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- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3, we need to see details of your computation.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
14		10
Total:		150

Problem 1) True/False questions (20 points). No justifications are needed.

1) T **F** The function $f(x,y,z) = x^2 - y^2 - z^2$ increases in the direction $\langle -3,-1,2\rangle/\sqrt{14}$ at the point $(1,1,1)$.

Solution:
Because the directional derivative is $\nabla f(1,1,1) \cdot \langle -3,-1,2\rangle/\sqrt{14} = \langle 2,-2,-2\rangle \cdot \langle -3,-1,2\rangle/\sqrt{14} = -6$, it decreases.

2) **T** F The unit tangent vector of the curve $\vec{r}(t) = \langle 3t, 4t, t^2\rangle$ at time $t = 0$ is $\langle 3/5, 4/5, 0\rangle$.

Solution:
The velocity is $\langle 3, 4, 2t\rangle$ and at time $t = 0$ this has length 5. Normalize the velocity vector and get the unit tangent vector.

3) T **F** There exist two nonzero vectors $\vec{a}$ and $\vec{b}$ such that the length of the vector projection of $\vec{a}$ to $\vec{a} \times \vec{b}$ is $\frac{1}{2}|\vec{b}|$.

Solution:
The projection from $\vec{a}$ onto the vector $\vec{a} \times \vec{b}$ perpendicular to $\vec{a}$ is always 0.

4) T **F** The arc length of the curve $\vec{r}_1(t) = \langle e^{3t^3} - 1, t^6 + 2, \sin(2t^3)\rangle$, $0 \leq t \leq 1$ is larger than that of $\vec{r}_2(t) = \langle e^{3t} - 1, t^2 + 2, \sin(2t)\rangle$, $0 \leq t \leq 1$.

Solution:
The arc length is the same because the second curve is just a reparametrization of the first.

5) T **F** The tangent plane of the graph of $f(x,y) = \sin(x) + y^3$ at $(0,1,1)$ is $x + 3y = 3$.

Solution:
In order to find the tangent plane, first write the graph as a level curve of a function of 3 variables: $g(x,y,z) = \sin(x) + y^3 - z = 0$. Its gradient is $\langle \cos(x), 3y^2, -1\rangle = \langle 1, 3, -1\rangle$.

6) T **F** There exists a curve C on the level surface of $f(x,y,z) = x^3 + e^{yz} + \cos(y) = 2$ such that the line integral $\int_C \nabla f \cdot d\vec{r} > 0$.

Solution:
The velocity vector to the curve is always perpendicular to the gradient vector so that the line integral is zero.

7) T (highlighted) F

If Q is the point away from the plane $3x + 5y + z = 7$ and P is the point on the plane closest to Q, then $\vec{PQ}$ is parallel to $\langle 3, 5, 1 \rangle$.

Solution:
The vector PQ is normal to the plane and so parallel to the gradient of the linear function defining the plane.

8) T F (highlighted)

The vector field $\vec{F}(x, y, z) = \langle y^2 - xz + e^y, -yz, x^4 + y^2 - z^2 \rangle$ is the curl of a vector field $\vec{G}$.

Solution:
If $F = \text{curl}(G)$, then $\text{div}(\text{curl}(G) = 0$. But the difvergence of $\vec{F}$ is not zero.

9) T F (highlighted)

Let $\vec{F}(x, y) = \langle P(x, y), Q(x, y) \rangle = \langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \rangle$ and C be the unit circle oriented counterclockwise. Since $Q_x = P_y$ everywhere, Green implies $\int_C \vec{F} \cdot d\vec{r} = 0$.

Solution:
Green's theorem can not be applied because at the origin, the curl is not defined.

10) T (highlighted) F

By linear approximation of the function $f(x, y, z) = e^{x+y+z}$ we can estimate $f(0.1, 0.01, 0.001)$ as 1.111.

Solution:
The gradient of f at $(0, 0, 0)$ is $\langle 1, 1, 1 \rangle$. The approximation is $f(0, 0, 0) + 1 \cdot 0.1 + 1 \cdot 0.01 + 1 \cdot 0.001$ which is what we have given.

