Vectors & The Dot Product

1 Are the following better described by vectors or scalars?

(a) The cost of a Super Bowl ticket.

(b) The wind at a particular point outside.

(c) The number of students at Harvard.

(d) The velocity of a car.

(e) The speed of a car.

2 Bert and Ernie are trying to drag a large box on the ground. Bert pulls the box toward the north with a force of 30 N, while Ernie pulls the box toward the east with a force of 40 N. What is the resultant force on the box?

Definition. The *dot product* $\mathbf{v} \cdot \mathbf{w}$ of two vectors $\mathbf{v}$ and $\mathbf{w}$ is defined as follows.

- If $\mathbf{v}$ and $\mathbf{w}$ are two-dimensional vectors, say $\mathbf{v} = \langle v_1, v_2 \rangle$ and $\mathbf{w} = \langle w_1, w_2 \rangle$, then their dot product is $v_1 w_1 + v_2 w_2$.
- If $\mathbf{v}$ and $\mathbf{w}$ are three-dimensional vectors, say $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ and $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$, then their dot product is $v_1 w_1 + v_2 w_2 + v_3 w_3$.

It is not possible to dot a two-dimensional vector with a three-dimensional vector!

3 Compute the following dot products.

(a) $\langle 1, 2 \rangle \cdot \langle 3, 4 \rangle$

(b) $\langle 1, 2, 3 \rangle \cdot \langle 4, -5, 6 \rangle$

(c) $6\mathbf{j} \cdot 4\mathbf{k}$

(d) $\mathbf{i} \cdot (\mathbf{j} + \mathbf{k})$

Basic Properties: Here are some basic algebraic properties of the dot product. If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors of the same dimension and c is a scalar, then

1. $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$.
2. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$.
3. $(c\mathbf{v}) \cdot \mathbf{w} = c(\mathbf{v} \cdot \mathbf{w}) = \mathbf{v} \cdot (c\mathbf{w})$.

4 True or false: if $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors of the same dimension, then $\mathbf{u} \cdot (\mathbf{v} \cdot \mathbf{w}) = (\mathbf{u} \cdot \mathbf{v}) \cdot \mathbf{w}$.

5 What is the relationship between $\mathbf{v} \cdot \mathbf{v}$ and $|\mathbf{v}|$?

Main Property: Often there is an alternate definition of the dot product given. Two vectors $\mathbf{v}$ and $\mathbf{w}$ determine an angle θ in their common plane, and then their dot product is simply

$$\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}|\, |\mathbf{w}| \cos(\theta).$$

Look back at Problems 3(cd) and 5. In each problem, what is the angle θ? Does that make sense for your answer?

6 Find the angle between $\langle 1, 2, 1 \rangle$ and $\langle 1, -1, 1 \rangle$.

7 Find the vector projection of $\langle 0, 0, 1 \rangle$ onto $\langle 1, 2, 3 \rangle$.

8 True or false: If $\mathbf{v}$ and $\mathbf{w}$ are parallel, then $|\mathbf{v} - \mathbf{w}| = |\mathbf{v}| - |\mathbf{w}|$.

9 If $\mathbf{v}$ and $\mathbf{w}$ are vectors with the property that $|\mathbf{v} + \mathbf{w}|^2 = |\mathbf{v}|^2 + |\mathbf{w}|^2$, which of the following must be true?

(a) $\mathbf{v} = \mathbf{w}$

(b) $\mathbf{v} = \mathbf{0}$

(c) $\mathbf{v}$ is orthogonal to $\mathbf{w}$

(d) $\mathbf{v}$ is parallel to $\mathbf{w}$

Vectors & The Dot Product – Answers and Solutions

1
(a) Scalar – the cost is just a number.
(b) Vector – the wind has both a speed and a direction.
(c) Scalar.
(d) Vector. The velocity is defined to be both the speed of the car (how fast it's going) and the direction it's going.
(e) Scalar. The speed refers only to how fast the car is going; it is the magnitude of the velocity vector.

2 The force Bert is applying can be described by the vector $\langle 0, 30\rangle$, while the force Ernie is applying is $\langle 40, 0\rangle$. Therefore, the resultant force is $\langle 40, 30\rangle$.

3
(a) $\langle 1,2\rangle \cdot \langle 3,4\rangle = 1\cdot 3 + 2\cdot 4 = 11$
(b) $\langle 1,2,3\rangle \cdot \langle 4,-5,6\rangle = 1\cdot 4 + 2\cdot -5 + 3\cdot 6 = 12$
(c) $6\mathbf{j}\cdot 4\mathbf{k} = \langle 0,6,0\rangle \cdot \langle 0,0,4\rangle = 0\cdot 0 + 6\cdot 0 + 0\cdot 4 = 0$
(d) $\mathbf{i}\cdot(\mathbf{j}+\mathbf{k}) = \langle 1,0,0\rangle \cdot \langle 0,1,1\rangle = 1\cdot 0 + 0\cdot 1 + 0\cdot 1 = 0$

4 Completely false. In fact, the statement doesn't even make sense! $\mathbf{v}\cdot\mathbf{w}$ is a scalar, and we can't dot a vector with a scalar.

5 $\mathbf{v}\cdot\mathbf{v}$ is equal to $|\mathbf{v}|^2$. Again, this is easy to see from the component definition. For a two-dimensional vector $\mathbf{v} = \langle v_1, v_2\rangle$, $\mathbf{v}\cdot\mathbf{v} = v_1^2 + v_2^2 = |\mathbf{v}|^2$. For a three-dimensional vector $\mathbf{v} = \langle v_1, v_2, v_3\rangle$, $\mathbf{v}\cdot\mathbf{v} = v_1^2 + v_2^2 + v_3^2 = |\mathbf{v}|^2$.

6 Let $\mathbf{v} = \langle 1,2,1\rangle$ and $\mathbf{w} = \langle 1,-1,1\rangle$, and let θ be the angle between $\mathbf{v}$ and $\mathbf{w}$. Then, we know that $\mathbf{v}\cdot\mathbf{w} = |\mathbf{v}||\mathbf{w}|\cos\theta$. We calculate that $\mathbf{v}\cdot\mathbf{w} = 1\cdot 1 + 2\cdot -1 + 1\cdot 1 = 0$, so $0 = |\mathbf{v}||\mathbf{w}|\cos\theta$. Since the lengths $|\mathbf{v}|$ and $|\mathbf{w}|$ are both positive, $\cos\theta = 0$, so $\theta = \frac{\pi}{2}$.

7 Let $\mathbf{v} = \langle 0,0,1\rangle$ and $\mathbf{w} = \langle 1,2,3\rangle$. We saw in class that the projection of $\mathbf{v}$ onto $\mathbf{w}$ is $\frac{\mathbf{v}\cdot\mathbf{w}}{\mathbf{w}\cdot\mathbf{w}}\mathbf{w}$. In this case, $\mathbf{v}\cdot\mathbf{w} = 3$ and $\mathbf{w}\cdot\mathbf{w} = 1^2 + 2^2 + 3^2 = 14$, so the projection is $\frac{3}{14}\langle 1,2,3\rangle = \left\langle \frac{3}{14}, \frac{6}{14}, \frac{9}{14}\right\rangle$.

8 False. For example, let $\mathbf{v} = \langle 1,0,0\rangle$ and $\mathbf{w} = -\langle 1,0,0,\rangle$. Then, $\mathbf{v}-\mathbf{w} = \langle 2,0,0\rangle$, which has length 2. On the other hand, $\mathbf{v}$ and $\mathbf{w}$ both have length 1, so $|\mathbf{v}| - |\mathbf{w}| = 0$.

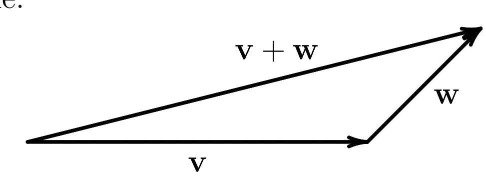
9 (c).

We can rewrite the left-hand side of the equation $|\mathbf{v}+\mathbf{w}|^2 = |\mathbf{v}|^2 + |\mathbf{w}|^2$ using the relationship between lengths and dot products. Then, we have

$$|\mathbf{v}+\mathbf{w}|^2 = (\mathbf{v}+\mathbf{w})\cdot(\mathbf{v}+\mathbf{w}) = \mathbf{v}\cdot\mathbf{v} + 2\mathbf{v}\cdot\mathbf{w} + \mathbf{w}\cdot\mathbf{w} = |\mathbf{v}|^2 + 2\mathbf{v}\cdot\mathbf{w} + |\mathbf{w}|^2.$$

Plugging this into our original equation and cancelling like terms on both sides, we get $2\mathbf{v}\cdot\mathbf{w} = 0$ or $\mathbf{v}\cdot\mathbf{w} = 0$. This is exactly what it means for $\mathbf{v}$ and $\mathbf{w}$ to be orthogonal.

You could also think about this problem geometrically. If $\mathbf{v}$ and $\mathbf{w}$ are not parallel, then $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{v}+\mathbf{w}$ form a triangle:



The equation $|\mathbf{v}+\mathbf{w}|^2 = |\mathbf{v}|^2 + |\mathbf{w}|^2$ says that the sides of the triangle satisfy the Pythagorean Theorem, so the triangle must be a right triangle with $\mathbf{v}+\mathbf{w}$ as the hypotenuse and $\mathbf{v}$ and $\mathbf{w}$ as the two legs. In other words, $\mathbf{v}$ and $\mathbf{w}$ must be orthogonal.

The Cross Product

Definition: The *cross product* of two vectors $\mathbf{a}$ and $\mathbf{b}$ is the vector $\mathbf{a} \times \mathbf{b}$ with

- length $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\ |\mathbf{b}|\ \sin(\theta)$, where θ is the (smaller) angle between $\mathbf{a}$ and $\mathbf{b}$, and
- direction $\mathbf{n}$, where $\mathbf{n}$ is the unit vector orthogonal (perpendicular) to both $\mathbf{a}$ and $\mathbf{b}$ so that $\{\mathbf{a}, \mathbf{b}, \mathbf{n}\}$ is oriented by the right-hand rule.

1 Use this definition to compute the following cross products. I've tried to make the vectors simple so you can find $|\mathbf{a}|$, $|\mathbf{b}|$, $\mathbf{n}$ and θ without much work.

(a) $\mathbf{i} \times \mathbf{j}$ (b) $\mathbf{i} \times (\mathbf{i} + \mathbf{j})$ (c) $\mathbf{j} \times \mathbf{i}$ (d) $(\mathbf{i} + \mathbf{j}) \times (\mathbf{i} - \mathbf{j})$

2 What is the relationship between $\mathbf{a} \times \mathbf{b}$ and $\mathbf{b} \times \mathbf{a}$? Are they the same? Are they a scalar multiple of each other (what scalar?)? Or are they not parallel at all?

3 (a) Calculate $\mathbf{i} \cdot (\mathbf{i} \times \mathbf{j})$. (You should have found $\mathbf{i} \times \mathbf{j} = \mathbf{k}$ in Problem 1, so this asks you to compute $\mathbf{i} \cdot \mathbf{k}$.)

(b) Calculate $(\mathbf{i} + \mathbf{j}) \cdot [\mathbf{i} \times (\mathbf{i} + \mathbf{j})]$. (You should have found $\mathbf{i} \times (\mathbf{i} + \mathbf{j}) = \mathbf{k}$ in Problem 1 as well, so this asks you to find $(\mathbf{i} + \mathbf{j}) \cdot \mathbf{k}$.)

In Problem 2, you were meant to notice that $\mathbf{a} \times \mathbf{b} \neq \mathbf{b} \times \mathbf{a}$ (in fact the relationship is $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$). In Problem 3, the conclusion is that $\mathbf{a} \times \mathbf{b}$ is orthogonal (perpendicular) to both $\mathbf{a}$ and $\mathbf{b}$. But that's obvious from the definition, isn't it?

More generally, we can compute a cross product in component form as follows. Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$. Then we usually write this using *determinants* (as they simplify the formulas tremendously):

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k}$$

$$= (u_2 v_3 - u_3 v_2)\mathbf{i} - (u_1 v_3 - u_3 v_1)\mathbf{j} + (u_1 v_2 - u_2 v_1)\mathbf{k}$$

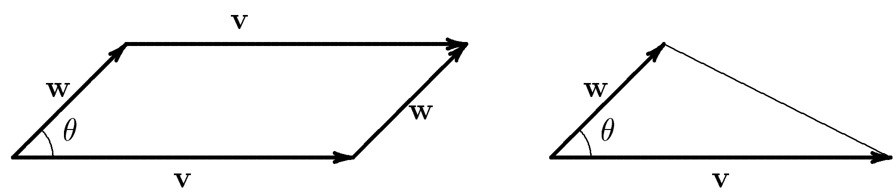
$$= \langle u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1 \rangle.$$

You have your choice: remember the determinant expression (fairly simple) or remember the last expression (ick!).

4 Practice a few cross products where it isn't straightforward to find the angle between the original vectors or the proper orthogonal vector (so the methods above won't work). Here are a few:

(a) $\langle 1, 2, 1 \rangle \times \langle 0, -1, 3 \rangle$ (b) $\langle 2, -2, 1 \rangle \times \langle 2, 1, -1 \rangle$

5 Any two non-zero vectors $\mathbf{v}$ and $\mathbf{w}$ determine a parallelogram (left) and triangle (right):



(a) What is the relationship between the area of the parallelogram and $\mathbf{v} \times \mathbf{w}$?

(b) What is the relationship between the area of the triangle and $\mathbf{v} \times \mathbf{w}$?

6 If the triple scalar product $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$ is 0, what can you say about the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$?

7 Does the expression $\mathbf{u} \times \mathbf{v} \times \mathbf{w}$ make sense? Pick vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$; then compute both

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} \qquad \text{and} \qquad \mathbf{u} \times (\mathbf{v} \times \mathbf{w})\,.$$

(Choose wisely so that you've already done some of the work!) Did you get the same thing in both computations?

8 Two true-false questions:

(a) True or false: If $\mathbf{u} \times \mathbf{v} = \mathbf{u} \times \mathbf{w}$, then $\mathbf{v} = \mathbf{w}$.

(b) True or false: If $\mathbf{v} \times \mathbf{w} = \mathbf{0}$ and $\mathbf{v} \cdot \mathbf{w} = 0$, then one of $\mathbf{v}$ and $\mathbf{w}$ is $\mathbf{0}$.

The Cross Product – Answers and Solutions

1 (a) $\mathbf{i} \times \mathbf{j} = \mathbf{k}$ (b) $\mathbf{i} \times (\mathbf{i} + \mathbf{j}) = \mathbf{k}$

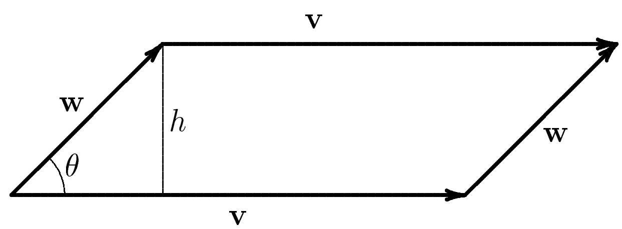
(c) $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$ (d) $(\mathbf{i} + \mathbf{j}) \times (\mathbf{i} - \mathbf{j}) = -2\mathbf{k}$

2 $\mathbf{a} \times \mathbf{b}$ and $\mathbf{b} \times \mathbf{a}$ have the same length but opposite directions, so we have simply $\mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b}$.

3 Both of these (scalar) quantities are zero. One can see this either from doing the computation, or noticing simply that the cross product $\mathbf{a} \times \mathbf{b}$ is chosen to be orthogonal to both vectors $\mathbf{a}$ and $\mathbf{b}$. Thus their dot product must be zero.

4 (a) $\langle 7, -3, -1 \rangle$ (b) $\langle 1, 4, 6 \rangle$

5 (a) The area of the parallelogram is $|\mathbf{v} \times \mathbf{w}|$. Here's a picture that might help explain:



By simple trigonometry, we have that the height is $h = |\mathbf{w}| \sin(\theta)$, so the area of the parallelogram is (the base times the height) $|\mathbf{v}|h = |\mathbf{v}|\ |\mathbf{w}| \sin(\theta)$, or $|\mathbf{v} \times \mathbf{w}|$.

(b) The area of the triangle is $\frac{1}{2}|\mathbf{v} \times \mathbf{w}|$ as the triangle is precisely half of the parallelogram of part (a).

6 This means the $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are coplanar. The dot product vanishing implies that $\mathbf{w}$ is perpendicular to $\mathbf{u} \times \mathbf{v}$. By the definition of the cross product, both $\mathbf{u}$ and $\mathbf{v}$ are perpendicular to $\mathbf{u} \times \mathbf{v}$ as well. Thus $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ all have a common perpendicular, so they all lie on the same plane.

7 The expression $\mathbf{u} \times \mathbf{v} \times \mathbf{w}$ doesn't make sense, since the two expressions

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} \qquad \text{and} \qquad \mathbf{u} \times (\mathbf{v} \times \mathbf{w})$$

can be different than each other. For example, if $\mathbf{u} = \mathbf{v} = \mathbf{i}$ and $\mathbf{w} = \mathbf{j}$, then

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (\mathbf{i} \times \mathbf{i}) \times \mathbf{j} = \mathbf{0} \times \mathbf{j} = \mathbf{0} \qquad \text{but} \qquad \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \mathbf{i} \times (\mathbf{i} \times \mathbf{j}) = \mathbf{i} \times \mathbf{k} = -\mathbf{j}.$$

8 (a) This is false. For example, $\mathbf{u} \times (\mathbf{u} + \mathbf{v}) = \mathbf{u} \times \mathbf{v}$ for any $\mathbf{u}$ and $\mathbf{v}$. This was done in Problem 1(a): $\mathbf{i} \times \mathbf{j} = \mathbf{i} \times (\mathbf{i} + \mathbf{j}) = \mathbf{k}$.

(b) This is true. An easy way to see it is to write the two equations as $|\mathbf{u}|\ |\mathbf{v}| \sin(\theta) = 0$ and $|\mathbf{u}|\ |\mathbf{v}| \cos(\theta) = 0$. We can't have both $\sin(\theta) = 0$ and $\cos(\theta) = 0$, so we must have either $|\mathbf{u}| = 0$ or $|\mathbf{v}| = 0$. Hence one of $\mathbf{u}$ or $\mathbf{v}$ is the zero vector.

Lines and Planes

Lines in Space How can we express the equation(s) of a line through a point (x_0, y_0, z_0) and parallel to the vector $\mathbf{u} = \langle a, b, c\rangle$? Many ways: as *parametric (scalar) equations:*

$$x = x_0 + ta \qquad y = y_0 + tb \qquad z = z_0 + tc;$$

as a *parametric vector equation:*

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{u} \qquad (\text{where } \mathbf{r} = \langle x, y, z\rangle \text{ and } \mathbf{r}_0 = \langle x_0, y_0, z_0\rangle);$$

or by *symmetric* equations:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

1 Let L be the line which passes through the points $(1, -2, 3)$ and $(4, -4, 6)$.

(a) Find a parametric vector equation for L.

(b) Find parametric (scalar) equations for L.

(c) Find symmetric equations for L.

2 How could we write symmetric equations for a line with, say, $c = 0$? Try this for the line through the points $(5, 2, 2)$ and $(3, -1, 2)$.

Planes in Space The equation of a plane through the point (x_0, y_0, z_0) and perpendicular (or normal or orthogonal) to the vector $\mathbf{n} = \langle a, b, c\rangle$ also has many (equivalent) equations:

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0 \qquad \text{or} \qquad \mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r}_0$$

(where again $\mathbf{r} = \langle x, y, z\rangle$ and $\mathbf{r}_0 = \langle x_0, y_0, z_0\rangle$); or

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \qquad \text{or} \qquad ax + by + cz + d = 0$$

(where d is a constant).

3 Find an equation describing the plane which goes through the point $(1, 3, 5)$ and is perpendicular to the vector $\langle 2, 1, -3\rangle$.

4 Find an equation describing the plane which passes through the points $P(2, 2, 1)$, $Q(3, 1, 0)$, and $R(0, -2, 1)$.

5 Let L_1 be the line with parametric vector equation $\mathbf{r}_1(t) = \langle 7, 1, 3\rangle + t\langle 1, 0, -1\rangle$ and L_2 be the line described parametrically by $x = 5, y = 1 + 3t,\ z = t$. How many planes are there which contain L_2 and are parallel to L_1? Find an equation describing one such plane.

Distances Between Points, Lines, and Planes

6 Find the distance from the point $(0, 1, 1)$ to the plane $2x + 3y + 4z = 15$.

7 Find the distance from the point $(1, 3, -2)$ to the line $\frac{x}{3} = y - 1 = z + 2$.

8 Two true-false questions:

(a) True or false: The line $x = 2t$, $y = 1+3t$, $z = 2+4t$ is parallel to the plane $x-2y+z = 7$.

(b) True or false: Let S be a plane normal to the vector $\mathbf{n}$, and let P and Q be points not on the plane S. If $\mathbf{n} \cdot \overrightarrow{PQ} = 0$, then P and Q lie on the same side of S.

Lines and Planes – Answers and Solutions

1 (a) $\mathbf{r}(t) = \langle 1, -2, 3\rangle + t\langle 3, -2, 3\rangle$

(b) $x = 1 + 3t$, $y = -2 - 2t$, $z = 3 + 3t$

(c) $\dfrac{x-1}{3} = \dfrac{y+2}{-2} = \dfrac{z-3}{3}$

2 The parameterization involves simply $z = 2$, so the symmetric equation reduces to

$$\frac{x-5}{-2} = \frac{y-2}{-3} \qquad z = 2.$$

3 Many possibilities:

$$\langle 2, 1, -3\rangle \cdot \langle x-1, y-3, z-5\rangle = 0$$
$$\langle 2, 1, -3\rangle \cdot \langle x, y, z\rangle = \langle 2, 1, -3\rangle \cdot \langle 1, 3, 5\rangle$$
$$2(x-1) + (y-3) - 3(z-5) = 0$$
$$2x + y - 3z + 10 = 0$$

Any one is fine.

