \langle 0, x \rangle$ along the curve

$$2\int_{-1}^{1} \langle 0, t - t^3 \rangle \cdot \langle 1 - 3t^2, 6t^2 - 10t^4 \; dt = 1/6 \; .$$