11) T (highlighted) F

If $\vec{F}(x, y, z)$ is a vector field defined on $0 < x^2 + y^2 + z^2 < 4$ and $\text{curl}(\vec{F}) = 0$ everywhere on this solid, then $\vec{F} = \nabla f$ for some function f.

Solution:
It is simply connected because we can pull together any closed loop to a point.

12) T F The tangent plane of the surface $x^2+y^4+z^6=6$ at $(2,1,1)$ is perpendicular to the line $\vec{r}(t)=\langle 1+2t, 3+2t, -4+3t\rangle$.

Solution:
The tangent plane is perpendicular to the gradient which is $\langle 4,4,6\rangle$.

13) T F Given two curves $C_1 : \vec{r}_1(t)=\langle t,t^3\rangle, 0\le t\le 1$ and $C_2 : \vec{r}_2(s)=\langle s,s^5\rangle, 0\le s\le 1$ $f(x,y)=\sin(x^2y)$. Then $\int_{C_1}\nabla f\cdot d\vec{r}=\int_{C_2}\nabla f\cdot d\vec{r}$.

Solution:
Use the fundamental theorem of line integrals.

14) T F If $f(x,y)$ has a global maximum, then the discriminant function $D(x,y)=f_{xx}f_{yy}-f_{xy}^2$ has a global maximum.

Solution:
There is no relation between critical points of f and D. Take $-x^2-y^2+x^2y$ for example. It has a global maximum but $D(x,y)=4-4y-4x^2$ does not.

15) T F Let $\vec{F}(x,y,z)=\langle x,y,z\rangle$ and S the surface boundary of the cube $0\le x\le 1, 0\le y\le 1, 0\le z\le 1$ oriented by outward normal vectors. Then $\int\int_S \vec{F}\cdot d\vec{S}=0$.

Solution:
Use the divergence theorem. It says that the result is 3, not zero.

16) T F Let $\vec{F}(x,y,z)=\langle x/3, y/3, z/3\rangle$ and S the unit sphere oriented by the outward normal vectors. Then $\int\int_S \text{curl}(\vec{F})\cdot d\vec{S}$ is the volume of the unit ball.

Solution:
By Stokes theorem, the result is zero. One can also see it with the divergence theorem.

17) T F In three dimensional space there exist two nonzero vector fields $\vec{F}$ and $\vec{G}$ such that $\text{curl}(\vec{F})=\text{div}(\vec{G})$.

Solution:
One is a vector, the other is a scalar

18) T F

The vector field $\vec{F}(x, y, z) = \langle \cos(y), \cos(z), \cos(x) \rangle$ has the property that $\vec{F} = \text{curl}(\text{curl}(\vec{F}))$.

Solution:
This is a direct computation: $\vec{G} = \text{curl}\vec{F} = \langle \sin(z), \sin(x), \sin(y) \rangle$ and $\text{curl}(\vec{G}) = \vec{F}$.

19) T F

There exists a vector field $\vec{F}(x, y, z)$ defined on $\mathbf{R}^3$ such that every line integral $\int_C \vec{F} \cdot d\vec{r}$ of $\vec{F}$ over a closed curve C is equal to 0, but not every surface integral $\int \int_S \vec{F} \cdot dS$ over a closed surface S is equal to 0.

Solution:
Take a gradient field which is not divergence free like $\langle x, y, z \rangle$.

20) T F

Whenever $\vec{F} = \nabla f$, for some function $f(x, y)$ defined on the annulus $\frac{1}{2} \leq x^2 + y^2 \leq 2$, then $\int_C \vec{F} \cdot d\vec{r} = 0$, where C is the circle $x^2 + y^2 = 1$.

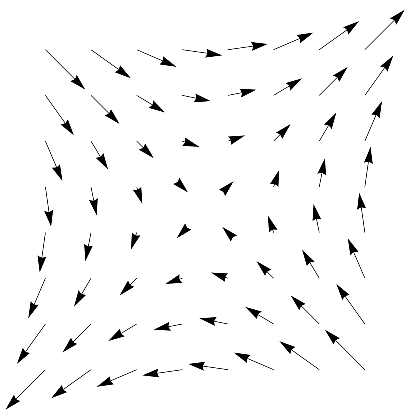
Solution:
Even so the function is not defined at the origin, the fundamental theorem of line integrals applies.