4 Here we need an extra step to find the normal $\mathbf{n}$. We find this by finding two vectors in the plane and computing their cross product. We will write $\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR}$, so

$$\begin{aligned}\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR} &= \langle 1, -1, -1\rangle \times \langle -2, -4, 0\rangle \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & -1 \\ -2 & -4 & 0 \end{vmatrix} = \begin{vmatrix} -1 & -1 \\ -4 & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & -1 \\ -2 & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & -1 \\ -2 & -4 \end{vmatrix}\mathbf{k} \\ &= \langle -4, 2, -6\rangle.\end{aligned}$$

Thus the plane is

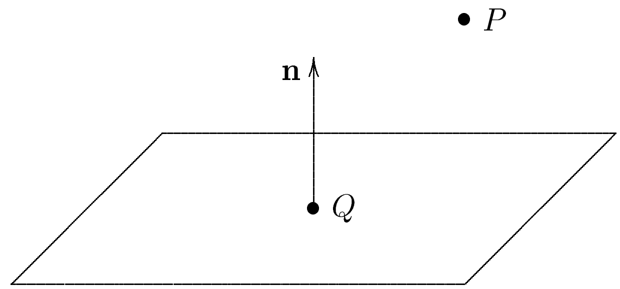
$$\langle -4, 2, -6\rangle \cdot \langle x-2, y-2, z-1\rangle = 0 \qquad \text{or} \qquad -4x + 2y - 6z + 10 = 0.$$

5 Observe that L_1 goes through the point $(7, 1, 3)$ and is parallel to the vector $\langle 1, 0, -1\rangle$ while L_2 goes through $(5, 1, 0)$ and is parallel to the vector $\langle 0, 3, 1\rangle$.

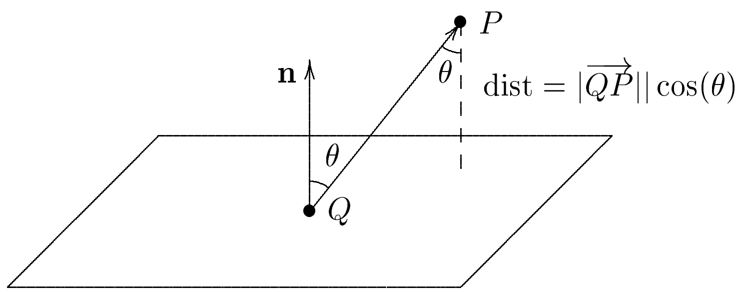
Since L_1 and L_2 are not parallel (which we can see because the vectors $\langle 1, 0, -1\rangle$ and $\langle 0, 3, 1\rangle$ are not parallel), there is only one such plane.

Therefore, the plane in question must be parallel to both $\langle 1, 0, -1\rangle$ and $\langle 0, 3, 1\rangle$, so it is orthogonal to $\langle 1, 0, -1\rangle \times \langle 0, 3, 1\rangle = \langle 3, -1, 3\rangle$. That is, $\mathbf{n} = \langle 3, -1, 3\rangle$ is a normal vector for the plane. In addition, the plane goes through $(5, 1, 0)$. So, the plane has equation $\langle 3, -1, 3\rangle \cdot \langle x-5, y-1, z\rangle = 0$ or $3x - y + 3z - 14 = 0$.

6 We have a point $P(0,1,1)$ and a plane, and we want to find the distance between the two. Here is one method. Suppose Q is *any* point in the plane and $\mathbf{n}$ is a normal vector for the plane.

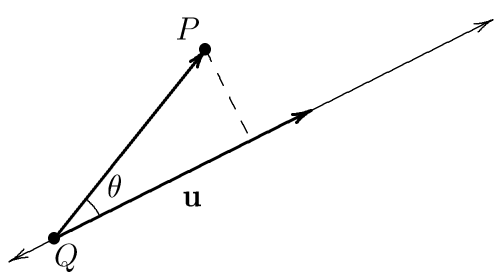


Then the distance from P to the plane is simply the (absolute value of the) scalar projection $|\operatorname{comp}_{\mathbf{n}} \overrightarrow{QP}| = |\overrightarrow{QP}||\cos(\theta)| = \left|\frac{|\mathbf{n}\cdot\overrightarrow{QP}|}{|\mathbf{n}|}\right|$:

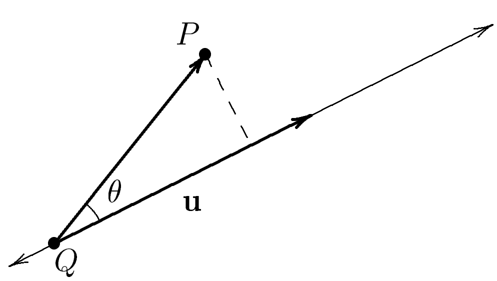


In this particular problem, we can take $Q = (0,5,0)$ as our point on the plane, and $\mathbf{n} = \langle 2,3,4\rangle$ is the normal vector for the plane. Then $\overrightarrow{QP} = \langle 0,-4,1\rangle$ and $\operatorname{comp}_{\mathbf{n}} \overrightarrow{QP} = \frac{\mathbf{n}\cdot\overrightarrow{QP}}{|\mathbf{n}|} = \frac{-8}{\sqrt{29}}$. Thus the distance is $\frac{8}{\sqrt{29}}$.

7 We have a point $P(1,3,-2)$ and a line, and we want to find the distance between the two:



Here's one way to do that. Find a point Q on the line and a vector $\mathbf{u}$ parallel to the line. The distance is then the height of of the right triangle with hypotenuse $\overrightarrow{QP}$ and base on the line:



This height is simply $|\overrightarrow{QP}|\sin(\theta)$, which we recognize as most of the length of the cross product of $\overrightarrow{QP}$ with $\vec{u}$: $|\overrightarrow{QP}|\sin(\theta) = \frac{|\overrightarrow{QP}\times\mathbf{u}|}{|\mathbf{u}|}$.

For the given line, it will be easier to find a point on the line and a vector parallel to the line if we rewrite it using a parametric vector equation. To do this, let's set t equal to $\frac{x}{3} = y - 1 = z + 2$. Then, $x = 3t$, $y = 1 + t$, and $z = -2 + t$, so we can describe the line by the parametric vector equation $\langle 0, 1, -2\rangle + t\langle 3, 1, 1\rangle$. From this, we can see that $Q(0, 1, -2)$ is a point on the line and $\mathbf{u} = \langle 3, 1, 1\rangle$ is a vector parallel to the line.

Now, we just compute $\frac{|\overrightarrow{QP}\times\mathbf{u}|}{|\mathbf{u}|}$: $\overrightarrow{QP} = \langle 1, 2, 0\rangle$, so $\overrightarrow{QP}\times\mathbf{u} = \langle 2, -1, -5\rangle$ and $\dfrac{|\overrightarrow{QP}\times\mathbf{u}|}{|\mathbf{u}|} = \sqrt{\dfrac{30}{11}}$.

8 (a) True.

A normal vector for the plane is $\mathbf{n} = \langle 1, -2, 1\rangle$. The line $x = 2t$, $y = 1 + 3t$, $z = 2 + 4t$ is parallel to the vector $\langle 2, 3, 4\rangle$, and this vector is orthogonal to $\mathbf{n}$, so this vector must be parallel to the plane.

Another way to see that the line and plane are parallel is to try to compute the intersection. If (x, y, z) is in both the line and plane, then the four equations $x = 2t$, $y = 1 + 3t$, $z = 2 + 4t$, and $x - 2y + z = 7$ must all be satisfied. Plugging the first three equations into the fourth, $2t - 2(1 + 3t) + (2 + 4t) = 7$, which simplifies to $0 = 7$, so there are no solutions to all four equations. This means that the line and plane do not intersect, so they must be parallel.

(b) True. The fact that $\mathbf{n}\cdot\overrightarrow{PQ} = 0$ means that $\mathbf{n}$ is orthogonal to $\overrightarrow{PQ}$, so $\overrightarrow{PQ}$ is parallel to the plane.

Math 21a — Spring, 2009

Graphs in Space

Here are six functions whose graphs we ought to be able to visualize in space:

1. $z = f(x, y) = 6 - 3x - 2y$
2. $z = f(x, y) = x^2 + y^2$
3. $z = f(x, y) = x^2 - y^2$
4. $z = f(x, y) = x^2 + y + 1$
5. $z = f(x, y) = (x - y)^2$
6. $z = f(x, y) = \dfrac{1}{1 + x^2 + y^2}$

Traces: One way to visualize surfaces that are three-dimensional graphs is to view pieces of them as two-dimensional graphs. If we intersect the graph of $z = f(x, y)$ with a plane (such as $x = k$ or $y = k$), we get a graph in a two-dimensional plane (the kind we're used to). This is called a *trace*. If we intersect the graph of $z = f(x, y)$ with the plane $z = k$, we get *level curves* (not necessarily graphs, although sometimes also called traces).

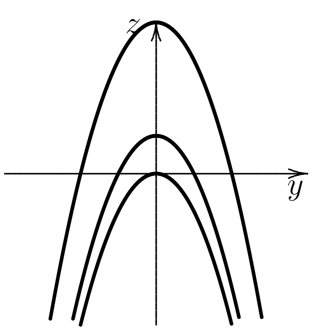
1. Here are traces for two of the surfaces above. Your job is to:

   (a) identify which graph these traces belong with,

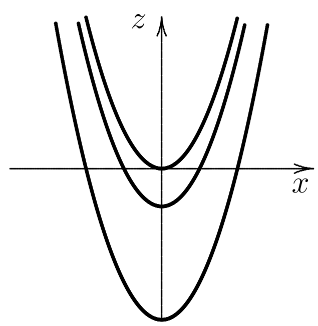
   (b) graph traces of the remaining graphs of surfaces, and

   (c) try to visualize the original surface.

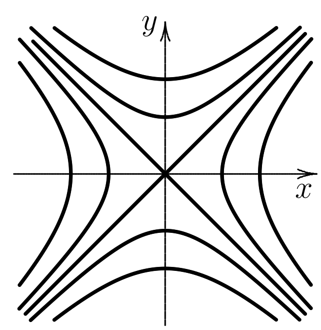
First set of traces (each with $k = 0$, ± 1, and ± 2, when possible):



$x = k$

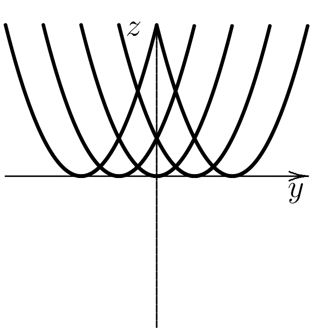


$y = k$

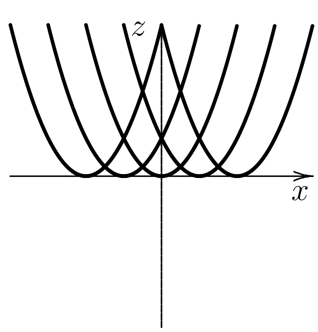


$z = k$

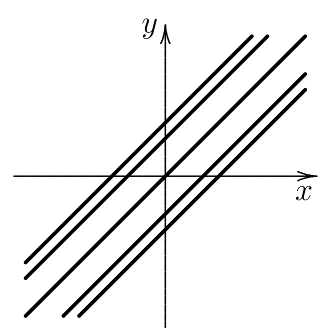
Second set of traces (each with $k = 0$, ± 1, and ± 2, when possible):



$x = k$

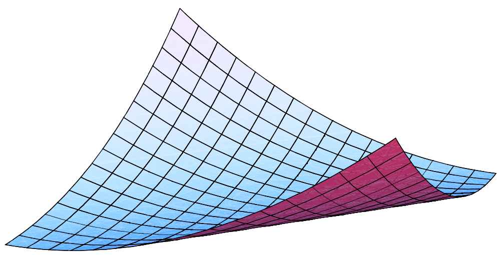


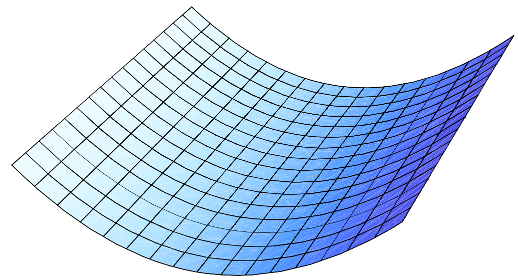
$y = k$

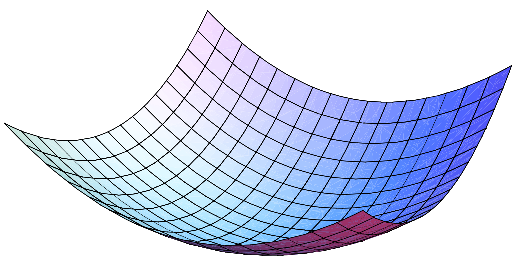


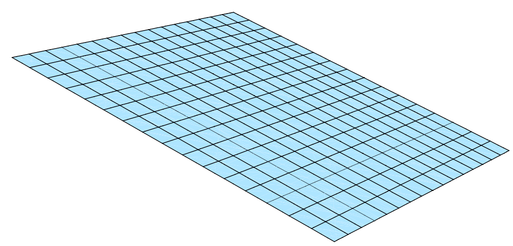
$z = k$

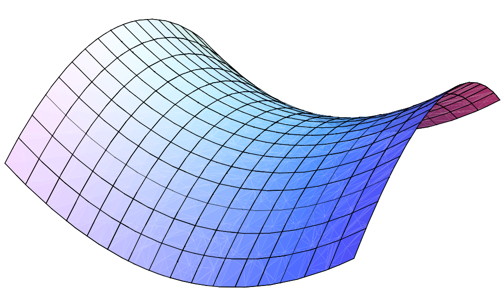
2 Now that you have a good idea of what each of these graphs look like, you should have no problem identifying which of the following (axes-less) graphs go with each equation for the previous page. Your reasoning should involve the graphs of traces you drew as well.

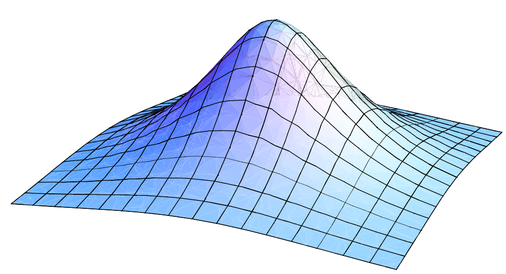












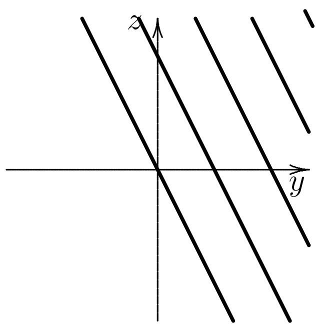
Graphs in Space – Answers and Solutions

For solutions, we'll show the traces and appropriate graph for each of the equations.

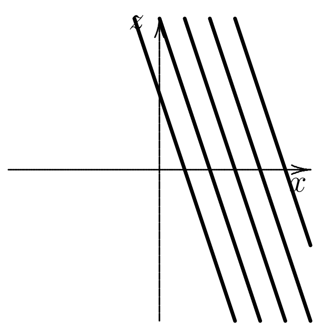
Warning: The graphs are not necessarily all shown from the same perspective! They have been rotated to show the appropriate features, so don't assume that the coordinate axes are positioned in any particular way!

1 $z = f(x, y) = 6 - 3x - 2y$

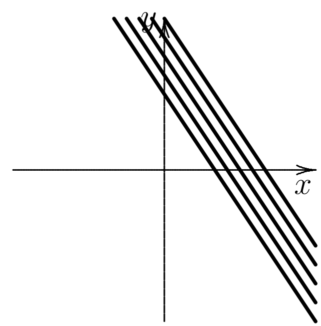
Here are the traces for $k = -2$, -1, 0, 1, and 2:



$x = k$

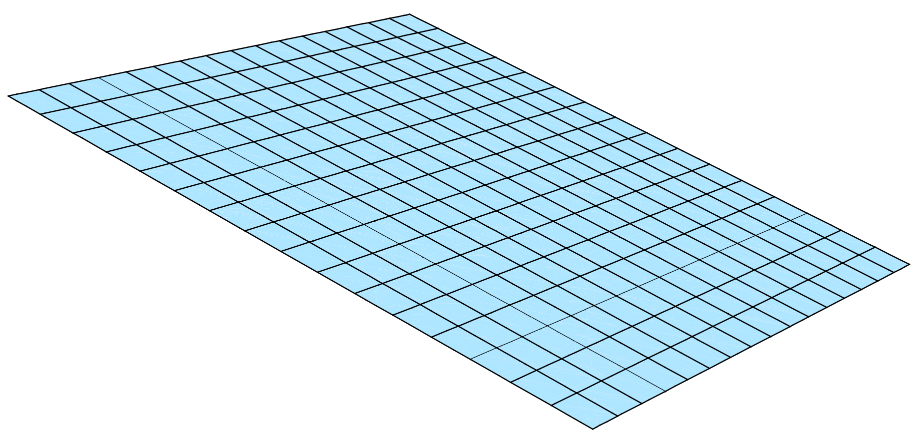


$y = k$



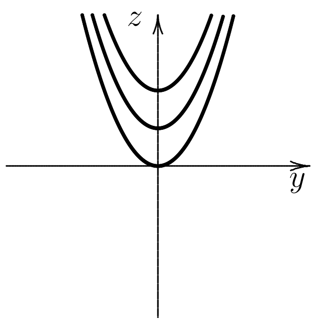
$z = k$

All these traces are lines, which makes sense as the original equation is simply $3x+2y+z = 6$, a plane!

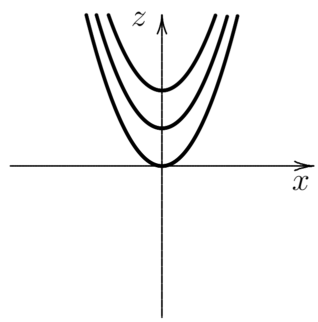


$\boxed{2}$ $z = f(x, y) = x^2 + y^2$

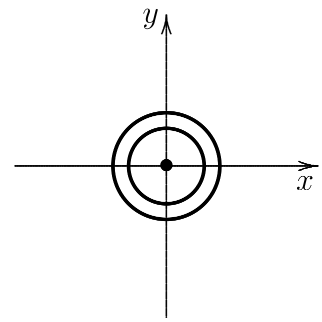
Here are the traces for $k = -2, -1, 0, 1$, and 2:



$x = k$

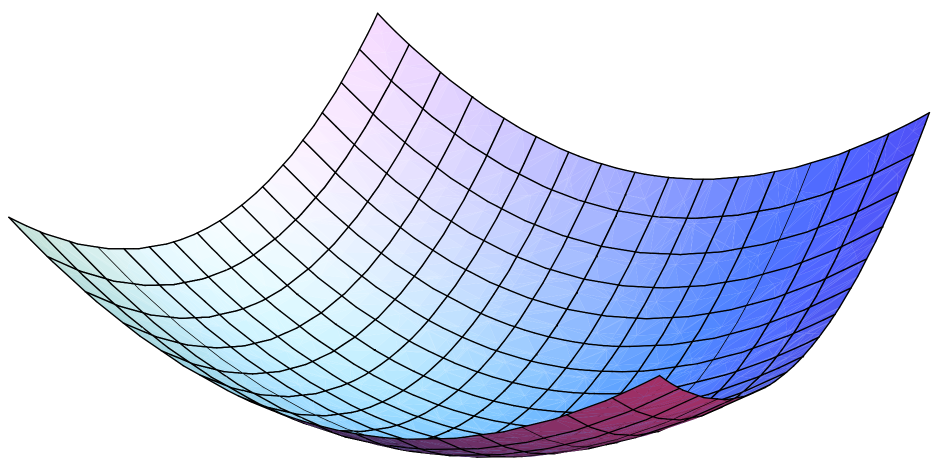


$y = k$



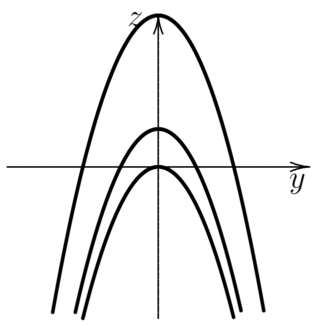
$z = k$

The $z = k$ traces are circles $x^2 + y^2 = k$ of radius $\sqrt{k}$ (so there is no trace at all if $k < 0$).

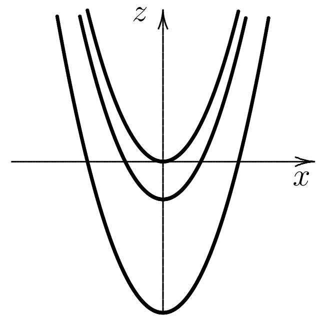


$\boxed{3}$ $z = f(x, y) = x^2 - y^2$

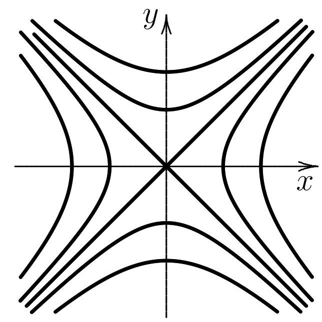
This is the first set of traces (from the first page of the worksheet):



$x = k$

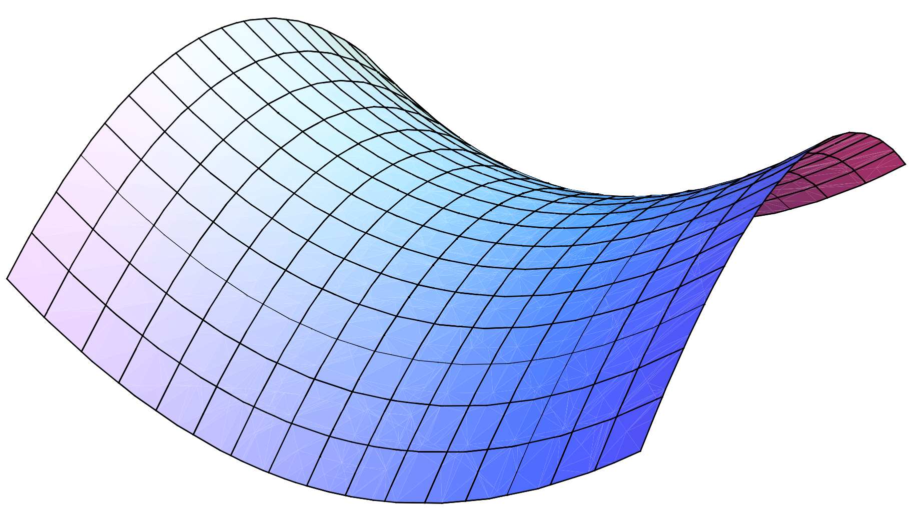


$y = k$



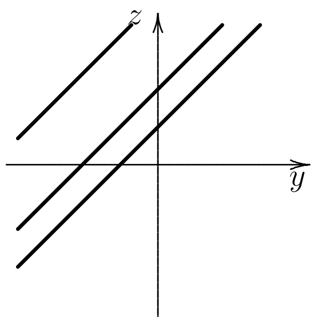
$z = k$

Note the hyperbolas in the $z = k$ traces.

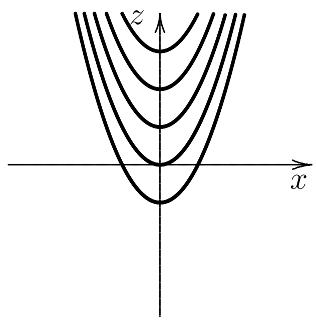


$\boxed{4}$ $z = f(x, y) = x^2 + y + 1$

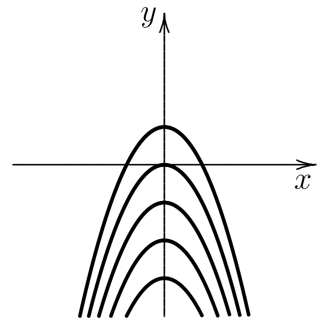
Here are the traces for $k = -2$, -1, 0, 1, and 2:



$x = k$

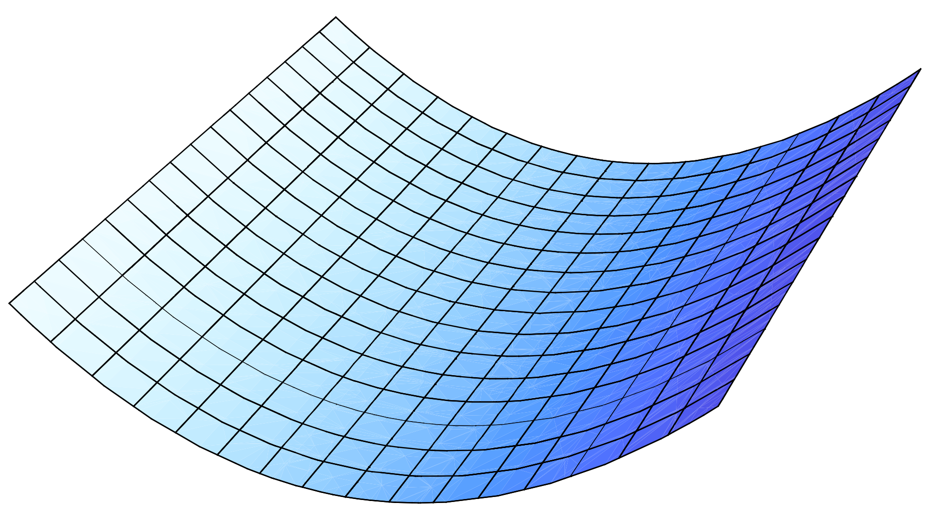


$y = k$



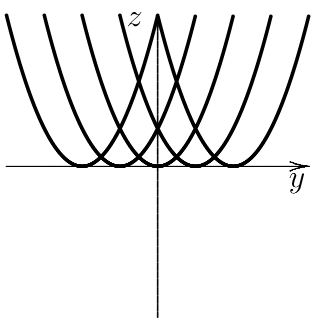
$z = k$

The $z = k$ traces are the lines $x - y = \pm\sqrt{k}$ or $y = x \mp \sqrt{k}$.

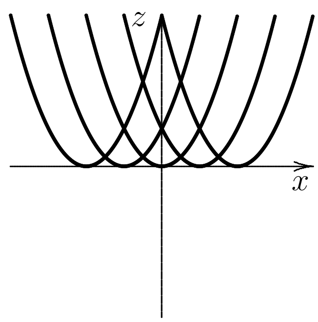


5 $z = f(x, y) = (x - y)^2$

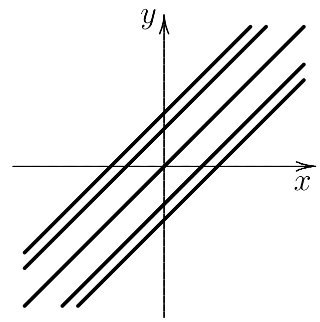
This is the second set of traces (from the first page of the worksheet):



$x = k$

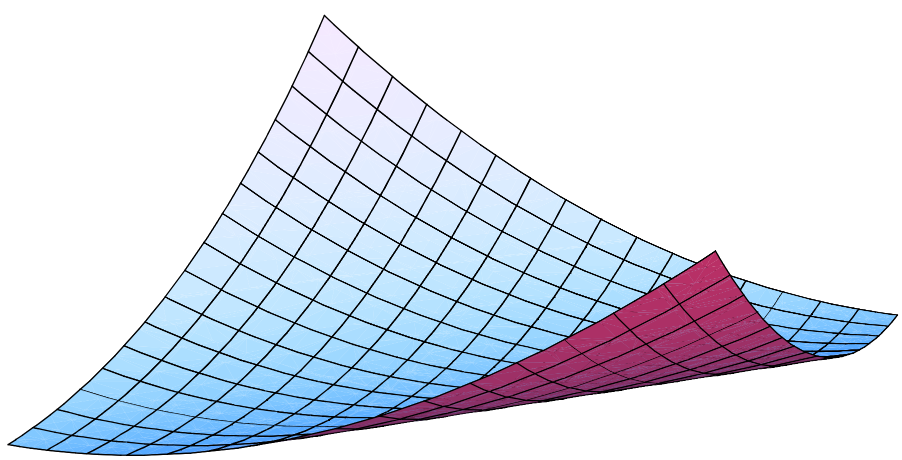


$y = k$



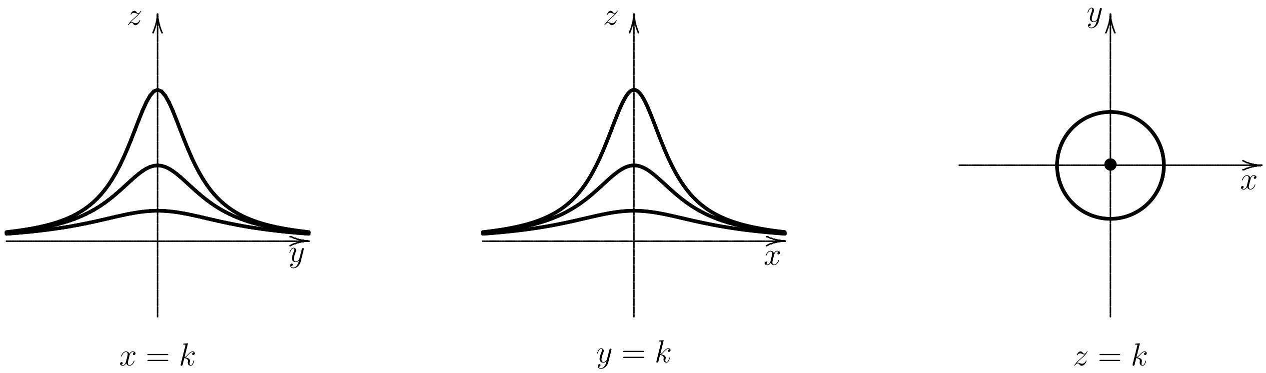
$z = k$

The $z = k$ traces are the lines $x - y = \pm\sqrt{k}$ or $y = x \mp \sqrt{k}$.

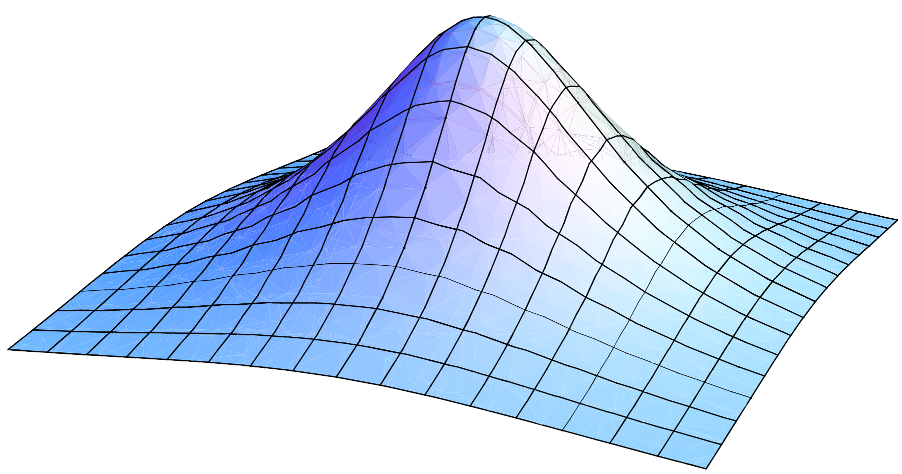


6 $z = f(x, y) = \dfrac{1}{1 + x^2 + y^2}$

Here are the traces for $k = -2, -1, 0, 1$, and 2.



Notice that the $z = k$ traces are only sensible for $k > 0$, so there are only two traces. These are each a circle of radius $\sqrt{1 - \frac{1}{k}}$, so when $k = 1$ this circle has degenerated into a point. The $x = k$ and $y = k$ traces, on the other hand, are the same graphs and don't depend on the sign of k.



Other Coordinate Systems

For each of the following, we'd like four descriptions of the point, surface, or solid:

(a) An equation in Cartesian coordinates (the usual x, y, z).

(b) An equation in cylindrical coordinates (r, θ, z).

(c) An equation in spherical coordinates (ρ, θ, ϕ).

(d) A description or graph of the surface or curve.

For each of the following points, surfaces and solids, expand the description to include all four of the above approaches.

Points:

1. $(x, y, z) = (1, 2, 3)$

2. $(r, \theta, z) = \left(3, \frac{\pi}{4}, 3\right)$

3. $(\rho, \theta, \phi) = \left(2, \frac{\pi}{3}, \frac{\pi}{6}\right)$

Surfaces:

4. $\theta = \frac{\pi}{4}$
(This is in cylindrical coordinates. Does it matter?)

5. $x^2 + y^2 + z^2 - 2z = 0$

6. $r = 2\cos(\theta)$

7. $\phi = \frac{\pi}{4}$

8. $z^2 - x^2 - y^2 = 1$

9. $\rho = 2\cos(\phi)$

Solids:

10. $3 \leq r \leq 5$

11. $\frac{\pi}{6} < \phi < \frac{\pi}{4}$

12. $\cos(\theta) < r < 1$

13. The solid formed by a sphere of radius 2 centered at the origin with a cylinder removed. Assume the cylinder has radius 1 and is centered on the z-axis.

Other Coordinate Systems – Answers and Solutions

Points:

1 $(r, \theta, z) = (\sqrt{5}, \tan^{-1}(2), 3) \approx$

$(\rho, \theta, \phi) = (\sqrt{14}, \tan^{-1}(2), \cos^{-1}(3/\sqrt{14})) \approx$

2 $(x, y, z) = (3\cos(\pi/4), 3\sin(\pi/4), 3) = \left(\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}, 3\right) \approx$

$(\rho, \theta, \phi) = (3\sqrt{2}, \frac{\pi}{4}, \frac{\pi}{4})$

3 $(x, y, z) = (2\sin(\pi/6)\cos(\pi/3), 2\sin(\pi/6)\sin(\pi/3), 2\cos(\pi/6)) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}, \sqrt{3}\right)$

$(r, \theta, z) = \big(2\sin(\pi/6), \frac{\pi}{3}, 2\cos(\pi/6)\big) = \big(1, \frac{\pi}{3}, \sqrt{3}\big)$

Surfaces:

4 (a) $y = x$ but also $x \geq 0$.
(b) $\theta = \frac{\pi}{4}$
(c) $\theta = \frac{\pi}{4}$ (the same in spherical as in cylindrical coordinates)
(d) This is a half-plane over (and under) the part of the line $y = x$ in the first quadrant (including the origin)

5 (a) $x^2 + y^2 + (z-1)^2 = 1$
(b) $r^2 + (z-1)^2 = 1$
(c) $\rho^2 = 2\rho\cos(\phi)$ or simply $\rho = 2\cos(\phi)$
(d) This is a sphere of radius 1 centered at $(x, y, z) = (0, 0, 1)$.

6 (a) $(x-1)^2 + y^2 = 1$
(b) $r = 2\cos(\theta)$
(c) $\rho\sin(\phi) = 2\cos(\theta)$ or perhaps $\rho = \frac{2\cos(\theta)}{\sin(\phi)}$
(d) This is a cylinder of radius 1 centered about the line $\mathbf{r}(t) = \langle 1, 0, 0\rangle + t\langle 0, 0, 1\rangle$. That is, it is a vertical cylinder above the circle of radius 1 centered in the xy-plane at the point $(x, y) = (1, 0)$.

7 (a) $z = \sqrt{x^2 + y^2}$. (The top half only of $z^2 = x^2 + y^2$.)
(b) $z = r$
(c) $\phi = \frac{\pi}{4}$
(d) This is the top half of a cone. It opens up, centered along the z-axis, with "point" at the origin. The trace $z = k$ for positive k is a circle of radius k (centered at the origin of the xy-plane).

$\boxed{8}$ (a) $z^2 - x^2 - y^2 = 1$

(b) $z^2 - r^2 = 1$

(c) $\rho = \dfrac{1}{\sqrt{2\cos^2(\phi) - 1}}$ or perhaps just $\rho^2\left(2\cos^2(\phi) - 1\right) = 1$.

(d) This is a hyperboloid of two sheets.

$\boxed{9}$ This is the same as Problem $\boxed{5}$.

Solids:

$\boxed{10}$ (a) $9 < x^2 + y^2 < 25$

(b) $3 < r < 5$

(c) $\frac{3}{\sin(\phi)} < \rho < \frac{5}{\sin(\phi)}$ or simply $3 < \rho\sin(\phi) < 5$ or simply

(d) This is a "solid cylinder" centered around the z-axis with interior radius 3 and exterior radius 5.

$\boxed{11}$ (a) The "top half" of $x^2 + y^2 < z^2 < 3\left(x^2 + y^2\right)$; that is, $\sqrt{x^2 + y^2} < z < \sqrt{3}\sqrt{x^2 + y^2}$.

(b) The "top half" of $r^2 < z^2 < 3r^2$; that is, $r < z < \sqrt{3}r$.

(c) $\frac{\pi}{6} < \phi < \frac{\pi}{4}$

(d) This is a "thickened cone" centered around the z-axis where the exterior angle is $\pi/4$ and the interior angle is $\pi/6$.

$\boxed{12}$ (a) In the xy-plane, this is the intersection of $\left(x - \frac{1}{2}\right)^2 > \frac{1}{4}$ (the points outside the circle of radius $1/2$ centered at $(x, y) = (1/2, 0)$) and $x^2 + y^2 < 1$ (the points inside the unit circle centered at the origin). In space, therefore, it is the solid unit cylinder (centered around the z-axis) with a cylinder removed (the removed cylinder is centered around the line $x = 1/2$, $y = 0$ and has radius $1/2$).

(b) $\cos(\theta) < r < 1$.

(c) $\cos(\theta) < \rho\sin(\phi) < 1$ or $\frac{\cos(\theta)}{\sin(\phi)} < \rho < \frac{1}{\sin(\phi)}$.

(d) See the answer to (a).

$\boxed{13}$ This is very similar to Example 8 on page 688 of the text, so you might prefer to read that.

(a) $1 \le x^2 + y^2 \le 4 - z^2$

(b) $1 \le r^2 \le 4 - z^2$ or $1 \le r \le \sqrt{4 - z^2}$

(c) $\sqrt{1 + \rho^2\cos^2(\phi)} \le \rho \le 2$ or $1 + \rho^2\cos^2(\phi) \le \rho^2 \le 4$.

(d) See the statement. The assumption here is that it is a ball (a "solid" sphere) rather than simply a sphere.

Vector-Valued Functions

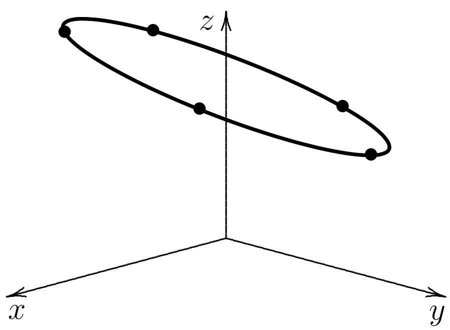
We're now going to study *vector-valued functions* or *parameterized curves*. We can write this as either

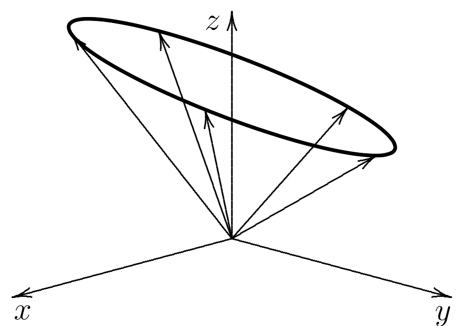
$$x = f(t) \qquad y = g(t) \qquad z = h(t)$$

or

$$\mathbf{r}(t) = \langle x, y, z \rangle = \langle f(t), g(t), h(t) \rangle.$$

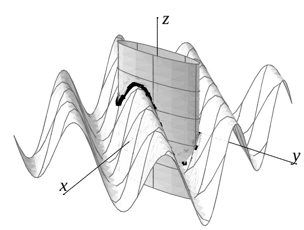
The difference is as illustrated in the two pictures below. Both are the same curve, but written as a parameterized curve (on the left, with the equations from the left) it is a collection of points while as a vector-valued function the curve is the trace of the heads of a collection of position vectors (vectors with their feet at the origin).



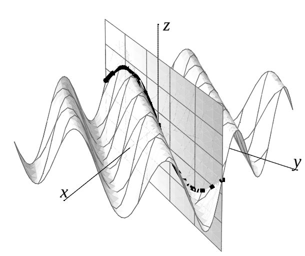


1 Problem 1 is the matching problem on the back of this page. Start there!

2 (a) The surfaces $9x^2 + \frac{y^2}{4} = 1$ and $z = \sin(x - y)$ intersect in a curve. Find a parameterization of the curve.



(b) The surfaces $z = \sin(x - y)$ and $y = 2x$ intersect in a curve. Find a parameterization of the curve.



1 Match each vector-valued function to the curve it parameterizes.

(a) $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$

(b) $\mathbf{r}(t) = \langle t \cos t, t \sin t, t \rangle$

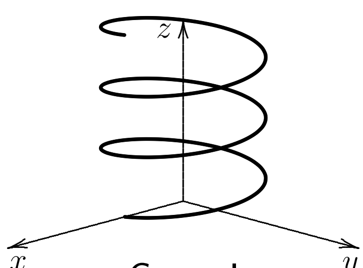
(c) $\mathbf{r}(t) = \langle \cos t, \sin t, t^3 \rangle$

(d) $\mathbf{r}(t) = \langle \cos t^3, \sin t^3, t^3 \rangle$

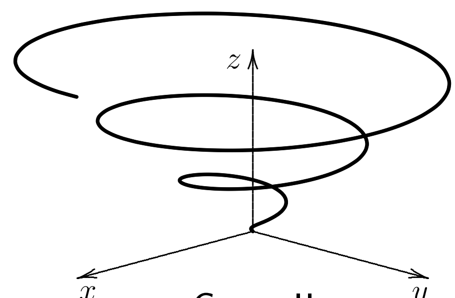
(e) $\mathbf{r}(u) = \langle \cos u, \sin u, 1 + \sin 4u \rangle$

(f) $\mathbf{r}(u) = \langle \cos u, \sin u, 1 + 4 \sin u \rangle$

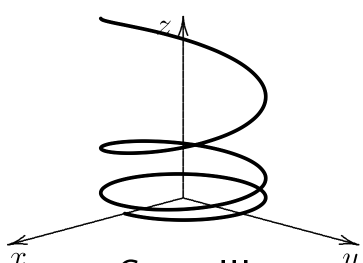
(g) $\mathbf{r}(t) = \langle 2 \cos t, 1 + 4 \cos t, 3 \cos t \rangle$



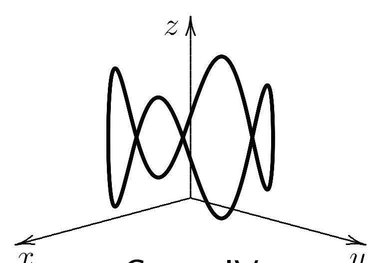
Curve I



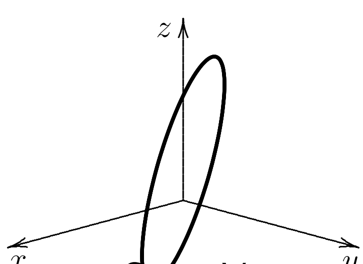
Curve II



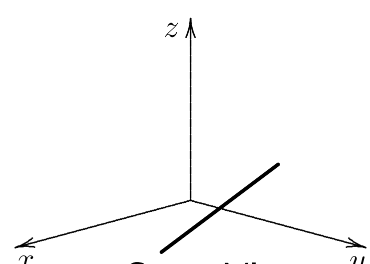
Curve III



Curve IV



Curve V



Curve VI

Vector-Valued Functions – Answers and Solutions

1 (a) This is Curve II, the *helix*. When looked on from above (from the positive z direction), this curve is simply a circle in the xy-plane. The $z = t$ component lifts the circle into the helix spinning above the circle in the plane.