Problem 2) (10 points)

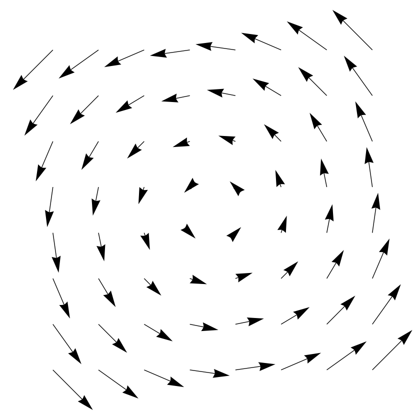
a) (5 points) We match in this problems vector fields with properties of vector fields and formulas for vector fields. A field $\vec{F}$ is **divergence free** if $\text{div}(\vec{F}) = 0$ everywhere in the plane. A field $\vec{F}$ is **irrotational**, if $\text{curl}(\vec{F}) = \vec{0}$ everywhere in the plane. In the last two columns of the following table, check the boxes which apply.

field	enter I-IV	divergence free	irrotational
$\vec{F}(x,y) = \langle -y, x \rangle$			
$\vec{F}(x,y) = \langle y, x \rangle$			
$\vec{F}(x,y) = \langle -x-y, x-y \rangle$			
$\vec{F}(x,y) = \langle x+y, x+y \rangle$			

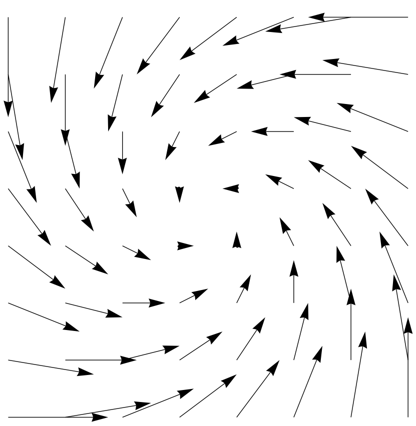
I



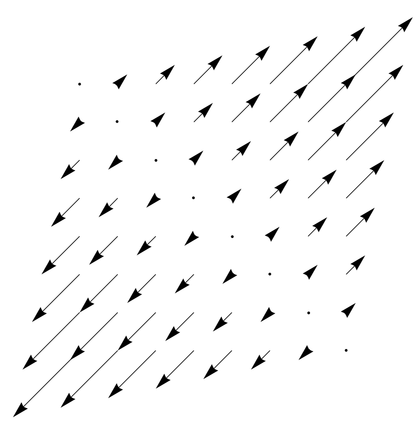
II



III



IV



b) (5 points) Match the following names of partial differential equations with functions $u(t,x)$ which satisfy the differential equation and with formulas defining these equations.

equation	A-D	1-4
wave		
heat		
transport		
Laplace		

A	$u(t,x) = t^2 + x^2$
B	$u(t,x) = t^2 - x^2$
C	$u(t,x) = \sin(x+t)$
D	$u(t,x) = x^2 + 2t$

1	$u_t(t,x) = u_x(t,x)$
2	$u_{tt}(t,x) = u_{xx}(t,x)$
3	$u_{tt}(t,x) = -u_{xx}(t,x)$
4	$u_t(t,x) = u_{xx}(t,x)$

Solution:
2a) II divergence free, I divergence free and irrotational, III , IV irrotational, 2b) ADCB, 2413

Problem 3) (10 points)

a) (6 points) Select 6 of the integrals $A-H$ in the lower tables and match them with their names in the following table:

name	label A-H
line integral	
flux integral	
surface area	
arc length	
volume	
area	

$\int\int_R x^2 - y^2 \; dxdy$	A
$\int\int_R 1 \; dxdy$	B
$\int\int\int_R 1 \; dxdydz$	C
$\int\int\int_R x^2 + z^2 \; dxdydz$	D