If we visualize this as a particle at the tip of the position vector $\mathbf{r}(t)$, then from above it looks like the particle is simply spinning in a circle. But we also know that $z = t$, so the particle is rising at a constant rate. Hence the helix.

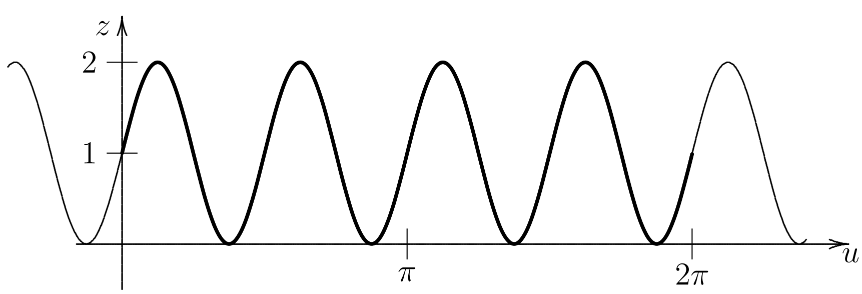
(b) This is Curve II. This is very similar to part (a), except now the x and y components have a changing magnitude (namely, $|t|$). Projected onto the xy-plane (or viewed from above), this is a spiral $\mathbf{r}(t) = \langle t\cos t, t\sin t\rangle$, and in space it is this elongated spiral.

(c) This is Curve III. This is again very similar to part (a), except now the z coordinate rises very slowly at first (so the lower rings are close together), then very quickly (so the higher rings are farther apart).

(d) This is Curve I again. If we make the substitution $s = t^3$, then the curve is $\mathbf{r}(s) = \langle \cos s, \sin s, s\rangle$ (the same as part (a), although with a differently named parameter). If we visualize this vector-valued function as telling us not only the path we traverse, but also how quickly we travel along it, we find the difference between (a) and (d). In part (a), we rise at a constant speed (since $z = t$), but in part (d) we rise slowly at first then very quickly (here $z = t^3$).

We trace out exactly the same curve in parts (a) and (d), but the rate at which we trace out the curve differs between (a) and (d). Put another way, the particle (of the answer to part (a)) travels at a constant rate in part (a) but a varying (slower at first, then faster and faster) rate in part (d).

(e) This is Curve IV. Again as we look from above we have a unit circle in the xy-plane. But what happens to the height z as u passes from $u = 0$ to $u = 2\pi$ (through one circle in the xy-plane)? Here's a graph of $z = 1 + \sin(4u)$ in the uz-plane, with the graph over $[0, 2\pi]$ highlighted:



Thus we can see that when our particle makes one orbit in the xy-plane, the height z rises and falls a total of four cycles. Thus we're looking at Curve IV.

(f) This is Curve V. We can make a similar argument here as we did in part (e), except now z makes one cycle (instead of four) for every cycle that x and y make. Another approach is to consider the z coordinates as bounded between -3 and 5 while x and y coordinates proceed in a circle. Both these approaches point to Curve V.

Another interesting approach is to notice that $y = \sin u$ and $z = 1+4\sin u$, so $z = 1+4y$. Thus our curve must lie on the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $z = 1 + 4y$. In fact our curve is the ellipse of intersection of these two surfaces.

(g) This is Curve VI. If we make the substitution $u = \cos t$, we get

$$\mathbf{r} = \langle 2u, 1 + 4u, 3u\rangle = \langle 0, 1, 0\rangle + u\langle 2, 4u, u\rangle.$$

This is the equation of a line.

We should note that $-1 \le u \le 1$ (since $u = \cos t$), so we actually have only a segment of the line. Thus Curve VI is our line segment.

2 (a) Notice that we already know how to express z in terms of x and y because $z = \sin(x-y)$. Therefore, if we can express both x and y in terms of a parameter t, we will automatically be able to express z in terms of that parameter as well.

So, let's focus on the relationship between x and y, which is given by the equation $9x^2 + \frac{y^2}{4} = 1$. If we rewrite this as $(3x)^2 + (y/2)^2 = 1$, then we see that we can write $3x = \cos t$, $y/2 = \sin t$, or $x = \frac{1}{3}\cos t$ and $y = 2\sin t$.

Since $z = \sin(x - y)$, we now have $z = \sin\left(\frac{1}{3}\cos t - 2\sin t\right)$. We can also write this as the vector-valued function

$$\mathbf{r}(t) = \left\langle \tfrac{1}{3}\cos t, 2\sin t, \sin(\tfrac{1}{3}\cos t - 2\sin t)\right\rangle.$$

(b) As in the previous part, it's easy to express z in terms of x and y, so we should focus on writing x and y in terms of a parameter t. Notice, however, that this time it's also easy to write y in terms of x, since $y = 2x$. Therefore, we can simply let x be the parameter, $x = t$. Then, $y = 2x = 2t$, and $z = \sin(x - y) = \sin(t - 2t) = \sin(-t) = -\sin t$. Written as a vector-valued function, $\mathbf{r}(t) = \langle t, 2t, -\sin t\rangle$.

Parametric Surfaces

For each of the following surfaces,

(a) write the surface in parameterized form as $\mathbf{r}(u, v)$, and

(b) describe the "grid curves" $u = k$ and $v = k$ for your parameterization.

1 $3x + 2y + z = 6$

2 $x^2 + y^2 + z^2 = 4$ but $z \geq 0$

3 $\dfrac{x^2}{4} - \dfrac{y^2}{9} + z^2 = 1$

4 The graph of the curve $y = 1 + \sin(x)$ revolved around the x-axis

5 $-\dfrac{x^2}{9} - \dfrac{y^2}{16} + z^2 = 1$

6 The graph of the curve $y = x^2 + x \qquad (0 \leq x \leq 2)$ revolved around the x-axis

Identify each of the following parameterized surfaces:

7 $\mathbf{r}(u, v) = \langle 2 + u, 3 - u - v, v + 5\rangle$

8 $\mathbf{r}(u, v) = \langle u, \cos(v), \sin(v)\rangle$

9 $\mathbf{r}(u, v) = \langle u, v, v^2 - u^2\rangle$

10 $\mathbf{r}(u, v) = \langle u^2, 4\cos(v), 5\sin(v)\rangle$

11 $\mathbf{r}(u, v) = \langle u, u\sin(v), u\cos(v)\rangle$

12 $\mathbf{r}(u, v) = u\cos(v), \sqrt[3]{u}, u\sin(v)\rangle$

Parametric Surfaces – Matching

On this page, match the equation of the surface to the appropriate graph. Give reasons!

(a) $\mathbf{r}(u,v) = \langle u, \sin(u)\cos(v), \sin(u)\sin(v)\rangle$

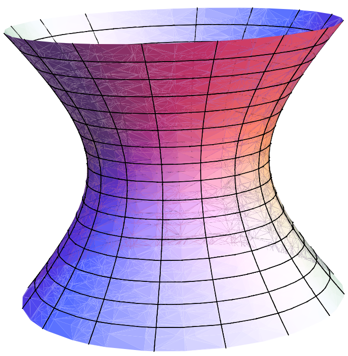
(b) $\mathbf{r}(u,v) = \langle u+1, v-2, 3-u-2v\rangle$

(c) $\mathbf{r}(u,v) = u\mathbf{i} + v\mathbf{j} + \sqrt{u^2+v^2}\,\mathbf{k}$

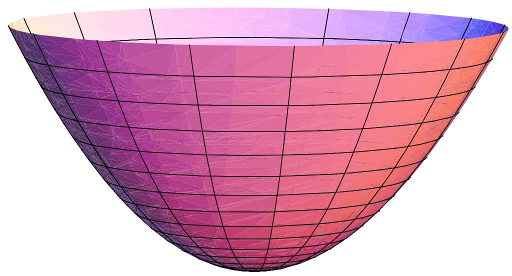
(d) $\mathbf{r}(u,v) = \langle \sin(u)\cos(v), \cos(u), \sin(u)\sin(v)\rangle$
$0 \le u \le \pi/2$, $0 \le v \le \pi$

(e) $\mathbf{r}(u,v) = \langle av\cos(u), bv\sin(u), 2v^2\rangle$

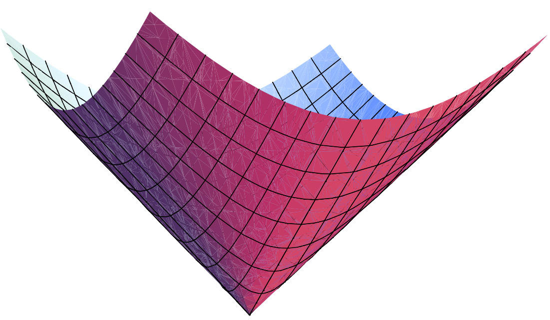
(f) $\mathbf{r}(u,v) = \langle \sqrt{1+u^2}\cos(v), \sqrt{1+u^2}\sin(v), u\rangle$



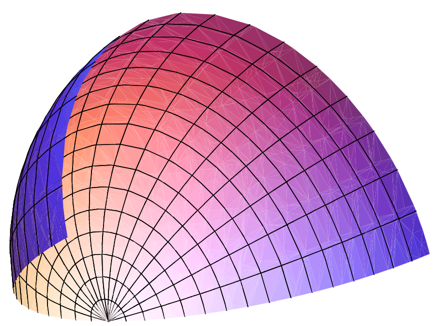
Graph I



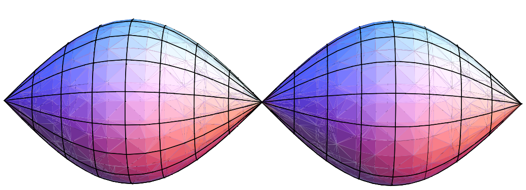
Graph II



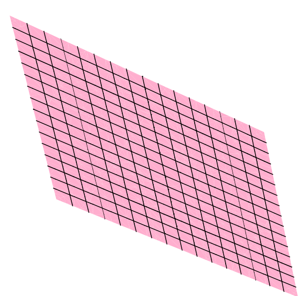
Graph III



Graph IV



Graph V



Graph VI

Parametric Surfaces – Answers and Solutions

1 (a) One choice is $\mathbf{r}(u,v) = \langle u, v, 6-3u-2v\rangle$. Many many others are possible.

(b) When $u = k$ we get the line $\mathbf{r}(v) = \langle k, v, 6-3k-2v\rangle$. For example, $\mathbf{r}(v) = \langle 1, v, 3-2v\rangle$ or $\mathbf{r} = \langle 1,0,3\rangle + v\langle 0,1,-2\rangle$ when $k=1$. Similarly, we get the line $\mathbf{r}(u) = \langle 0,k,6-2k\rangle + u\langle 1,0,-3\rangle$ when $v = k$.

2 (a) Two reasonable choices are $\mathbf{r}(u,v) = \langle u, v, \sqrt{4-u^2-v^2}\rangle$ or (using spherical coordinates):

$$\mathbf{r}(\phi,\theta) = \langle 2\sin(\phi)\cos(\theta), 2\sin(\phi)\sin(\theta), 2\cos(\phi)\rangle \qquad 0 \le \phi \le \frac{\pi}{2},\quad 0 \le \theta < 2\pi.$$

(b) The grid curves for the first parameterization are the lines on the hemisphere directly above the lines $x = k$ and $y = k$. In the spherical coordinate parameterization, the grid curves are latitude lines (circles at a fixed height) for $\phi = k$ and arcs of great circles from the "north pole" to the "equator" for $\theta = k$.

3 (a) Here we agreed to assume that $y \ge 0$ (although see (f) on page 2 for a similar example with a single parameterization). We could start out $x = u$ and $z = v$, but instead we use the parameterization

$$\mathbf{r}(u,v) = \langle 2v\cos(u), 3\sqrt{v^2-1}, v\sin(u)\rangle \qquad 0 \le u < 2\pi,\quad v \ge 1.$$

(b) The grid curves corresponding to $v = k$ are (provided $k \ge 1$) ellipses in the $y = 3\sqrt{k^2-1}$ plane, parallel to the xz-plane. The $u = k$ grid curves are hyperbolas in space over the ray corresponding to the angle $\theta = u$ in the xy-plane.

4 (a) The usual parameterization for this surface of revolution is

$$\mathbf{r}(u,v) = \langle u, (1+\sin(u))\cos(v), (1+\sin(u))\sin(v)\rangle, \qquad 0 \le v < 2\pi.$$

This is simply $\mathbf{r}(x,\theta) = \langle x, f(x)\cos(\theta), f(x)\sin(\theta)\rangle$.

(b) The $u = k$ grid curves are circles lying in planes parallel to the yz-plane, while the $v = k$ grid curves are copies of the graph of $y = 1+\sin(x)$ rotated through an angle of k about the x-axis.

5 This is a double-sheeted hyperboloid, so we will restrict ourselves to only one sheet. Let's assume that $z \ge 0$.

(a) One way to do this is to simply use the $x = u$, $y = v$ parameterization, from which we find that

$$\mathbf{r}(u,v) = \left\langle u, v, \sqrt{1+\frac{u^2}{9}+\frac{v^2}{16}}\right\rangle.$$

Another approach is to notice that the $z = k$ trace is an ellipse and to start with that ellipse. Let's let $x = 3u\cos(v)$ and $y = 4u\sin(v)$, so $z^2 = 1+\frac{(3u\cos(v))^2}{9}+\frac{(4u\sin(v))^2}{16} = 1+u^2$. Thus our parameterization becomes

$$\mathbf{r}(u,v) = \left\langle 3u\cos(v), 4u\sin(v), \sqrt{1+u^2}\right\rangle.$$

We'll use this one in part b.

(b) When $u = k$, our grid curve is an ellipse in the plane $z = \sqrt{1+k^2}$. When $v = k$, we again (as in Problem 3) have a portion of a hyperbola lying over the ray with $\theta = k$ in the xy-plane.

6 (a) Following the example of Problem 4, we have as our parameterization

$$\mathbf{r}(u,v) = \left\langle u, (u^2+u)\cos(v), (u^2+u)\sin(v) \right\rangle, \qquad 0 \le u \le 2, \quad 0 \le v < 2\pi.$$

(b) The $u = k$ grid curves are circles in planes parallel to the yz-plane, while the $v = k$ grid curves are copies of the graph $y = x^2 - x$ rotated through an angle $\theta = k$ around the x-axis. Note the similarity of this answer to the answer in Problem 4(b).

7 This is a plane. Solve the equations

$$x = 2 + u \qquad\qquad y = 3 - u - v \qquad\qquad z = v + 5$$

for u and v. We get $u = x - 2$ and $v = z - 5$. Plugging these into the equation for y, we get $y = 3 - (x-2) - (z-5) = 10 - x - z$, or $x + y + z = 10$.

8 This is a cylinder. The traces $u = k$ are circles in planes parallel to the yz-plane, whereas the the $v = k$ traces are lines parallel to the x-axis through the point $(0, \cos(k), \sin(k))$.

9 This is the graph of $z = y^2 - x^2$. This is a hyperbolic paraboloid (see the table on page 682 of Stewart).

10 This is very similar to Problem 8. The differences: we now have ellipses (rather than circles) as the $u = k$ traces parallel to the yz-plane, and the $v = k$ traces are only rays, not lines (since $x = u^2 \ge 0$. Thus we get what might be called an elliptical half-cylinder.

11 The $u = k$ traces here circles of radius k centered on the x-axis in the plane $x = k$. This is a cone. (The $v = k$ traces are lines that project line with an angle $\theta = v$ in the yz-plane. These lines make a 45° angle with the x-axis.

12 The $u = k$ traces are circles of radius k centered on the y-axis in the plane $y = \sqrt[3]{k}$. The $v = k$ traces are curves that are actually rotations of the $v = 0$ curve ($\mathbf{r}(u) = \langle u, \sqrt[3]{u}, 0\rangle$ is the $v = 0$ curve) about the y-axis. In fact we have a surface of revolution.

(If we make the change $t = \sqrt[3]{u}$, we get the parameterization $\mathbf{r}(t,v) = \langle t^3\cos(v), t, t^3\sin(v)\rangle$. This looks more like the surface of revolution example from the text, except the axis of revolution here is the y-axis, not the x-axis.

Matching – Answers and Solutions

The answers are:

(a) Graph V	Graph I: (f)
(b) Graph VI	Graph II: (e)
(c) Graph III	Graph III: (c)
(d) Graph IV	Graph IV: (d)
(e) Graph II	Graph V: (a)
(f) Graph I	Graph VI: (b)

Here are some explanations:

(a) This is a surface of revolution, rotating $y = \sin(x)$ around the x-axis. Thus Graph V.

(b) This is a plane, similar to Problem 7. Hence Graph VI.

(c) This is Graph III, the half-cone. One way to see this is to stare at $\mathbf{r}(u, v) = \langle u, v, \sqrt{u^2 + v^2}\rangle$ and recognize that this means $x = u$, $y = v$, and $z = \sqrt{u^2 + v^2} = \sqrt{x^2 + y^2}$. Thus $z^2 = x^2+y^2$, or (in cylindrical coordinates) $z = r$. This is a cone. Since our z coordinate is always non-negative, we have only the top half of the cone. Thus Graph III.

(d) This parameterization is simply spherical coordinates with $\rho = 1$, $u = \phi$, and $v = \theta$ (and x, y, and z permuted). The restrictions on u and v mean that it's only a quarter of the sphere of radius 1, or Graph IV.

(e) Here the $v = k$ grid curves are ellipses at a height $z = 2k^2$. The $u = k$ grid traces are half-parabolas over the ray in the xy-plane at angle $\theta = k$. Another approach is to notice that this curve satisfies the equation

$$\frac{z}{2} = \frac{x^2}{a^2} + \frac{y^2}{b^2},$$

which is an elliptic paraboloid. Both approaches point to Graph II.

(f) This is another surface of revolution. We're revolving $x = \sqrt{1 + z^2}$ around the z-axis. This curve is a hyperbola, so grid curves are either hyperbolas or circles. Thus we end up with the one-sheeted hyperboloid shown in Graph I.

Partial Derivatives

Define the partial derivative of $f(x, y)$ with respect to x by

$$f_x = \frac{\partial}{\partial x}(f) = \frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

and similarly the partial derivative of $f(x, y)$ with respect to y by

$$f_y = \frac{\partial}{\partial y}(f) = \frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

For each of the following functions, compute both first partial derivatives f_x and f_y (or f_t):

1 $f(x, y) = e^x \cos(y)$

2 $f(x, y) = x^3 - 3xy^2$

3 $f(x, t) = e^{-(x+t)^2}$

4 $f(x, t) = \sin(x - t) + \sin(x + t)$

We can compute higher order derivatives by simply repeating the process. For example,

$$f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

and

$$f_{xx} = (f_x)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}.$$

Compute the four second partial derivatives f_{xx}, f_{xy}, f_{yx}, and f_{yy} for the four functions above.

5 These four functions above were selected because they solve some *partial differential equations* or PDEs. Listed below are four common PDEs, some of which you will see in the homework. Determine which function is a solution of which PDE by substituting in derivatives.

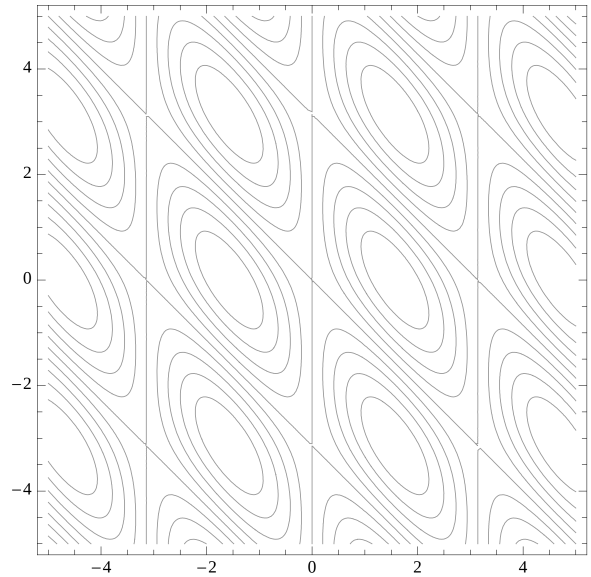
Laplace Equation: $f_{xx} + f_{yy} = 0$

Advection (Transport) Equation: $f_t = f_x$

Wave Equation: $f_{tt} = f_{xx}$

Heat Equation: $f_t = f_{xx}$

6 Here is a contour plot for the function $f(x, y) = \sin(x)\sin(x + y)$.



Without actually computing the derivatives, answer the following questions:

(a) What is the sign of f_x at $(x, y) = (1, 0)$?

(b) What is the sign of f_y at $(x, y) = (1, 0)$?

(c) What is the sign of f_{xx} at $(x, y) = (\frac{1}{2}, 1)$?

(d) What is the sign of f_{xy} at $(x, y) = (\frac{1}{2}, 1)$?

7 You may notice that $f_{xy} = f_{yx}$ in all of the above cases. This is a consequence of *Clairaut's Theorem.*

(a) Use Clairaut's Theorem to compute the requested derivatives of the following functions:

(i) f_{xyxyxy} if $f(x, y) = x^2 \cos(e^y + y^2)$

(ii) f_{xxxyy} if $f(x, y) = x^3y^2 - \frac{y}{x+\ln(x)}$

Partial Derivatives – Answers and Solutions

1 For $f(x,y) = e^x \cos(y)$, we get first derivatives

$$f_x = \frac{\partial f}{\partial x} = e^x \cos(y) \qquad \text{and} \qquad f_y = \frac{\partial f}{\partial y} = -e^x \sin(y).$$

The second derivatives are

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = e^x \cos(y), \qquad f_{yy} = \frac{\partial^2 f}{\partial y^2} = -e^x \cos(y),$$

and

$$f_{xy} = f_{yx} = -e^x \sin(y) \qquad \left(\text{where } f_{xy} = \frac{\partial^2 f}{\partial y \partial x} \text{ and } f_{yx} = \frac{\partial^2 f}{\partial x \partial y}\right).$$

Notice that $f_{xx} + f_{yy} = e^x \cos(y) - e^x \cos(y) = 0$, so this function $f(x,y)$ satisfies the Laplace equation.

2 For $f(x,y) = x^3 - 3xy^2$, we get first derivatives

$$f_x = 3x^2 - 3y^2 \qquad \text{and} \qquad f_y = -6xy$$

and second derivatives

$$f_{xx} = 6x, \qquad f_{xy} = f_{yx} = -6y, \qquad \text{and} \qquad f_{yy} = -6x.$$

Notice again that $f_{xx} + f_{yy} = 6x - 6x = 0$, so this $f(x,y)$ is a solution to the Laplace equation.

3 For $f(x,t) = e^{-(x+t)^2}$, we get first derivatives

$$f_x = f_t = -2(x+t)e^{-(x+t)^2}$$

and second derivatives

$$f_{xx} = f_{xt} = f_{tx} = t_{tt} = \left(4(x+t)^2 - 2\right) e^{-(x+t)^2}.$$

Notice that $f_t = f_x$ and $f_{tt} = f_{xx}$, so this $f(x,y)$ is a solution to both the advection equation and the wave equation.

4 For $f(x,t) = \sin(x-t) + \sin(x+t)$, we get first derivatives

$$f_x = \cos(x-t) + \cos(x+t) \qquad \text{and} \qquad f_t = -\cos(x-t) + \cos(x+t)$$

and second derivatives

$$f_{xx} = -\sin(x-t) - \sin(x+t), \qquad f_{xt} = f_{tx} = \sin(x-t) - \sin(x+t),$$

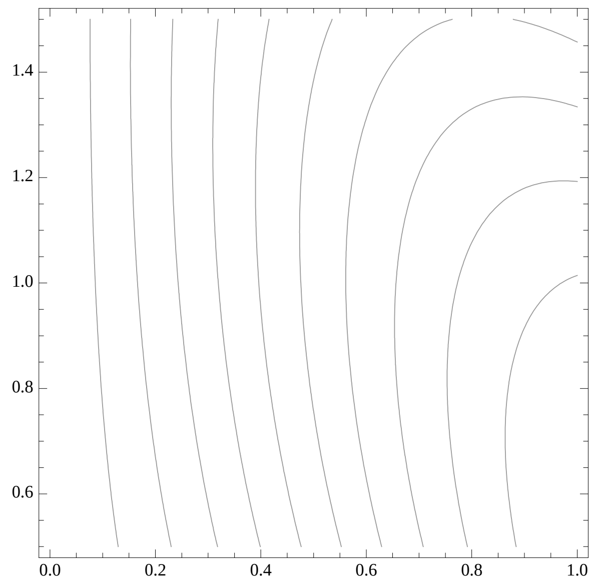
and

$$f_{xx} = -\sin(x-t) - \sin(x+t).$$

Notice again that $f_{tt} = f_{xx}$, so this $f(x,y)$ is also a solution to the wave equation.

6 (a) and (b): Both $f_x > 0$ and $f_y > 0$ at $(x, y) = (1, 0)$. The idea is simply that there is a maximum at $(x, y) = (\frac{\pi}{2}, 0) \approx (1.57, 0)$ (where $f(\frac{\pi}{2}, 0) = 1$), which is in the circled region on the contour plot. Since the the vertical lines $x = k\pi$ and diagonal lines $y = -x + k\pi$ are the only places where $f(x, y) = 0$, we see that the function is increasing in both the positive x and positive y directions at $(1, 0)$.

(c) Here is a "zoomed-in" contour plot near the point $(x, y) = (\frac{1}{2}, 1)$:



As above, $f_x > 0$ at this point. As we move to the right, however, the contour lines become spaced farther apart. Thus f is increasing slower, so f_x is decreasing (although still positive). Thus $f_{xx} < 0$.

(d) Looking at the same picture as in part (c), we see that the contour lines again become spaced farther apart (horizontally) as we move up vertically. Thus, again, $f_{xy} < 0$ at $(x, y) = (\frac{1}{2}, 1)$.

7 The point of both these problems is to re-order the derivatives so that you take the "easier" derivatives first.

(a) Here we take the x derivatives first:

$$\begin{aligned} f &= x^2 \cos\left(e^y + y^2\right) \\ f_x &= 2x \cos\left(e^y + y^2\right) \\ f_{xx} &= 2 \cos\left(e^y + y^2\right) \\ f_{xxx} &= 0; \end{aligned}$$

so $f_{xyxyxy} = f_{xxxyyy} = 0$.

(b) Here we take the y derivatives first. We get

$$\begin{aligned} f &= x^3y^2 - \frac{y}{x+\ln(x)} \\ f_y &= 2x^3y - \frac{1}{x+\ln(x)} \\ f_{yy} &= 2x^3, \end{aligned}$$

and so (after more derivatives) $f_{yyxxx} = 2 \cdot 3 \cdot 2 = 12$. Thus $f_{xxxyy} = 12$ as well.

Approximately Tangent Planes

1 To find the *tangent plane* to the graph $z = f(x, y)$ of a surface at the point $(x, y) = (a, b)$ (really $(x, y, z) = (a, b, f(a, b))$), we begin by finding vectors tangent to the surface at this point.

(a) The x grid curve (or $y = a$ trace) through this point is parameterized as $\mathbf{r}_1(x) = \langle x, b, f(x, b)\rangle$. Find the tangent vector to this curve at $x = a$. (This is a tangent vector to the surface at the point $(x, y, z) = (a, b, f(a, b))$.)

(b) Find another tangent vector to the surface at the point $(x, y, z) = (a, b, f(a, b))$ using the y grid curve $(x = a)$.

(c) Find the normal vector to the tangent plane using the two tangent vectors from parts (a) and (b).

(d) Now find the equation of the tangent plane. This is the plane through $(a, b, f(a, b))$ and perpendicular to the normal vector from part (c).

2 Mostly the tangent plane is used for approximation. Here is a simple example.

(a) Find the equation of the tangent plane to the elliptic paraboloid $z = x^2 + 2y^2$ at the point $(x, y, z) = (1, 1, 3)$.

(b) Your answer to part (a) (and part (d) of Problem 1) should involve the formula

$$z \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

Use this to approximate the values of $z = f(x, y) = x^2 + 2y^2$ at the points $(x, y) = (0.9, 1.1)$ and $(x, y) = (0.95, 0.95)$.

(c) Compare your answers to part (b) to the *actual* values of $f(0.9, 1.1)$ and $f(0.95, 0.95)$.

The *linearization* of the function $f(x, y)$ is the function

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

(this is the same right-hand side as above). The *linear approximation* (or *tangent plane approximation*) is then $f(x, y) \approx L(x, y)$ for (x, y) near (a, b).

3 Here is a more complicated example: $f(x, y) = ye^{xy}$. Calculate $f(1, 0)$ and use the *linear approximation* or *tangent plane approximation* from the previous problem to approximate $f(0.9, 0.1)$ and $f(1.1, -0.05)$. Compare your answers to the actual values (if you have a calculator).

4 Sometimes the linear approximation isn't actually very good. Here's an example. Consider the function

$$f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

(a) Compute $f_x(0, 0)$ and $f_y(0, 0)$. You'll need to use the definitions

$$f_x(a, b) = \lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h} \qquad \text{and} \qquad f_y(a, b) = \lim_{h \to 0} \frac{f(a, b + h) - f(a, b)}{h}.$$

(b) Use the values from part (a) to compute the linear approximation to the surface at the point $(x, y, z) = (0, 0, 0)$.

(c) Use this approximation to approximate the values of $f(x, y)$ at the points $(x, y) = (a, a)$.

(d) Use this approximation to approximate the values of $f(x, y)$ at the points $(x, y) = (b, -b)$.

(e) Compare the approximations from parts (c) and (d) to the actual values of $f(a, a)$ and $f(b, -b)$. (And notice that these points can be taken to be *very* close to the origin.)

5 We avoid the problem of the last problem by assuming that it doesn't happen. We say that f is *differentiable* if (more or less) the approximation tends to the exact value. There is a theorem that says that f is differentiable at (x_0, y_0) if both f_x and f_y exist near (x_0, y_0) and are continuous at (x_0, y_0). Thus, if the derivatives are nice enough, we always get a good approximation.

Show that this theorem doesn't apply to the previous example by finding general formulas for f_x and f_y and determining that they are not continuous at $(x, y) = (0, 0)$.

6 We also want to be able to find tangent planes for parametric surfaces $\mathbf{r}(u,v)$. This problem will step through this for the example

$$\mathbf{r}(u,v) = \langle \cos u, 3\sin u \cos v, 4\sin u \sin v\rangle$$

at the point $(x,y,z) = (0,0,4)$ corresponding to $(u,v) = (\frac{\pi}{2}, \frac{\pi}{2})$.

(a) The u grid curve ($v = \frac{\pi}{2}$) is given by the parametric curve

$$\mathbf{r}(u, \tfrac{\pi}{2}) = \langle \cos u, 3\sin u \cos \tfrac{\pi}{2}, 4\sin u \sin \tfrac{\pi}{2}\rangle = \langle \cos u, 0, 4\sin u\rangle.$$

Find the tangent vector to this curve at $u = \frac{\pi}{2}$.

(b) Repeat part (a) for the v grid curve. That is, the v grid curve ($u = \frac{\pi}{2}$) is given by the parametric curve

$$\mathbf{r}(\tfrac{\pi}{2}, v) = \langle \cos \tfrac{\pi}{2}, 3\sin \tfrac{\pi}{2} \cos v, 4\sin \tfrac{\pi}{2} \sin v\rangle = \langle 0, 3\cos v, 4\sin v\rangle.$$

Find the tangent vector to this curve at $v = \frac{\pi}{2}$.

(c) Use your answers to parts (a) and (b) to find the normal to the tangent plane at $(x,y,z) = (0,0,4)$.

(d) Now find the tangent plane at at $(x,y,z) = (0,0,4)$.

(e) The linear approximation is: for (u,v) near (u_0,v_0),

$$\mathbf{r}(u,v) \approx \mathbf{r}(u_0,v_0) + \mathbf{r}_u(u_0,v_0)(u-u_0) + \mathbf{r}_v(u_0,v_0)(v-v_0).$$

Verify that this is the tangent plane approximation you found in part (d).

7 Repeat the previous problem at the point $(u,v) = (\frac{\pi}{2}, 0)$. What point (x,y,z) does this represent?

Approximately Tangent Planes – Answers and Solutions

1 (a) $\mathbf{r}_1'(x_0) = \langle 1, 0, f_x(x_0, y_0)\rangle$

(b) $\mathbf{r}_2'(y_0) = \langle 0, 1, f_y(x_0, y_0)\rangle$

(c) $\langle 1, 0, f_x(x_0, y_0)\rangle \times \langle 0, 1, f_y(x_0, y_0)\rangle = \langle -f_x(x_0, y_0), -f_y(x_0, y_0), 1\rangle$

(d) $\langle -f_x(x_0, y_0), -f_y(x_0, y_0), 1\rangle \cdot \langle x - x_0, y - y_0, z - z_0\rangle = 0$
or $z = z_0 + f_x(x_0, y_0)\,(x - x_0) + f_y(x_0, y_0)\,(y - y_0)$.

2 (a) $z = 3 + 2(x - 1) + 4(y - 1)$

(b) $f(0.9, 1.1) \approx 3.2$ $\quad f(0.95, 0.95) \approx 2.7$

(c) $f(0.9, 1.1) = 3.23$ $\quad f(0.95, 0.95) = 2.7075$

3 (Using (a), (b), and (c) in parallel with question 2.)

(a) $z = y$

(b) $f(0.9, 0.1) \approx 0.1$ $\quad f(1.1, -0.05) \approx -0.05$

(c) $f(0.9, 0.1) = 0.1e^{0.09} \approx 0.10941742837052$
$f(1.1, -0.05) = -0.05e^{-0.055} \approx -0.047324257397674$

4 (a) $f_x(0, 0) = f_y(0, 0) = 0$

(b) Thus $L(x, y) = 0$, so the approximation is $z \approx 0$

(c) On this line ($y = x$), we get $f(x, y) = 1/2$ (provided $(x, y) \neq (0, 0)$)

(d) On this line ($y = -x$), we get $f(x, y) = -1/2$ (provided $(x, y) \neq (0, 0)$)

(e) The point is that this is a *bad* approximation, since the approximation error doesn't decrease to zero as $(x, y) \to (0, 0)$.

5 We can differentiate the function in Problem 4 away from the origin pretty easily: it's simply

$$f_x(x, y) = \frac{y\,(x^2 + y^2) - xy(2x)}{(x^2 + y^2)^2} = \frac{y^3 - x^2 y}{(x^2 + y^2)^2}.$$

Thus

$$f_x(x, y) = \begin{cases} \frac{y^3 - x^2 y}{(x^2 + y^2)^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

(since we've already calculated the $f_x(0, 0)$ derivative). Similarly

$$f_y(x, y) = \begin{cases} \frac{x^3 - x y^2}{(x^2 + y^2)^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Neither of these is continuous at the origin. For example, along the path $y = 0$, the function $f_y(x, 0) = 1/x$ (for $x \neq 0$) which has no finite limit as $x \to 0$.

6 (a) The u grid curve ($v = \frac{\pi}{2}$) is given by the parametric curve

$$\mathbf{r}(u, \tfrac{\pi}{2}) = \langle \cos u, 0, 4\sin u \rangle,$$

so the tangent vector is the derivative of this with respect to u:

$$\frac{\partial \mathbf{r}}{\partial u}(u, \tfrac{\pi}{2}) = \mathbf{r}_u(u, \tfrac{\pi}{2}) = \langle -\sin u, 0, 4\cos u \rangle.$$

When $u = \frac{\pi}{2}$, this is the vector $\mathbf{r}_u(\frac{\pi}{2}, \frac{\pi}{2}) = \langle -\sin\frac{\pi}{2}, 0, 4\cos\frac{\pi}{2} \rangle = \langle -1, 0, 0 \rangle$.

(b) Since $\mathbf{r}_v(\frac{\pi}{2}, v) = \langle 0, -3\sin v, 4\cos v \rangle$, the tangent vector to the v grid curve at $v = \frac{\pi}{2}$ is $\mathbf{r}_v(\frac{\pi}{2}, \frac{\pi}{2}) = \langle 0, -3\sin\frac{\pi}{2}, 4\cos\frac{\pi}{2} \rangle = \langle 0, -3, 0 \rangle$.

(c) The normal to the tangent plane at $(x, y, z) = (0, 0, 4)$ is

$$\mathbf{r}_u(\tfrac{\pi}{2}, \tfrac{\pi}{2}) \times \mathbf{r}_v(\tfrac{\pi}{2}, \tfrac{\pi}{2}) = \langle -1, 0, 0 \rangle \times \langle 0, -3, 0 \rangle = \langle 0, 0, 3 \rangle.$$

(d) The tangent plane at at $(x, y, z) = (0, 0, 4)$ is $\langle 0, 0, 3 \rangle \cdot \langle x, y, z - 4 \rangle = 0$ or $3(z - 4) = 0$ or $z = 4$.

(e) The linear approximation is: for (u, v) near $(\frac{\pi}{2}, \frac{\pi}{2})$,

$$\mathbf{r}(u, v) \approx \mathbf{r}(\tfrac{\pi}{2}, \tfrac{\pi}{2}) + \mathbf{r}_u(\tfrac{\pi}{2}, \tfrac{\pi}{2})(u - \tfrac{\pi}{2}) + \mathbf{r}_v(\tfrac{\pi}{2}, \tfrac{\pi}{2})(v - \tfrac{\pi}{2}).$$

or

$$\mathbf{r}(u, v) \approx \langle 0, 0, 4 \rangle + \left(u - \tfrac{\pi}{2}\right) \langle -1, 0, 0 \rangle + \left(v - \tfrac{\pi}{2}\right) \langle 0, -3, 0 \rangle.$$

This is the tangent plane approximation you found in part (d): this says $x = -(u - \frac{\pi}{2})$, $y = -3(v - \frac{\pi}{2})$, and $z = 4$. Thus x and y are arbitrary and $z = 4$.

7 The point $(u, v) = (\frac{\pi}{2}, 0)$ represents the point $(x, y, z) = (0, 3, 0)$.

(a) The u grid curve ($v = 0$) is given by the parametric curve

$$\mathbf{r}(u, 0) = \langle \cos u, 3\sin u, 0 \rangle,$$

so the tangent vector is the derivative of this with respect to u:

$$\frac{\partial \mathbf{r}}{\partial u}(u, 0) = \mathbf{r}_u(u, 0) = \langle -\sin u, 3\cos u, 0 \rangle.$$

When $u = \frac{\pi}{2}$, this is the vector $\mathbf{r}_u(\frac{\pi}{2}, 0) = \langle -\sin\frac{\pi}{2}, 3\cos\frac{\pi}{2}, 0 \rangle = \langle -1, 0, 0 \rangle$.

(b) Since $\mathbf{r}(\frac{\pi}{2}, v) = \langle 0, 3\cos v, 4\sin v \rangle$, we get $\mathbf{r}_v(\frac{\pi}{2}, v) = \langle 0, -3\sin v, 4\cos v \rangle$ and $\mathbf{r}_v(\frac{\pi}{2}, 0) = \langle 0, 0, 4 \rangle$.

(c) The normal to the tangent plane at $(x, y, z) = (0, 3, 0)$ is

$$\mathbf{r}_u(\tfrac{\pi}{2}, 0) \times \mathbf{r}_v(\tfrac{\pi}{2}, 0) = \langle -1, 0, 0\rangle \times \langle 0, 0, 4\rangle = \langle 0, 4, 0\rangle.$$

(d) The tangent plane at at $(x, y, z) = (0, 3, 0)$ is $\langle 0, 4, 0\rangle \cdot \langle x, y - 3, z\rangle = 0$ or $4(y - 3) = 0$ or $y = 3$.

(e) The linear approximation is: for (u, v) near $(\frac{\pi}{2}, 0)$,

$$\mathbf{r}(u, v) \approx \mathbf{r}(\tfrac{\pi}{2}, 0) + \mathbf{r}_u(\tfrac{\pi}{2}, 0)(u - \tfrac{\pi}{2}) + \mathbf{r}_v(\tfrac{\pi}{2}, 0)(v - 0)$$

or

$$\mathbf{r}(u, v) \approx \langle 0, 3, 0\rangle + (u - \tfrac{\pi}{2})\langle -1, 0, 0\rangle + v\langle 0, 0, 4\rangle.$$

This is the tangent plane approximation you found in part (d): this says $x = -(u - \frac{\pi}{2})$, $y = 3$, and $z = 4v$. Thus x and z are arbitrary and $y = 3$.

Gradients & Level Surfaces

Recall from Monday that the *gradient* of a function f is the vector ∇f defined by

$$\nabla f(x,y) = \text{grad } f = \langle f_x(x,y), f_y(x,y)\rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

in the plane or

$$\nabla f(x,y,z) = \text{grad } f = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

in three-space.

1 For each of the following functions $f(x,y)$ in the plane, do the following:

(i) Compute the gradient ∇f (or grad f).

(ii) Identify the level curve $f(x,y) =$ constant through the point $(x,y) = (1,1)$.

(iii) Find a parameterization $\mathbf{r}(t)$ of this curve.

(iv) Verify that the tangent vector $\mathbf{r}'(t)$ and the gradient vector ∇f are perpendicular at the point $(x,y) = (1,1)$.

(a) $f(x,y) = 3x - y$

(b) $f(x,y) = 2x^2 + 3y^2$

2 For each of the following functions $F(x,y,z)$ in the plane, do the following:

(i) Compute the gradient ∇F (or grad F).

(ii) Identify the level surface $F(x,y,z) =$ constant through the point $(x,y,z) = (1,1,1)$.

(iii) Find the tangent plane to the level surface from part (ii). Recall that the tangent plane to a surface $z = f(x,y)$ at the point (x_0,y_0,z_0) is

$$z - z_0 = f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0).$$

(iv) Verify that the gradient vector ∇F is perpendicular to (tangent plane of) the level surface at the point $(x,y,z) = (1,1,1)$.

(a) $F(x,y,z) = 3x + 2y + z$

(b) $F(x,y,z) = x^2 + y^2 - z^2$

These problems illustrate the following point:

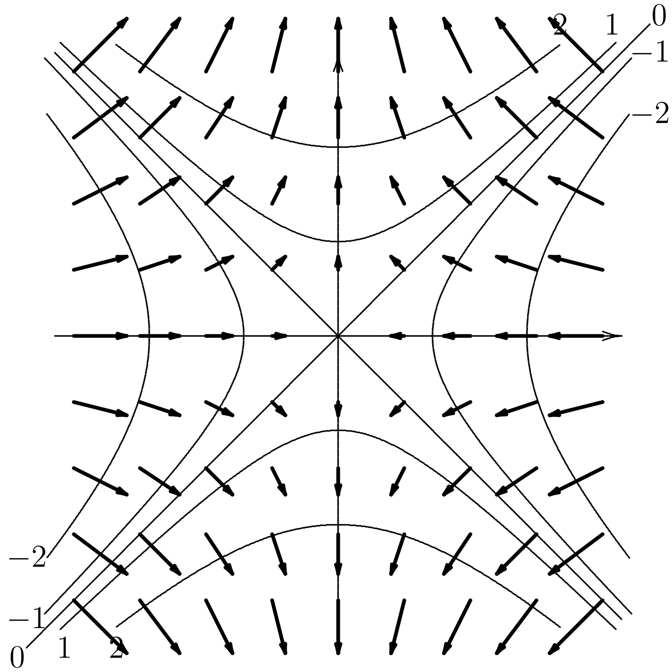
The gradient of a function is always perpendicular to the level curves (or level surfaces) of the function!