$\int\int_R \vec{F}(\vec{r}(u,v)) \cdot \vec{r}_u \times \vec{r}_v \; dudv$	E
$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) \; dt$	F
$\int_a^b \|\vec{r}\,'(t)\| \; dt$	G
$\int\int_R \|\vec{r}_u \times \vec{r}_v\| \; dudv$	H

b) (4 points)

derivative	enter A-D
divergence	
curl	
gradient	
directional derivative	

The middle column of the following table is obtained by applying a derivative operation to the object in the left column. Fill in the correct label (A-D) of that operation into the above table.

object	derivative	label
$\vec{F}(x,y,z) = \langle -y, x, x\rangle$	$\langle 0, -1, 2\rangle$	A
$\vec{F}(x,y,z) = \langle x^2, y, x\rangle$	$2x+1$	B
$f(x,y,z) = x^2 + y^2 + z$	$\langle 2x, 2y, 1\rangle$	C
$f(x,y,z) = x^3 + 5y^2$	$10y$	D

Solution:
a) FEHGCB, b) BACD

Problem 4) (10 points)

Consider the tetrahedron with vertices

$$A = (0,1,-1), B = (4,0,-1), C = (2,1,3), \text{ and } D = (2,2,0) .$$

a) (3 points) What is the area of the parallelogram spanned by $\vec{AB}$ and $\vec{AD}$?

b) (3 points) Find the volume of the parallelepiped spanned by $\vec{AC}, \vec{AB}$ and $\vec{AD}$.

c) (4 points) Determine the distance between the two skew lines AB and CD.

Solution:
$\vec{AC} = \langle 2,0,4\rangle$.
$\vec{AB} = \langle 4,-1,0\rangle$.
$\vec{AD} = \langle 2,1,1\rangle$.
a) The area of the parallelogram is $|\langle -1,-4,6\rangle| = \sqrt{53}$.
b) The volume of the parallelepiped is 22.
c) The distance is $|\vec{AD} \cdot (\vec{AB} \times \vec{CD})|/|\vec{AB} \times \vec{CD}| = 22/13$.

Problem 5) (10 points)

a) (5 points) The curl of $\vec{F}(x,y) = \langle -e^{xy}, y\rangle$ is equal to a scalar function $f(x,y)$. Estimate $f(1.1, 0.001)$ by linear approximation.

b) (5 points) Using the same function as in a), the equation $f(x,y) = \text{curl}(\vec{F})(x,y) = 1$ defines y as a function $g(x)$ of x near $x = 1$. Find $g'(1)$.

Solution:
The function is $f(x,y) = xe^{xy}$. We have $\nabla f(x,y) = \langle e^{xy} + xye^{xy}, x^2e^{xy}\rangle$. Compute $f(1,0) = 1$ and $\nabla f(1,0) = \langle 1,1\rangle$.
a) We have $L(1.1, 0.001) = 1 + 0.1 + 0.001 = 1.101$.
b) We use the implicit differentiation formula which comes from $f(x,g(x)) = 1$. We have $g'(x) = -f_x(1,0)/f_y(1,0) = -1$.

Problem 6) (10 points)

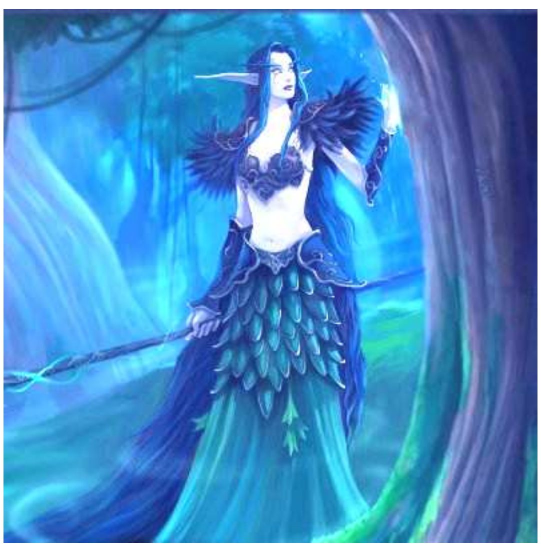
Find all the critical points of the function $f(x, y) = y^3 - 3y^2 + 4x + x^2 - 3$ and classify them by telling whether they are local maxima, local minima or saddle points.

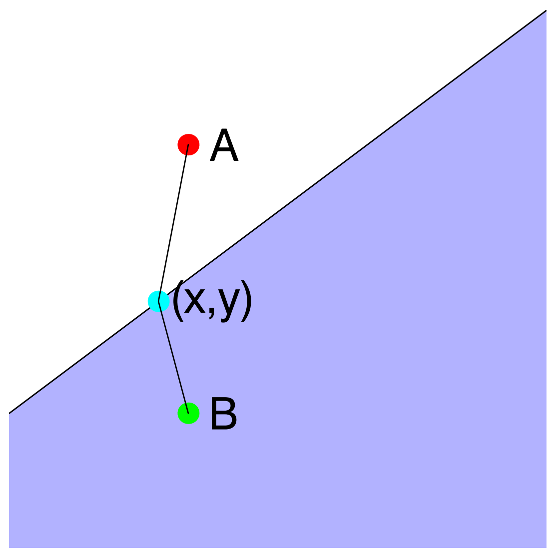
Solution:
The critical points are $P = (-2, 0)$ and $Q = (-2, 2)$. The Discriminant at P is $D = -12$ so that P is a saddle point. The Hessian at Q is 12 and $f_{xx} = 2$ which is a local minimum.

$P = (-2, 0)$	D=-12	Saddle point
$Q = (-2, 2)$	D=12, $f_{xx} = 2$	local minimum

Problem 7) (10 points)

A nightelf in the game World of Warcraft runs from $A = (0, 2)$ to $B = (0, 0)$ along a straight line segment from A to (x, y) and swims through the lake $x - y \geq -1$ from (x, y) to a gold chest located at $B = (0, 0)$ again on a straight line segment. The effort from A to (x, y) is the square of the distance from A to (x, y). Her effort from (x, y) to B is 2 times the squared distance from (x, y) to B. Using the Lagrange method, find the choice of a drop point (x, y) on the lake shore that minimizes her effort.





Solution:
This is a Lagrange problem. We have to extremize the total effort $f(x,y) = x^2 + (y-2)^2 + 2(x^2+y^2) = 3x^2 + 3y^2 - 4y + 4$ under the constraint $g(x,y) = x - y + 1 = 0$ that the drop point is on the shore. The Lagrange equations are

$$\begin{aligned} 6x &= \lambda \\ 6y - 4 &= -\lambda \\ x - y &= -1 \ . \end{aligned}$$

To solve it, the first two equations give $6x = 4 - 6y$. Together with the third, we end up with the solution $\boxed{x = -1/6, y = 5/6}$.

Problem 8) (10 points)

Evaluate the line integral

$$\int_C \vec{F} \cdot d\vec{r} \ ,$$

where C is the curve given by

$$\vec{r}(t) = \langle \frac{t\pi}{2}, 1-t, t^3 \rangle, 0 \leq t \leq 1$$

and

$$\vec{F}(x,y,z) = \langle e^{y^2} + z\cos(xz), 2xye^{y^2}, x\cos(xz) \rangle \ .$$

Solution:
The vector field is a gradient field with gradient $f(x,y) = xe^{y^2} + \sin(xz)$. Apply the fundamental theorem of line integrals. We have $\vec{r}(1) = \langle \pi/2, 0, 1 \rangle$ and $\vec{r}(0) = \langle 0, 1, 0 \rangle$. The result is $\boxed{\pi/2 + 1}$.

Problem (9) (10 points)

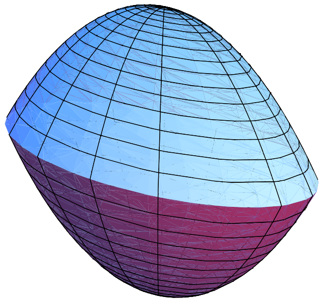
The picture shows an unidentified flying object (UFO). Although it is unidentified, we know its shape. One part of the surface

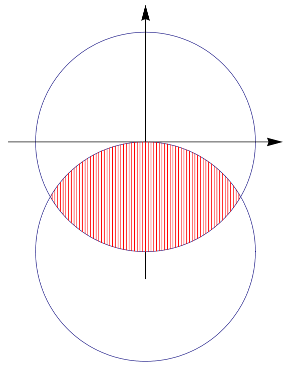
$$x^2 + y^2 + z^2 = 4$$

and the other part of the surface is

$$x^2 + y^2 + (z+2)^2 = 4 \ .$$

Find the surface area of the UFO.