Gradients & Level Surfaces

There are two important facts about the gradient vector:

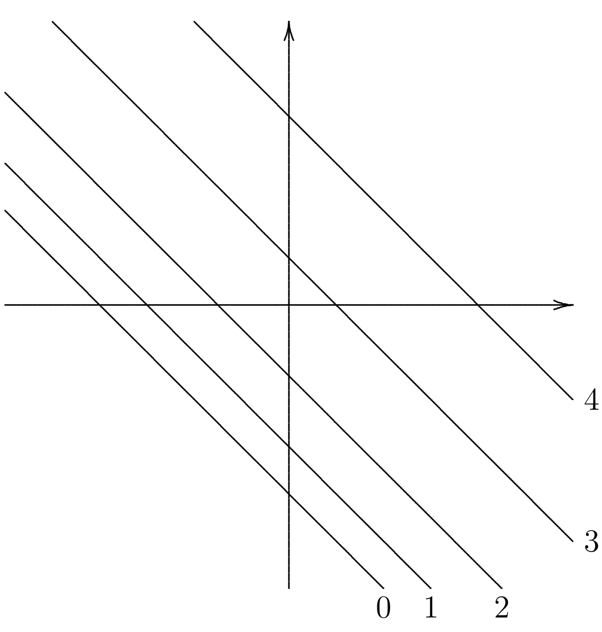
- grad f (or ∇f) is perpendicular to the level curves of f (as we saw on page one of this handout)
- $|\text{grad}\, f|$ (or the magnitude of ∇f) is the rate of change of f in the direction of grad f

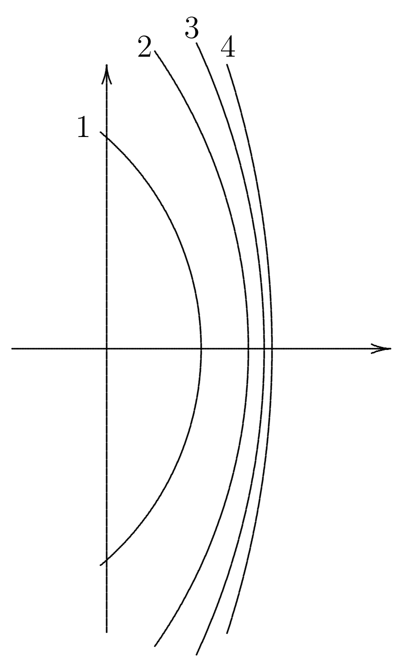
Here is an example sketch of the level curves of $f(x, y) = y^2 - x^2$ and the associated gradient vector field:



(The arrows shown here are in fact one-tenth the *actual* length of the gradient, but they're shrunk to make the picture cleaner. Here $\nabla f = \langle f_x, f_y \rangle = \langle -2x, 2y \rangle$, so $|\nabla f| = \sqrt{4x^2 + 4y^2} = 2r$.)

Use the two facts shown above to sketch the gradient vector field given the following contour plots (pictures of level curves):





Gradients & Level Surfaces – Answers / Solutions

1 (a) (i) $\nabla f = \langle 3, -1\rangle$ at every point, not just $(x, y) = (1, 1)$.

(ii) The curve $f(x, y) = f(1, 1)$ is $3x - y = 2$, a line.

(iii) A simple parameterization is $x = t$, so $y = 3t - 2$, or $\mathbf{r}(t) = \langle t, 3t - 2\rangle$.

(iv) The tangent vector in our parameterization is always $\mathbf{r}'(t) = \langle 1, 3\rangle$, so $\nabla f \cdot \mathbf{r}'(t) = 0$ for all t, not just for $t = 1$ (the point $(x, y) = (1, 1)$).

(b) (i) $\nabla f = \langle 4x, 6y\rangle$, so $\nabla f(1, 1) = \langle 4, 6\rangle$.

(ii) The curve $f(x, y) = f(1, 1)$ is $2x^2 + 3y^2 = 5$, an ellipse.

(iii) One parameterization is

$$\mathbf{r}(t) = \langle x(t), y(t)\rangle = \left\langle \sqrt{\tfrac{5}{2}}\cos(t), \sqrt{\tfrac{5}{3}}\sin(t) \right\rangle.$$

I found this by using the parameterization $\langle u, v\rangle = \langle \sqrt{5}\cos(t), \sqrt{5}\sin(t)\rangle$ for the circle $u^2 + v^2 = 5$, then writing our ellipse as $(\sqrt{2}x)^2 + (\sqrt{3}y)^2 = 5$ and making the substitutions $u = \sqrt{2}x$ and $v = \sqrt{3}y$.

(iv) The tangent vector in our parameterization is $\mathbf{r}'(t) = \left\langle -\sqrt{\tfrac{5}{2}}\sin(t), \sqrt{\tfrac{5}{3}}\sin(t)\right\rangle$. You might think we need to find t_0 when $(x, y) = (1, 1)$, but in reality we only need to find $\cos(t_0)$ and $\sin(t_0)$. We know that $\mathbf{r}(t_0) = \langle 1, 1\rangle$, so

$$\left\langle \sqrt{\frac{5}{2}}\cos(t_0), \sqrt{\frac{5}{3}}\sin(t_0) \right\rangle = \langle 1, 1\rangle \qquad \text{or} \qquad \cos(t_0) = \sqrt{\frac{2}{5}} \quad \text{and} \quad \sin(t_0) = \sqrt{\frac{3}{5}}.$$

Thus $\mathbf{r}'(t_0) = \left\langle -\sqrt{\tfrac{5}{2}} \cdot \sqrt{\tfrac{3}{5}}, \sqrt{\tfrac{5}{3}} \cdot \sqrt{\tfrac{2}{5}} \right\rangle = \left\langle -\sqrt{3/2}, \sqrt{2/3}\right\rangle$. Thus

$$\nabla f(1, 1) \cdot \mathbf{r}'(t_0) = \langle 4, 6\rangle \cdot \left\langle -\sqrt{3/2}, \sqrt{2/3}\right\rangle = -4\sqrt{3/2} + 6\sqrt{2/3} = 0.$$

Thus the two vectors are perpendicular.

2 (a) (i) $\nabla F = \langle 3, 2, 1\rangle$

(ii) The level surface $F(x, y, z) = F(1, 1, 1)$ is the plane $3x + 2y + z = 6$.

(iii) To use this formula, we solve for z: $z = f(x, y) = 6 - 3x - 2y$. Thus the tangent line is

$$z - 1 = -3(x - 1) - 2(y - 1) \qquad \text{or} \qquad 3x + 2y + z = 6.$$

Fancy that! The tangent plane to a plane is the plane itself!

(iv) The gradient $\nabla F = \langle 3, 2, 1\rangle$ is the same as the normal to the tangent plane (and the level surface itself); hence the gradient is perpendicular to the tangent plane of the level surface.

(b) (i) $\nabla F = \langle 2x, 2y, -2z\rangle$, so $\nabla F(1,1,1) = \langle 2, 2, -2\rangle$.

(ii) The level surface $F(x,y,z) = F(1,1,1)$ is the one-sheeted hyperboloid $x^2+y^2-z^2 = 1$.

(iii) To use this formula, we solve for z: $z = f(x,y) = \sqrt{x^2+y^2-1}$ (we want the positive square root since $z = 1$ at our point). At $(x,y) = (1,1)$, the derivative $f_x(1,1)$ is easy to find:

$$f_x = \frac{1}{2}\left(x^2+y^2-1\right)^{-1/2} \cdot 2x = \frac{x}{\sqrt{x^2+y^2-1}},$$

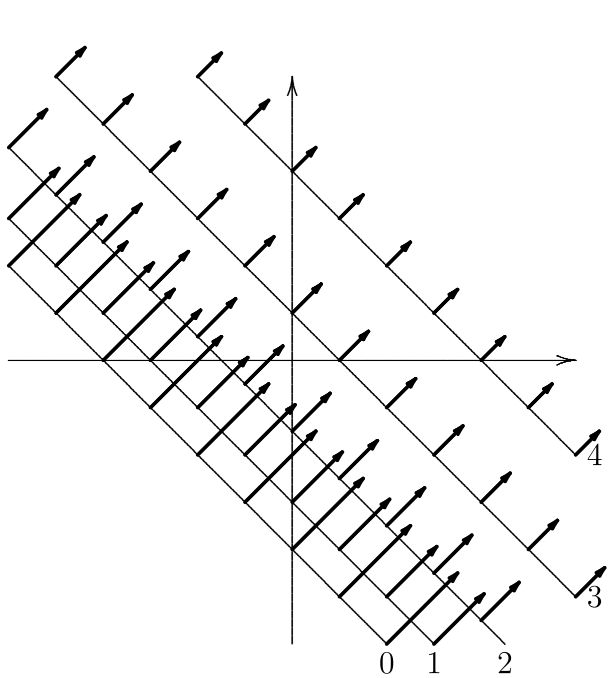
so $f_x(1,1) = \frac{1}{\sqrt{1^2+1^2-1}} = 1$ at the point $(x,y) = (1,1)$. Thus the tangent line is

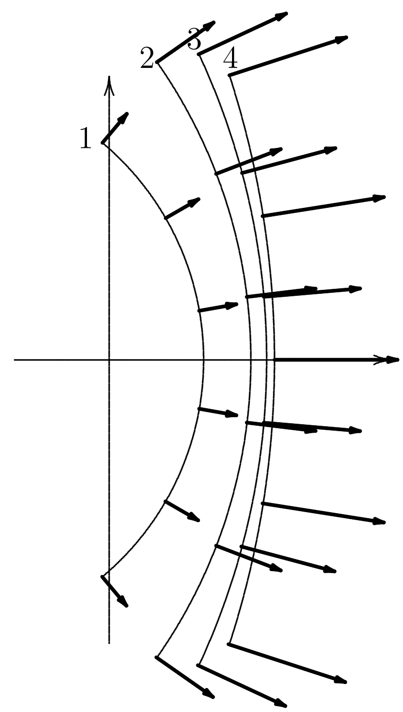
$$z - 1 = 1(x-1) + 1(y-1) \qquad \text{or} \qquad x + y - z = 1.$$

Note that the normal to this tangent plane is $\mathbf{n} = \langle 1, 1, -1\rangle$.

(iv) The gradient $\nabla F(1,1,1) = \langle 2, 2, -2\rangle$ is parallel to the normal $\mathbf{n} = \langle 1, 1, -1\rangle$ to the tangent plane. As before, therefore, the gradient is perpendicular to the tangent plane of the level surface.

Back Page: Here are the two graphs with some gradient vectors drawn in:





In both cases the gradient vectors have been scaled to make sure the picture is not overwhelmed (or underwhelmed) with arrows.

Directional Derivatives

1 If $f(x, y) = 3x + 7y^2$ and $\mathbf{u} = \langle \frac{3}{5}, \frac{4}{5} \rangle$, find the directional derivative $D_{\mathbf{u}} f$ at $(1, 1)$.

2 If $f(x, y) = \sin(xy)$, find the directional derivative $D_{\mathbf{u}} f$ where $\mathbf{u} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$.

3 You are skiing on a mountain which happens to be the graph of the function $f(x, y) = 10 - x^2 - y^4$. You are at the point $(1, 1, 8)$. If you want to ski down the steepest path, what direction should you head?

4 A fly is flying around a room in which the temperature is given by $T(x, y, z) = x^2 + y^4 + 2z^2$. The fly is at the point $(1, 1, 1)$ and realizes that he's cold. In what direction should he fly to warm up most quickly?

5 Alice the "A" student is debating Chuck the "C" student. Alice says that the direction of greatest ascent on the graph $z = f(x, y)$ is in the direction ∇f. Chuck says that instead we should look at the level surface $F(x, y, z) = z - f(x, y) = 0$ and go in direction $\operatorname{grad} F$. Who is right? How would you explain this to the student that is wrong?

6 A racehorse lives in a valley which happens to be the graph of $f(x, y) = 3x^2 + y^2$. He is doomed to wander his racetrack, which is the set of points in the valley where $x^2 + y^2 = 1$.

The racehorse secretly wishes to be a mountain climber, and his fantasy is to escape from his racetrack and take the steepest path up the mountain. At what point should he make his escape, and in what direction should he run? (There are actually two possible answers.)

Directional Derivatives – Solutions

1 We use the formula $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u}$. Here $\nabla f = \langle 3, 14y \rangle$, so $\nabla f(1,1) \cdot \mathbf{u} = \langle 3, 14 \rangle \cdot \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle = 13$.

2 Now $\nabla f = \langle y\cos(xy), x\cos(xy) \rangle$, so $\nabla f \cdot \mathbf{u} = \frac{x+y}{\sqrt{2}} \cos(xy)$.

3 Since the problem asks only for a direction, we should give our answer as a unit vector $\mathbf{u}$. We are looking for the unit vector $\mathbf{u}$ for which $D_{\mathbf{u}}f(1,1)$ is the most negative. We know that $D_{\mathbf{u}}f(1,1) = \nabla f(1,1) \cdot \mathbf{u}$. Let's first calculate ∇f: it is $\langle -2x, -4y^3 \rangle$, so $\nabla f(1,1) = \langle -2, -4 \rangle$. Therefore, we want to find the unit vector for which $\langle -2, -4 \rangle \cdot \mathbf{u}$ is the most negative.

Remember that the dot product of two vectors is given by the formula $\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}|\,|\mathbf{w}| \cos\theta$.

If θ is the angle between $\langle -2, -4 \rangle$ and $\mathbf{u}$, then $|\langle -2, -4 \rangle \cdot \mathbf{u}| = |\langle -2, -4 \rangle|\,|\mathbf{u}| \cos\theta$. Since $\mathbf{u}$ is a unit vector, $|\mathbf{u}| = 1$, so we really want to make $\cos\theta$ as negative as possible. This means we should take $\theta = \pi$, so we want to pick the unit vector which goes in the direction opposite of $\langle -2, -4 \rangle$. That is, we want a vector which goes in the direction of $\langle 2, 4 \rangle$ but which has length 1. To get such a vector, we just divide $\langle 2, 4 \rangle$ by its length, which is $2\sqrt{5}$. So, our answer is $\left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle$.

4 We want to find the unit vector $\mathbf{u}$ which maximizes $D_{\mathbf{u}}T(1,1,1)$. Using the same idea as in the last problem, this should be the unit vector in the direction of $\nabla T(1,1,1)$.

Since $\nabla T = \langle 2x, 4y^3, 4z \rangle$, we want the unit vector in the direction of $\langle 2, 4, 4 \rangle$, which is $\left\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right\rangle$.

5 Alice is correct: the direction of greatest ascent on the surface $z = f(x,y)$ is in the direction ∇f.

What is the problem with Chuck's answer? He's looking at the level surface $F(x,y,z) = 0$, where $F(x,y,z) = z - f(x,y)$. This is the same surface that Alice is looking at. But when he suggests moving in the ∇F direction, this means moving a direction in space that increases the value of $F(x,y,z)$ as quickly as possible. In particular, this means moving off the level surface $F = 0$. This is a different problem altogether.

6 What the problem is really asking us to do is find the points (x,y) and unit vectors $\mathbf{u}$ for which the directional derivative $D_{\mathbf{u}}f(x,y)$ is biggest, although we're only allowed to look at points where $x^2 + y^2 = 1$.

Let's first focus on a specific point (x,y) and figure out the biggest the directional derivative $D_{\mathbf{u}}f(x,y)$ could be at that point. We know that the directional derivative is largest in the direction of the gradient. The unit vector in the direction of the gradient is $\frac{\nabla f}{|\nabla f|}$, and the directional derivative in this direction is $\nabla f \cdot \frac{\nabla f}{|\nabla f|}$. Since $\nabla f \cdot \nabla f$ is just $|\nabla f|^2$, the directional derivative in this direction is really $|\nabla f|$. Thus, we could restate the problem as: find the points (x,y) for which $|\nabla f(x,y)|$ is largest.

Let's now calculate the gradient: $\nabla f(x,y) = \langle 6x, 2y \rangle$, so $|\nabla f(x,y)| = \sqrt{36x^2 + 4y^2}$.

Thus, we can again restate the problem as: maximize $36x^2 + 4y^2$ subject to the constraint that $x^2 + y^2 = 1$. Next week, we will learn a method for doing this called the method of Lagrange multipliers. However, we can figure this problem out without Lagrange multipliers.

We can write $36x^2 + 4y^2$ as $32x^2 + 4(x^2 + y^2)$; when $x^2 + y^2 = 1$, this is just equal to $32x^2 + 4$. So, we're really trying to maximize $32x^2 + 4$. We do this by making x^2 as large as possible.

Since (x, y) has to stay on the circle $x^2 + y^2 = 1$, this means we want $x = \pm 1$, which makes $y = 0$.

The unit vectors $\mathbf{u}$ are supposed to go in the direction of $\nabla f = \langle 6x, 2y \rangle$. In the case of the point $(1, 0)$, this means $\mathbf{u} = \langle 1, 0 \rangle$; in the case of the point $(-1, 0)$, this means $\mathbf{u} = \langle -1, 0 \rangle$.

So our answer is: the point $(1, 0)$ with the unit vector $\langle 1, 0 \rangle$ and the point $(-1, 0)$ with the unit vector $\langle -1, 0 \rangle$.

Extremal Points

Tests for Maxima, Minima, Saddle Points

Critical Points:

A point (a, b) is a *critical point* for the function $z = f(x, y)$ if $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

That is, critical points are those points where $\nabla f = \langle f_x, f_y \rangle = \mathbf{0}$.

Is A Critical Point A Max, Min, or Saddle?
Set $D = f_{xx} f_{yy} - f_{xy}^2$.

- $D < 0$ at (a, b) $\implies$ (a, b) is a saddle point.
- $D > 0$ at (a, b)
  - $f_{xx} < 0$ at (a, b) $\implies$ (a, b) a local max
  - $f_{xx} > 0$ at (a, b) $\implies$ (a, b) a local min
- $D = 0$ at (a, b) $\implies$ no info about (a, b)

For each of the following functions...

(a) Compute f_x, f_y, f_{xx}, f_{yy}, and f_{xy}. (You can assume that $f_{yx} = f_{xy}$.)

(b) Find the critical points. That is, find all points where both $f_x = 0$ and $f_y = 0$.

(c) Find the value of f_{xx} and D at each critical point.

(d) Using the test above, determine (if possible) whether each critical point is a local maximum, a local minimum, or a saddle point.

1 $f(x, y) = x^2 + 2xy + 2y^2 - 8y + 12$

2 $f(x, y) = x^2 - 2y^2 + xy - 4$

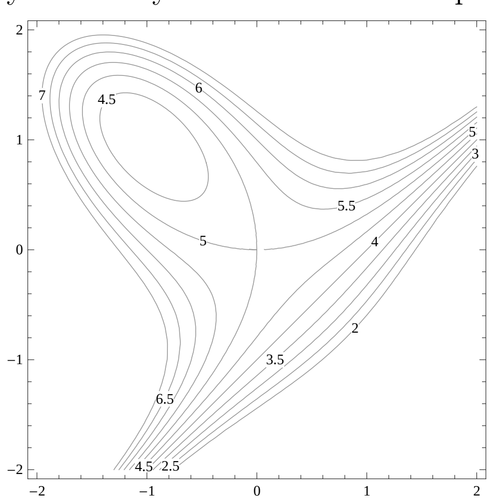
3 $f(x, y) = 8 - x^2 - xy - y^2$

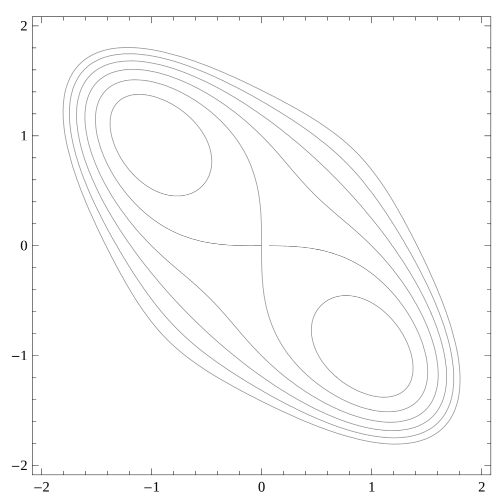
4 $f(x, y) = x^4 + y^4 + 4xy + 3$

5 $f(x, y) = y^2 + 2xy - x^2 + 2x - 2y + 3$

6 $f(x, y) = 5 - x^3 + y^3 + 3xy$

7 Two of the above functions are shown as contour plots below. For each plot, predict the location of the critical points and classify them as maxima, minima, or saddle points. Can you identify which functions are plotted here?





Answers:

1 (b) $(a, b) = (-4, 4)$ (c) $f_{xx} = 2$, $D = 4$ (d) min

2 (b) $(a, b) = (0, 0)$ (c) $f_{xx} = 2$, $D = -9$ (d) saddle

3 (b) $(a, b) = (0, 0)$ (c) $f_{xx} = -2$, $D = 3$ (d) max

4 (b) $(a, b) = (0, 0)$, $(1, -1)$, $(-1, 1)$
(d) minima at $\pm(1, -1)$, saddle at origin

5 (b) $(a, b) = (1, 0)$ (c) $f_{xx} = -2$, $D = -8$ (d) saddle

6 (b) $(a, b) = (0, 0)$ & $(-1, 1)$ (d) saddle & min

7 These plots are Problems 6 (left) and 4 (right)

Lagrange Multipliers

The method of Lagrange multipliers allows us to maximize or minimize functions with the constraint that we only consider points on a certain surface. To find critical points of a function $f(x, y, z)$ on a level surface $g(x, y, z) = C$ (or *subject to the constraint* $g(x, y, z) = C$), we must solve the following system of simultaneous equations:

$$\begin{aligned} \nabla f(x, y, z) &= \lambda \nabla g(x, y, z) \\ g(x, y, z) &= C \end{aligned}$$

Remembering that ∇f and ∇g are vectors, we can write this as a collection of four equations in the four unknowns x, y, z, and λ:

$$\begin{aligned} f_x(x, y, z) &= \lambda g_x(x, y, z) \\ f_y(x, y, z) &= \lambda g_y(x, y, z) \\ f_z(x, y, z) &= \lambda g_z(x, y, z) \\ g(x, y, z) &= C \end{aligned}$$

The variable λ is a dummy variable called a "Lagrange multiplier"; we only really care about the values of x, y, and z.

Once you have found all the critical points, you plug them into f to see where the maxima and minima. The critical points where f is greatest are maxima and the critical points where f is smallest are minima.

Solving the system of equations can be hard! Here are some tricks that may help:

1. Since we don't actually care what λ is, you can first solve for λ in terms of x, y, and z to remove λ from the equations.
2. Try first solving for one variable in terms of the others.
3. Remember that whenever you take a square root, you must consider both the positive and the negative square roots.
4. Remember that whenever you divide an equation by an expression, you must be sure that the expression is not 0. It may help to split the problem into two cases: first solve the equations assuming that a variable is 0, and then solve the equations assuming that it is not 0.

For problems 1-3,

(a) Use Lagrange multipliers to find all the critical points of f on the given surface (or curve).

(b) Determine the maxima and minima of f on the surface (or curve) by evaluating f at the critical values.

1 The function $f(x, y, z) = x + y + 2z$ on the surface $x^2 + y^2 + z^2 = 3$.

2 The function $f(x, y) = xy$ on the curve $3x^2 + y^2 = 6$.

3 The function $f(x, y, z) = x^2 - y^2$ on the surface $x^2 + 2y^2 + 3z^2 = 1$. (Make sure you find *all* the critical points!)

If the level surface is infinitely large, Lagrange multipliers will not always find maxima and minima.

4 (a) Use Lagrange multipliers to show that $f(x, y, z) = z^2$ has only one critical point on the surface $x^2 + y^2 - z = 0$.

(b) Show that the one critical point is a minimum.

(c) Sketch the surface. Why did Lagrange multipliers not find a maximum of f on the surface?

Lagrange Multipliers – Solutions

1 (a) We have $f(x, y, z) = x + y + 2z$ and $g(x, y, z) = x^2 + y^2 + z^2$, so $\nabla f = \langle 1, 1, 2\rangle$ and $\nabla g = \langle 2x, 2y, 2z\rangle$. The equations to be solved are thus

$$1 = 2\lambda x \tag{1}$$
$$1 = 2\lambda y \tag{2}$$
$$2 = 2\lambda z \tag{3}$$
$$x^2 + y^2 + z^2 = 3 \tag{4}$$

To solve these, note that λ cannot be 0 by the first three equations, so we get

$$x = \frac{1}{2\lambda}, \qquad y = \frac{1}{2\lambda} \qquad \text{and} \qquad z = \frac{1}{\lambda}.$$

Plugging these values into (4) gives

$$\frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} + \frac{1}{\lambda^2} = 3,$$

or $\lambda = \pm\frac{1}{\sqrt{2}}$. Plugging these values of λ into the equations above, the critical points are thus $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \sqrt{2})$ and $(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, -\sqrt{2})$.

(b) Since $f(x, y, z) = x + y + 2z$, we have $f(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \sqrt{2}) = 3\sqrt{2}$ and $f(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, -\sqrt{2}) = -3\sqrt{2}$. Thus $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \sqrt{2})$ is the maximum and $(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, -\sqrt{2})$ is the minimum.

2 (a) We have $f(x, y) = xy$ and $g(x, y, z) = 3x^2 + y^2$, so $\nabla f = \langle y, x\rangle$ and $\nabla g = \langle 6x, 2y\rangle$. The equations to be solved are thus

$$y = 6\lambda x \tag{5}$$
$$x = 2\lambda y \tag{6}$$
$$3x^2 + y^2 = 6 \tag{7}$$

Plugging the first equation into the second gives

$$y = 6\lambda(2\lambda y) = 12\lambda^2 y.$$

If y were 0, then x would be 0 too, which is impossible by (7). Thus we can divide by y to get that $12\lambda^2 = 1$. Now plug the first equations into (7) to get

$$\begin{aligned} 6 &= 3x^2 + (6\lambda x)^2 \\ &= 3x^2 + 36\lambda^2 x^2 \\ &= 3x^2 + 3(12\lambda^2)x^2 \\ &= 3x^2 + 3x^2. \end{aligned}$$

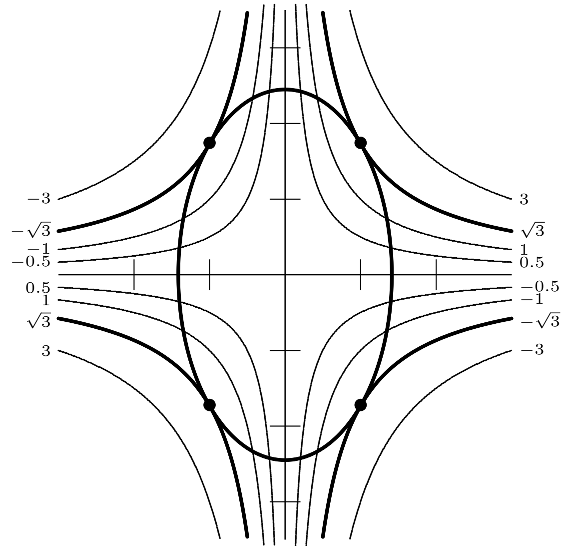
Thus $x = \pm 1$, and $y = \pm\sqrt{3}$ by (7). There are thus four critical points: $(1, \sqrt{3})$, $(1, -\sqrt{3})$, $(-1, \sqrt{3})$, and $(-1, -\sqrt{3})$.

(b) Since $f(x,y) = xy$, we have

$$f(1,\sqrt{3}) = f(-1,-\sqrt{3}) = \sqrt{3}$$
$$f(1,-\sqrt{3}) = f(-1,\sqrt{3}) = -\sqrt{3}.$$

Thus $(1,\sqrt{3})$ and $(-1,-\sqrt{3})$ are maxima and $(1,-\sqrt{3})$ and $(-1,\sqrt{3})$ are minima.

It is instructive to see the picture:



The ellipse is the level curve of $g(x,y)$. The other curves are all various level curves of $f(x,y)$, and the extreme values occur when these level curves share a tangent with the level curve of $g(x,y)$. (These tangent level curves are darker than the other level curves of f.)

3 (a) We have $f(x,y,z) = x^2 - y^2$ and $g(x,y,z) = x^2 + 2y^2 + 3z^2$, so $\nabla f = \langle 2x, -2y, 0\rangle$ and $\nabla g = \langle 2x, 4y, 6z\rangle$. The equations to be solved are thus

$$2x = 2\lambda x \tag{8}$$
$$-2y = 4\lambda y \tag{9}$$
$$0 = 6\lambda z \tag{10}$$
$$x^2 + 2y^2 + 3z^2 = 1 \tag{11}$$

To solve these equations, we look at several cases:

Case 1: $\lambda = 0$

By the first two equations, this implies $x = 0$ and $y = 0$. Thus by (11), $z = \pm\frac{1}{\sqrt{3}}$, and there are two critical points, $(0,0,\frac{1}{\sqrt{3}})$ and $(0,0,-\frac{1}{\sqrt{3}})$.

Case 2: $\lambda \neq 0$

By the third equation, this implies $z = 0$.

Case 2a: $x = 0$

Then by (11), $y = \pm\frac{1}{\sqrt{2}}$, and there are two critical points, $(0,\frac{1}{\sqrt{2}},0)$ and $(0,-\frac{1}{\sqrt{2}},0)$.

Case 2b: $x \neq 0$

By the first equation, this implies $\lambda = 1$. The second equation then becomes $-2y = 4y$, so $y = 0$. Thus by (11), $x = \pm 1$, and there are two critical points, $(1, 0, 0)$ and $(-1, 0, 0)$.

(b) Since $f(x, y, z) = x^2 - y^2$, we have

$$f(0, 0, \tfrac{1}{\sqrt{3}}) = f(0, 0, -\tfrac{1}{\sqrt{3}}) = 0$$
$$f(0, \tfrac{1}{\sqrt{2}}, 0) = f(0, -\tfrac{1}{\sqrt{2}}, 0) = -\tfrac{1}{2}$$
$$f(1, 0, 0) = f(-1, 0, 0) = 1.$$

Thus $(1, 0, 0)$ and $(-1, 0, 0)$ are maxima and $(0, \frac{1}{\sqrt{2}}, 0)$ and $(0, -\frac{1}{\sqrt{2}}, 0)$ are minima. It can be shown that $(0, 0, \frac{1}{\sqrt{3}})$ and $(0, 0, -\frac{1}{\sqrt{3}})$ are saddle points.

4 (a) We have $f(x, y, z) = z^2$ and $g(x, y, z) = x^2 + y^2 - z$, so $\nabla f = \langle 0, 0, 2z \rangle$ and $\nabla g = \langle 2x, 2y, -1 \rangle$. The equations to be solved are thus

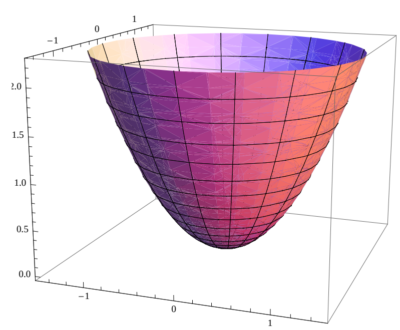
$$0 = 2\lambda x \tag{12}$$
$$0 = 2\lambda y \tag{13}$$
$$2z = -\lambda \tag{14}$$
$$x^2 + y^2 - z = 0 \tag{15}$$

If $\lambda \neq 0$, then $x = 0$ and $y = 0$ by the first two equations, so $z = 0$ by (15). This gives a critical point $(0, 0, 0)$.

If $\lambda = 0$, then $z = 0$ by (14), which implies $x = 0$ and $y = 0$ by (15). Thus we again just get the same critical point $(0, 0, 0)$.

(b) Since $f(x, y, z) = z^2$, $f(x, y, z) \geq 0$ for all (x, y, z). But at our point $(0, 0, 0)$, we have $f(0, 0, 0) = 0$. Thus $(0, 0, 0)$ is a minimum.

(c) This is our standard example of an elliptic paraboloid:



As we can see from the sketch, the surface is infinite, and in particular we can find points (x, y, z) on the surface with z as big as we want. Thus $f(x, y, z) = z^2$ can be as big as we want on the surface, so it has no maximum. That is, the reason Lagrange multipliers did not find a maximum is that there isn't any maximum!

Global Extrema; Lagrange Multipliers II

Here's the theorem we're going to take advantage of today:

> **Extreme Value Theorem**
> **(For Functions Of Two Variables)**
>
> If $f(x, y)$ is continuous on a closed, bounded region D in the plane, then f attains a maximum value $f(x_1, y_1)$ and a minimum value $f(x_2, y_2)$ at points (x_1, y_1) and (x_2, y_2) in D.

1 For which of the regions D described below is it true that every continuous function $f(x, y)$ must attain an absolute maximum value and absolute minimum value on D? (There may be more than one.)

(i) D is the set of points (x, y) such that $|x| \leq 4$ and $|y| < 2$.

(ii) D is the set of points (x, y) such that $|x + y| \leq 1$.

(iii) D is the set of points (x, y) such that $x^2 + 4y^2 \leq 1$.

(iv) D is the set of points (x, y) such that $x^2 + 4y \leq 1$.

(v) D is the set of points (x, y) such that $-x \leq y \leq x$.

Now do the following:

(a) For one region D that you picked, find the absolute minimum and absolute maximum value of $f(x, y) = x^2 - 4x + y^2$ on the region.

(b) For one region you didn't pick, find a function $f(x, y)$ which has either no maximum or no minimum on the region D.

2 Find the absolute maximum and minimum of the function $f(x, y) = x^2 + y^2$ in the elliptical region $2x^2 + 4y^2 \leq 1$.

3 Find the absolute maximum and minimum of the function $f(x, y) = x^2 - y^2$ in the circular region $x^2 + y^2 \leq 1$.

4 Find the point (or points) on the graph of the function $f(x, y) = 2x^2 - y^2 + 1$ that is closest to the origin. (You may make the reasonable assumption that $x^2 + y^2 \leq 10$.)

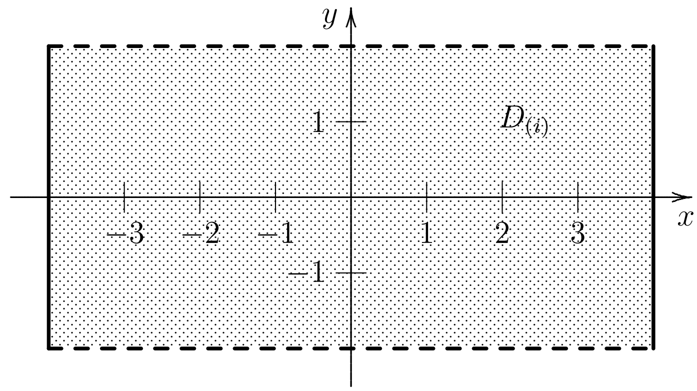
5 Find the absolute maximum and minimum of the function $f(x, y) = 3xy - y + 5$ in the circular region $x^2 + y^2 \leq 1$.

6 The temperature in a room is described by the function $T(x, y, z) = x^2y + z$. A bug is walking on a surface in the room, which can be described parametrically by $\mathbf{r}(u, v) = \langle u, e^v, u + v \rangle$, $0 \leq u \leq 1$, $0 \leq v \leq 1$. What is the warmest point the bug can reach? What is the coolest?

Solutions – Global Extrema

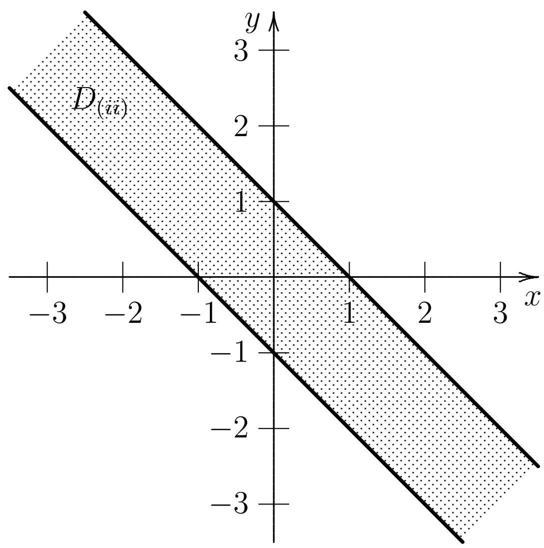
1 We begin by determining whether or not our theorem applies to each region. That is, we check to see whether each region is closed and bounded.

(i) Here D is the set of points in a rectangle:



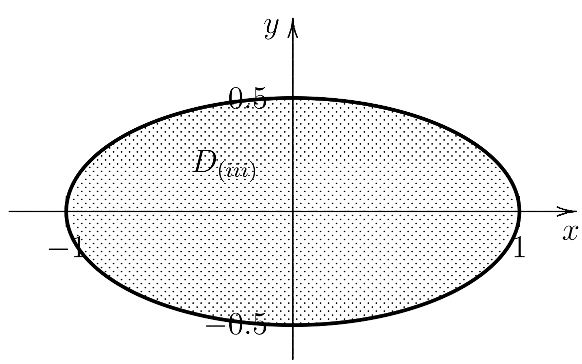
This is clearly bounded, but it is not closed as the boundary points along the line segments $y = 2$ and $y = -2$ are not part of D.

(ii) Here D is the set of points between the lines $x + y = 1$ and $x + y = -1$:



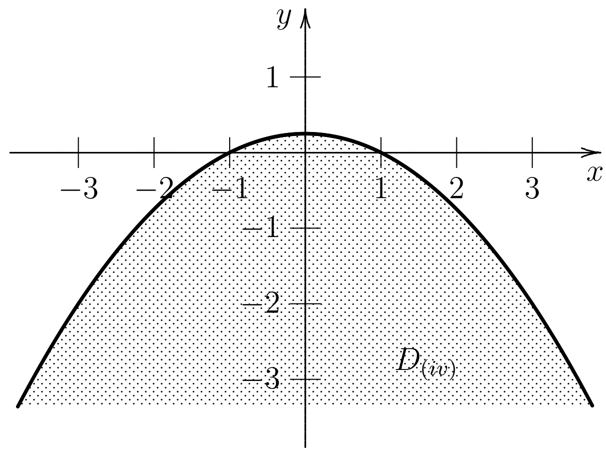
This is closed but not bounded.

(iii) Here D is the set of points on or in an ellipse:



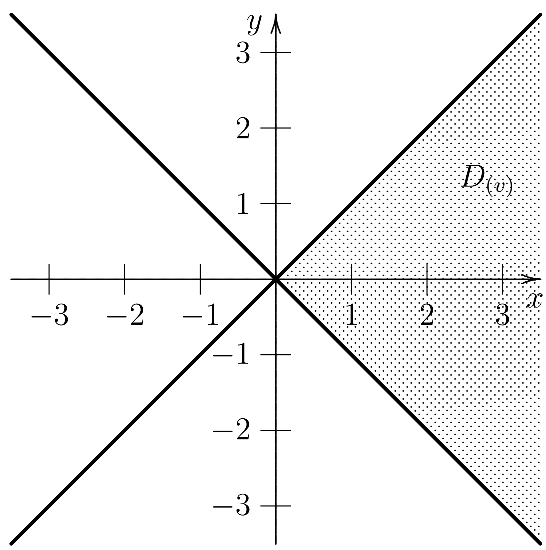
This is both closed and bounded.

(iv) Here D is the set of points on or below the parabola $y = \frac{1-x^2}{4}$:



This is closed but not bounded.

(v) Here D is the set of points above the line $y = -x$ and also below the line $y = x$:



This is closed but not bounded.

(a) Our only option is (iii). We'll look at a collection of points:

- Points inside the ellipse that might be local maxima or local minima. These critical points are where $\nabla f = \mathbf{0}$.
- Points on the ellipse that might be local maxima or local minima of $f(x, y)$ when constrained to the ellipse. To find these we use Lagrange multipliers with $g(x, y) = x^2 + 4y^2 = 1$.

Since $\nabla f = \langle 2x - 4, 2y \rangle$, the only point where $\nabla f = \mathbf{0}$ is the point $(x, y) = (2, 0)$. This point is outside the ellipse and so can be ignored.

We turn to finding points on the ellipse using Lagrange multipliers. From $\nabla f = \lambda \nabla g$ and $g(x, y) = 1$, we have three equations:

$$2x - 4 = \lambda 2x \qquad 2y = \lambda 8y \qquad x^2 + 4y^2 = 1.$$

From the second equation we see that $y = 0$ or $y \neq 0$. If $y = 0$, then the final equation (the constraint) tells us that $x = \pm 1$. If $y \neq 0$, then dividing the second equation by $8y$ gives us $\lambda = 1/4$. The first equation then becomes $2x - 4 = \frac{x}{2}$, or $x = \frac{8}{3}$. But with this value of x the last equation cannot be satisfied (as $x^2 > 1$ already). Thus we have only two points: $(x, y) = (\pm 1, 0)$. Since $f(1, 0) = -3$ and $f(-1, 0) = 5$, our absolute maximum is at $(x, y) = (-1, 0)$ and our absolute minimum is at $(1, 0)$.

(b) We'll find examples of functions without maxima or minima for each of the remaining regions.

(i) This one is very simple: the function $f(x, y) = y$ has no maximum or minimum in $D_{(i)}$. This function attains every value between -2 and $+2$, but neither of these endpoints.

(ii) Consider the function $f(x, y) = y - x$. The level sets for this f are lines parallel to the line $y = x$, each of which is perpendicular to the defining lines of the region $D_{(ii)}$. As we move up and to the left on $D_{(ii)}$, the values of f increase without bound; as we move down and to the right, f decreases without bound. (The level curve $f(x, y) = k$ is the line $y = x + k$, so the value k is simply the y-intercept of this line. We can make this as large or small as we like.) Thus f has no maximum or minimum on $D_{(ii)}$.

(iv) Here the function $f(x, y) = x$ has no maximum or minimum.

(v) Here the function $f(x, y) = y$ has no maximum or minimum. This example and the previous one are simple examples of why the theorem requires that D be bounded.

2 There is one critical point – the origin – in the interior where $\nabla f = \langle 2x, 2y \rangle = \mathbf{0}$. When constrained to the boundary $2x^2 + 4y^2 = 1$, there are four points where $\nabla f = \lambda \nabla g$, or $\langle 2x, 2y \rangle = \lambda \langle 4x, 8y \rangle$. These points are $(x, y) = (\pm 1/\sqrt{2}, 0)$ and $(x, y) = (0, \pm 1/2)$. Our maximum is $f\left(\pm 1/\sqrt{2}, 0\right) = 1/2$ and our minimum is $f(0, 0) = 0$.

Thomas K. (the apprentice in the 11:00 class) pointed out that this is equivalent to minimizing and maximizing the distance from the origin to points in the elliptical region.

3 In this problem the one critical point – again the origin – is a saddle point. The absolute maximum and minimum lie on the boundary where $\nabla f = \lambda \nabla g$ or $\langle 2x, -2y \rangle = \lambda \langle 2x, 2y \rangle$. This can occur only when $x = 0$ or $y = 0$, which gives us the points $(x, y) = (0, \pm 1)$ and $(x, y) = (\pm 1, 0)$. The maximum is $f(\pm 1, 0) = 1$ and the minimum is $f(0, \pm 1) = -1$.

4 Here we're looking for the points $(x, y, z) = (x, y, 2x^2 - y^2 + 1)$ that minimize $\sqrt{x^2 + y^2 + z^2}$ or, even better, minimize $x^2 + y^2 + z^2$. Thus we'd like to minimize $F(x, y) = x^2 + y^2 + (2x^2 - y^2 + 1)^2$. This can be done in the usual way.