Solution:
The two surfaces forming the UFO hull are parts of spheres of radius 2. We use spherical coordinates. The angle θ goes from 0 to 2π. The angle ϕ ranges from 0 to $\pi/3$. We know $|r_\theta \times r_\phi| = \rho^2 \sin(\phi) = 4\sin(\phi)$. Lets compute half of the surface:

$$2\pi \int_0^{\pi/3} 4\sin(\phi)\, d\phi = 2\pi 4\cos(\phi)|_0^{\pi/3} = (2\pi)4(1-1/2) = 4\pi \ .$$

Both surfaces together have surface area $\boxed{8\pi}$.

Problem 10) (10 points)

Evaluate the following integral

$$\int_0^2 \int_1^3 \int_{z^2}^4 xz\cos(y^2)\ dydxdz \ .$$

Solution:
We are stuck with the integral. Change the order of integration using Fubini (for the most inner two integrals, where z is a constant and does not depend on x, y):

$$\int_0^2 \int_{z^2}^4 \int_1^3 xz \cos(y^2) \; dxdydz \; .$$

Now we can solve the most inner integral using $\int_1^3 x \; dx = 4$

$$\int_0^2 \int_{z^2}^4 4z \cos(y^2) \; dydz \; .$$

Again, we are stuck and we change the order of integration again. It helps to make a 2D picture to find:

$$\int_0^4 \int_0^{\sqrt{y}} 4z \cos(y^2) \; dzdy \; .$$

Now we can evaluate the inner integral

$$\int_0^4 2y \cos(y^2) dy = \sin(y^2)|_0^4 = \sin(16) \; .$$

The final result is $\boxed{\sin(16)}$.

Problem 11) (10 points)

Let $\vec{F}(x, y, z) = \langle x + yz, xye^{-xz}, e^{-xz} \rangle$. Find

$$\int \int_S \vec{F} \cdot d\vec{S} \; ,$$

where S is the surface $z = 1 - x^2 - y^2, z \geq 0$ oriented so that the normal vector points upwards.

Solution:
We use the divergence theorem: let E be the solid enclosed by S and the disc D in the xy-plane oriented downwards. Then

$$\int\int_D \vec{F} \cdot d\vec{S} + \int\int_S \vec{F} \cdot d\vec{S} = \int\int\int_E \text{div}(\vec{F})\, dV \ .$$

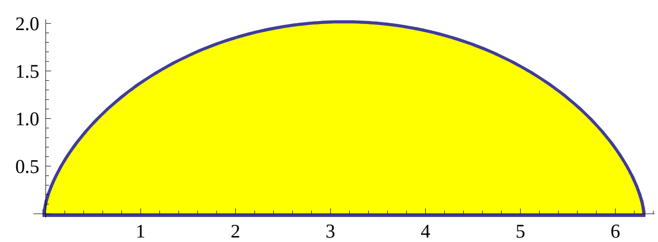
First the right hand side. The divergence of the vector field is 1.

$$\int\int\int_E \text{div}(\vec{F})\, dxdydz = \int_0^{2pi} \int_0^1 (1 - r^2) r\, drd\theta = 2\pi(\frac{r^2}{2} - \frac{r^4}{4})|_0^1\, d\theta = \frac{\pi}{2} \ .$$

On the xy plane, the field is $\langle x, xy, 1 \rangle$. For the parametrization $\vec{r}(u,v) = \langle u, v, 0 \rangle$ of the bottom surface, the normal vector is $\langle 0, 0, 1 \rangle$. The flux through the bottom surface D is π if the surface is oriented upwards but it is $-\pi$ if the surface is oriented downwards. We solve for the unknown $\int\int_D \vec{F} \cdot d\vec{S}$ in the equation given by the divergence theorem and get $\boxed{3\pi/2}$.

Problem 12) (10 points)

Find the area of the region on the plane enclosed by the curve $\vec{r}(t) = \langle t - \sin(t), 1 - \cos(t) \rangle$ with $0 \leq t \leq 2\pi$ and the x-axes.



Solution:
Use Greens theorem for the region D enclosed by the given curve and the x axes. We chose a vector field $\vec{F} = \langle -y, 0 \rangle$ which has curl 1. [While the computation can also be done with $\vec{F} = \langle 0, x \rangle$ for example, the first choice has the advantage that the field is zero on the x-axes.] Greens theorem tells that

$$-I + II = \int\int \text{curl}(\vec{F}) \; dxdy = \text{area}(D) \; .$$

where $I = \int_C \vec{F}(\vec{r}(t)) \; \vec{r}'(t) \; dt$ is the line integral along C and $II = \int_0^{2\pi} \vec{F}(t,0) \cdot \langle 1, 0 \rangle \; dt = 0$ is the line integral along the x axes from $(0,0)$ to $(2\pi, 0)$ with the curve $\vec{s}(t) = \langle t, 0 \rangle$. The minus sign for the first integral is because the curve $\vec{r}(t)$ goes in the clockwise direction. So,

$$\text{area}(D) = -I = -\int_C \vec{F}(\vec{r}(t)) \; \vec{r}'(t) \; dt = -\int_0^{2\pi} -(1-\cos(t))^2 = 3\pi \; .$$

The area of the region is $\boxed{3\pi}$.

Problem 13) (10 points)

Evaluate the integral

$$\int\int_S \text{curl}(\vec{F}) \cdot d\vec{S} \; ,$$

where $\vec{F}(x,y,z) = \langle xe^{y^2}z^3 + 2xyze^{x^2+z}, x + z^2e^{x^2+z}, ye^{x^2+z} + ze^x \rangle$ and where S is the part of the ellipsoid $x^2 + y^2/4 + (z+1)^2 = 2$, $z > 0$ oriented so that the normal vector points upwards.

Solution:
Stokes theorem assures that the flux integral we are looking for is equal to the line integral along the boundary of the surface. The boundary is the ellipse $\vec{r}(t) = \langle \cos(t), 2\sin(t) \rangle$, $0 \leq t \leq 2\pi$. The vector field on the xy-plane $z = 0$ is

$$\vec{F}(x,y,0) = \langle 0, x, ye^{x^2} \rangle \; .$$

To compute the line integral of this vector field along the boundary curve, compute $\vec{r}'(t) = \langle -\sin(t), 2\cos(t), 0 \rangle$ and $\vec{F}(\vec{r}(t)) = \langle 0, \cos(t), 2\sin(t)e^{\sin^2(t)} \rangle$. The dot product of these two vectors is the function $2\cos^2(t)$, the power. Integrating this over $[0, 2\pi]$ gives $\boxed{2\pi}$. [P.S. The problem could also have been solved with the divergence theorem by computing the flux of $\text{curl}(\vec{F})$ through the bottom surface. This requires however to compute the curl of $\vec{F}$ and produces substantially more work.]

Problem 14) (10 points)

Let E be the rectangular solid $0 \leq x \leq a, 0 \leq y \leq b, 0 \leq z \leq 1$ and let S be the boundary of E. The surface S consists of 6 planar pieces where each is oriented so that the normal vector points outwards. Given the vector field

$$\vec{F} = \langle -x^2 - 4xy, -yz, 12z \rangle \ ,$$

for which parameters a, b is the flux integral

$$\int\int_S \vec{F} \cdot \vec{dS}$$

a global maximum?

Solution:
We have $\text{div}(F) = -2x - 4y - z + 12$. By the divergence theorem, we can compute the flux through S as a triple integral of $\text{div}(F)$ through E. This is

$$\int_0^a \int_0^b \int_0^1 -2x - 4y - z + 12 \ dzdydx = 23ab/2 - a^2b - 2ab^2 \ .$$

Extremizing this function gives 3 saddle points and one local maximum.

a	b	D	f_{aa}	Nature	$f(a,b)$
0	0	< 0	-	saddle	0
0	23/4	< 0	-	saddle	0
23/6	23/12	> 0	< 0	max	12167/432
23/2	0	< 0	-	saddle	0

The parameters $(a, b) = (23/6, 23/12)$ lead to a global maximum.