Of course, it's probably easier to say: we'd like to minimize $F(x, y, z) = x^2 + y^2 + z^2$ subject to the constraint $g(x, y, z) = 2x^2 - y^2 + 1 - z = 0$. This is a problem built for Lagrange multipliers: $\nabla F = \lambda \nabla g$ when

$$\langle 2x, 2y, 2z \rangle = \lambda \langle 4x, -2y, -1 \rangle.$$

These are the equations (with the constraint)

$$\begin{aligned} 2x &= 4x\lambda \\ 2y &= -2y\lambda \\ 2z &= -\lambda \\ 2x^2 - y^2 + 1 - z &= 0. \end{aligned}$$

The first equation gives us two cases: $x = 0$ or $x \neq 0$ (in which case $\lambda = 1/2$). We'll consider these in turn.

Case 1: $x = 0$ In this case, the second equation now says that either $y = 0$ or $y \neq 0$ (so $\lambda = -1$). If $y = 0$, the last equation tells us that $z = 1$, so we get the point $(x, y, z) = (0, 0, 1)$. If $y \neq 0$, then we divide through by $-2y$ to get $\lambda = -1$. Then the third equation tells us that $z = 1/2$, so the last equation now implies $y = {}^{\pm 1}/_{\sqrt{2}}$. Thus we get the points $(x, y, z) = \left(0, {}^{\pm 1}/_{\sqrt{2}}, 1/2\right)$.

Case 2: $x \neq 0$ In this case, we divide the first equation by $4x$ to find $\lambda = 1/2$. Then the second and third equations allow us to solve for $y = 0$ and $z = {}^{-1}/_4$. The final equation now says that $x^2 = {}^{-5}/_8$, so we get no critical points at all.

We now compute the value of $F = x^2 + y^2 + z^2$ at these three points to find the minimum:

$$F(0, 0, -1) = 1 \qquad \text{and} \qquad F(0, {}^{\pm 1}/_{\sqrt{2}}, 1/2) = 3/4.$$

The smallest of these is the pair of points $(0, {}^{\pm 1}/_{\sqrt{2}}, 1/2)$.

5 I end up with four points:

$$\left(\frac{1+\sqrt{73}}{12}, \pm\sqrt{\frac{35-\sqrt{73}}{72}}\right) \approx (0.795, \pm 0.606)$$

and

$$\left(\frac{1-\sqrt{73}}{12}, \pm\sqrt{\frac{35+\sqrt{73}}{72}}\right) \approx (-0.629, \pm 0.778).$$

Note that

$$f\left(\frac{1+\sqrt{73}}{12}, \sqrt{\frac{35-\sqrt{73}}{72}}\right) \approx 5.840 \qquad \text{and} \qquad f\left(\frac{1+\sqrt{73}}{12}, -\sqrt{\frac{35-\sqrt{73}}{72}}\right) \approx 4.160,$$

while

$$f\left(\frac{1-\sqrt{73}}{12}, \sqrt{\frac{35+\sqrt{73}}{72}}\right) \approx 2.756 \qquad \text{and} \qquad f\left(\frac{1-\sqrt{73}}{12}, -\sqrt{\frac{35+\sqrt{73}}{72}}\right) \approx 7.244.$$

This means the maximum is almost 7.25 and the minimum is just over 2.75.

6 One simple way to do this is to maximize the function $f(u,v) = T(\mathbf{r}(u,v))$ on the unit square $0 \le u \le 1$, $0 \le v \le 1$. This turns out to be the function $f(u,v) = u^2 e^v + u + v$. Notice that $\nabla f = \langle 2ue^v + 1, u^2 e^v + 1 \rangle$. Since u and v cannot be negative, neither of these components can be zero. Thus there are no critical points inside this square.

On the boundary, we could try to use Lagrange multipliers, but this won't work. On our boundaries, ∇g will be parallel to $\mathbf{i}$ or $\mathbf{j}$, and ∇f cannot be parallel to these vectors. (This is for the same reason as above: the components of ∇f cannot be zero on our square.) What goes wrong? Well, our boundary has corners so the tangent line doesn't exist at these points. So while these corners might be maxima or minima, we won't have ∇f and ∇g parallel there.

It turns out that if we look carefully at the boundaries, we see that the absolute maximum occurs at $(u,v) = (1,1)$ and the absolute minimum occurs at $(u,v) = (0,0)$. These correspond to the points $\mathbf{r}(0,0) = \langle 0,1,0 \rangle$ and $\mathbf{r}(1,1) = \langle 1,e,2 \rangle$. The temperatures at these points are $T(0,1,0) = 0$ (the coolest point) and $T(1,e,2) = e+2$ (the warmest point).

How do we "look carefully at the boundaries"? If we restrict ourselves to each piece of the boundary in turn, we have four functions:

The $v = 0$ piece:	$f(u,0) = u^2 + u$	min: 0, max 2
The $v = 1$ piece:	$f(u,1) = eu^2 + u + 1$	min: 1, max $e+2$
The $u = 0$ piece:	$f(0,v) = v$	min: 0, max 1
The $u = 1$ piece:	$f(1,v) = e^v + 1 + v$	min: 2, max $e+2$

We can thus see that the maximum occurs when $(u,v) = (1,1)$ and the minimum occurs when $(u,v) = (0,0)$.

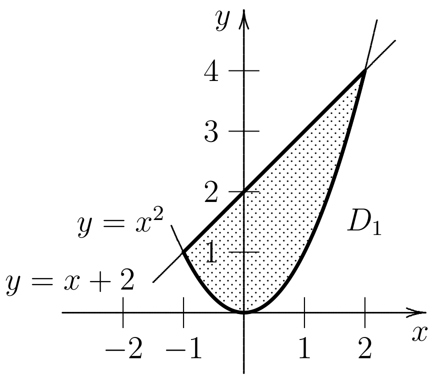
A region D is called Type I if it can be written in the following way:

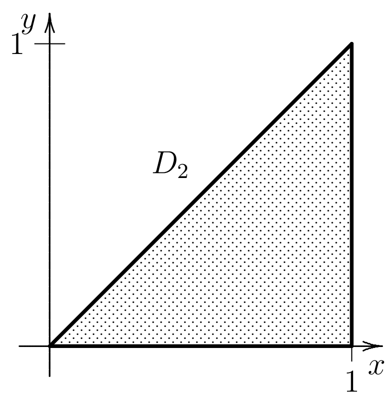
$$D = \{(x,y) : a \le x \le b,\ g_1(x) \le y \le g_2(x)\}.$$

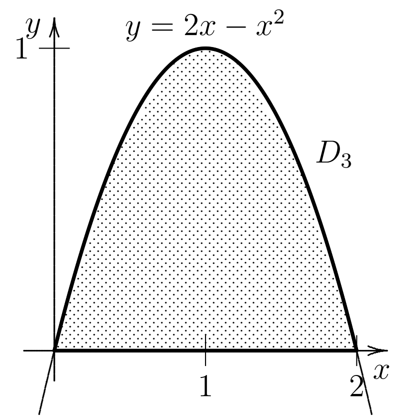
We can then compute a double integral as

$$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx.$$

Here are some Type I regions. Compute the integrals in the problems below.







[1] $\iint_{D_1} xy\,dA$

[2] $\iint_{D_2} e^{x^2}\,dA$

[3] $\iint_{D_3} (x-1)y^2\,dA$

[4] $\iint_{D_1} 1\,dA$

Similarly, a region D is called Type II if it can be written in the following way:

$$D = \{(x,y) : c \le y \le d,\ h_1(y) \le x \le h_2(y)\}.$$

We can then compute a double integral as

$$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy.$$

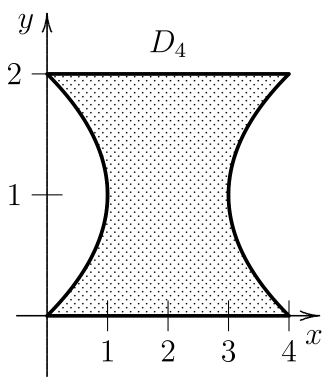
Compute the following integrals as Type II integrals. Some of the regions are shown on the next page.

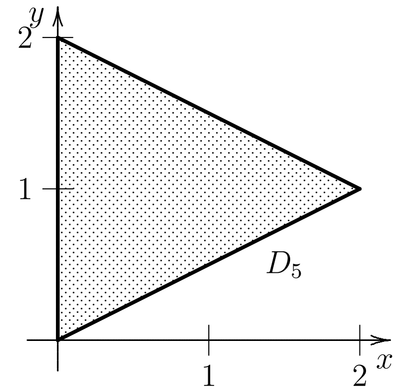
[5] $\iint_{D_2} (1-y)^3\,dA$

[6] $\iint_{D_4} (y-1)x^2\,dA$

[7] $\iint_{D_2} \cos(x^2)\,dA$

[8] $\iint_{D_5} (x-1)\,dA$





(The curves in D_4 are $x = 2y - y^2$ and $x = y^2 - 2y + 4$.)

Of course, sometimes it is necessary to draw the region and possibly even switch the order of integration! For each of the following integrals, draw the region in question, write down an integral with the reverse order of integration, then finally integrate.

9 $\displaystyle\int_0^2 \int_x^2 (x + y)\, dy\, dx$

10 $\displaystyle\int_0^2 \int_0^{\sqrt{x}} (x^2 - y^2)\, dy\, dx$

The real trouble begins when you can't integrate without switching the order of integration. Here are four examples of this. You should draw the region of integration!

11 $\displaystyle\int_0^1 \int_y^1 e^{x^2}\, dx\, dy$

12 $\displaystyle\int_0^{\pi/2} \int_y^{\pi/2} \frac{\sin(x)}{x}\, dx\, dy$

13 $\displaystyle\int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} \left(3x - x^3\right)^{10}\, dx\, dy$

14 $\displaystyle\int_1^{e^3} \int_{\ln(y)}^{3} \left(e^x - x\right)^5\, dx\, dy$

Double Integrals – Answers and Solutions

1 $$\iint_{D_1} xy\,dA = \int_{-1}^{2}\int_{x^2}^{x+2} xy\,dy\,dx = \frac{45}{8}$$

2 $$\iint_{D_2} e^{x^2}\,dA = \int_0^1\int_0^x e^{x^2}\,dy\,dx = \frac{1}{2}(e-1)$$

3 $$\iint_{D_3} (x-1)y^2\,dA = \int_0^2\int_0^{2x-x^2} (x-1)y^2\,dy\,dx = 0$$

As pointed out in class, the function $f(x,y)$ is anti-symmetric about the line $x=1$, so when we integrate over a region that is symmetric about $x=1$ we should expect to get zero.

4 $$\iint_{D_1} 1\,dA = \int_{-1}^{2}\int_{x^2}^{x+2} 1\,dy\,dx = \int_{-1}^{2}\left[(x+2)-x^2\right]\,dy\,dx = 4.5$$

Notice that this is the area of the region D_1. In the middle of this computation we see the single-variable calculus formula for the area between two curves (the integral of "the top curve minus the bottom curve").

5 $$\iint_{D_2} (1-y)^3\,dA = \int_0^1\int_y^1 (1-y)^3\,dx\,dy = \int_0^1 (1-y)^4\,dy = -\frac{1}{5}(1-y)^5\Big|_0^1 = \frac{1}{5}$$

6 $$\iint_{D_4} (y-1)x^2\,dA = \int_0^2\int_{2y-y^2}^{y^2-2y+4} (y-1)x^2\,dx\,dy$$
$$= \frac{1}{3}\int_0^2 (y-1)\left[\left(y^2-2y+4\right)^3 - \left(2y-y^2\right)^3\right]\,dy = 0$$

It's easy to compute this last integral using the substitution $u = y^2 - 2y$. Once we make this substitution we see that the limits of integration are from $u=0$ to $u=0$; hence the integral is zero. We could also have used an argument similar to that of Problem 3 to show that this is zero.

7 $$\iint_{D_2} \cos(x^2)\,dA = \frac{1}{2}\sin(1)$$

The point of this problem is to illustrate that if we write it as a Type II integral, as asked, we get an integral we just can't compute:

$$\iint_{D_2} \cos(x^2)\,dA = \int_0^1\int_y^1 \cos(x^2)\,dx\,dy.$$

On the other hand, if we re-write it as a Type I integral, everything works out nicely:

$$\iint_{D_2} \cos(x^2)\,dA = \int_0^1\int_0^x \cos(x^2)\,dy\,dx = \int_0^1 x\cos(x^2)\,dx = \frac{1}{2}\sin(x^2)\Big|_0^1 = \frac{1}{2}\sin(1).$$

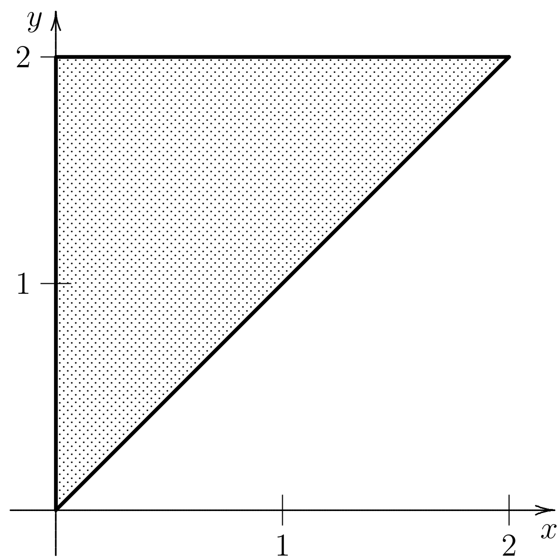
8 $\displaystyle\iint_{D_5}(x-1)\,dA=\int_1^2\int_0^{4-2y}(x-1)\,dx\,dy+\int_0^1\int_0^{2y}(x-1)\,dx\,dy=-\frac{1}{3}+\left(-\frac{1}{3}\right)=-\frac{2}{3}$

Note that it would be easy to write this as only one integral if we thought of D_5 as a Type I region:

$$\iint_{D_5}(x-1)\,dA=\int_0^2\int_{x/2}^{2-x/2}(x-1)\,dy\,dx=-\frac{2}{3}.$$

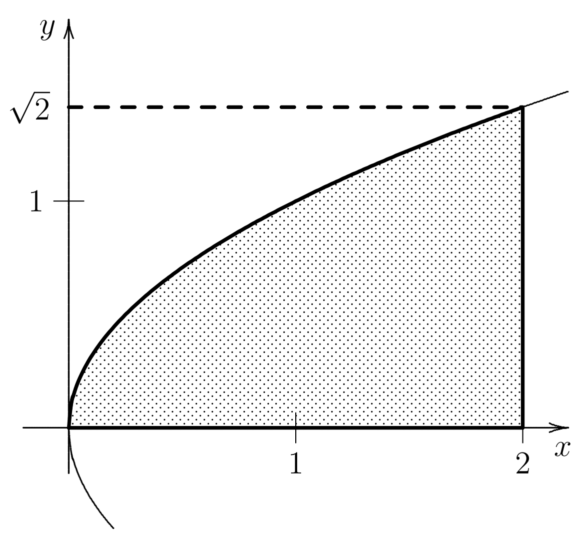
9 $\displaystyle\int_0^2\int_x^2(x+y)\,dy\,dx=\int_0^2\int_0^y(x+y)\,dx\,dy=4$

Here's a picture of the region over which we're integrating:

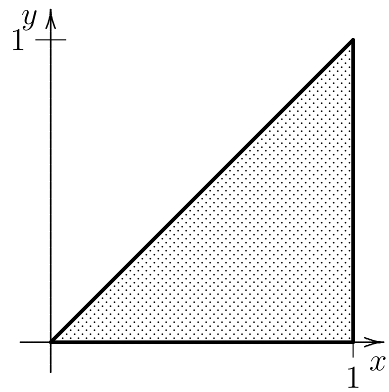


10 $\displaystyle\int_0^2\int_0^{\sqrt{x}}(x^2-y^2)\,dy\,dx=\int_0^{\sqrt{2}}\int_{y^2}^2(x^2-y^2)\,dx\,dy=\frac{184}{105}\sqrt{2}\approx 2.4782.$

Here's a picture of the region over which we're integrating:



11 The region of integration is

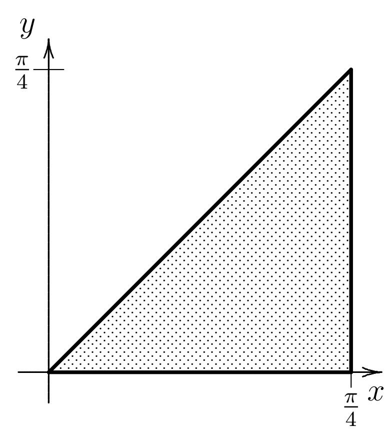


Thus the integral can be re-written as

$$\int_0^1 \int_0^x e^{x^2}\, dy\, dx = \int_0^1 x e^{x^2}\, dx = \frac{1}{2}\left(e - 1\right).$$

This integral turn out to be the same integral as in Problem 2.

12 The region of integration is

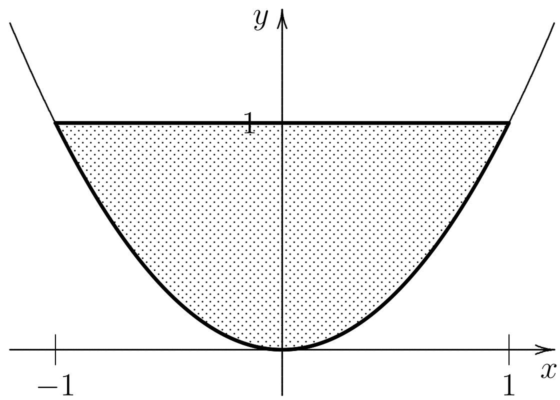


Thus the integral can be re-written as

$$\int_0^{\pi/2} \int_0^x \frac{\sin(x)}{x}\, dy\, dx = \int_0^1 \sin(x)\, dx = 1.$$

Note that the function $\frac{\sin(x)}{x}$ is undefined at $x = 0$, which in this region is simply the origin. But since $\frac{\sin(x)}{x} \to 1$ as $x \to 0$, we can extend this integrand to be continuous at this point. Let's assume that this is what we've done.

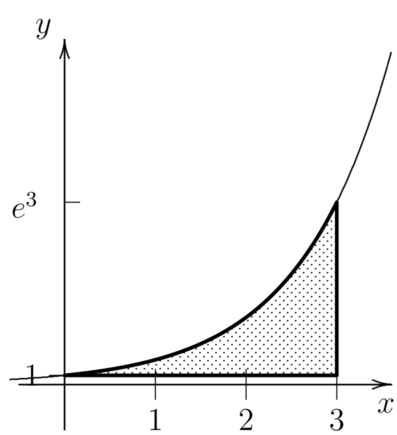
13 The region of integration is



Thus we can write the integral as

$$\int_{-1}^{1}\int_{x^2}^{1}\left(3x-x^3\right)^{10}\ dy\ dx=\int_{-1}^{1}\left(3x-x^3\right)^{10}\left(1-x^2\right)\ dx=\frac{2^{12}}{33}.$$

14 The region of integration is



Thus we can write the integral as

$$\int_{0}^{3}\int_{1}^{e^x}\left(e^x-x\right)^5\ dy\ dx=\int_{0}^{3}\left(e^x-x\right)^5\left(e^x-1\right)\ dx=\frac{1}{6}\left(e^3-4\right).$$

Double Integrals in Polar Coordinates

For Problems 1–6,

(a) Sketch the region of integration.

(b) Try to describe the region in polar coordinates and decide whether you should use polar coordinates or rectangular coordinates.

(c) Evaluate the integral. If you are using polar coordinates, remember that $dA = r\,dr\,d\theta$.

1 $\displaystyle\iint_R \sqrt{x^2+y^2}\,dA$, where R is the region $x^2+y^2 \le 1$.

2 $\displaystyle\iint_R x\,dA$, where R is the region $x^2+y^2 \le 1$, $x \ge 0$.

3 $\displaystyle\iint_R (x+y)^2\,dA$, where R is the region $1 \le x^2+y^2 \le 9$, $x \ge 0$, $y \ge 0$.

Hint: Use the identity $\sin 2\theta = 2\sin\theta\cos\theta$.

4 $\displaystyle\iint_R \left(x^2+y^2\right)\,dA$, where R is the region $0 \le x \le 1$, $0 \le y \le 1$.

5 $\displaystyle\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} e^{x^2+y^2}\,dy\,dx$

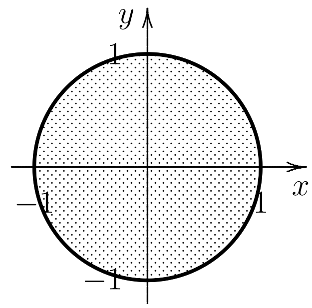
6 $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \sqrt{1-x^2-y^2}\,dx\,dy$

7 Compute the volume of the solid that is under the paraboloid $z = 9 - x^2 - y^2$ and above the xy-plane.

8 Compute the volume of the part of the sphere $x^2 + y^2 + (z+1)^2 = 4$ which is above the xy-plane.

Double Integrals in Polar Coordinates – Solutions

1 (a) The region of integration is the unit circle:

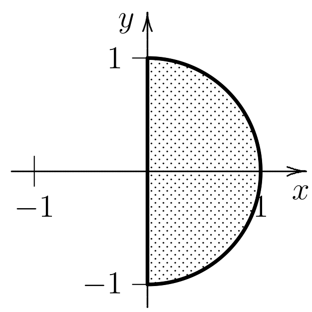


(b) In polar coordinates, the region is $0 \le \theta \le 2\pi$, $0 \le r \le 1$. This is simple, so we should use polar coordinates.

(c) In polar coordinates, the integral becomes

$$\int_0^{2\pi}\int_0^1 r\ r\ dr\ d\theta = \int_0^{2\pi}\int_0^1 r^2\ dr\ d\theta = \int_0^{2\pi} \frac{1}{3}\ d\theta = \frac{2\pi}{3}.$$

2 (a) The region of integration is half the unit circle:

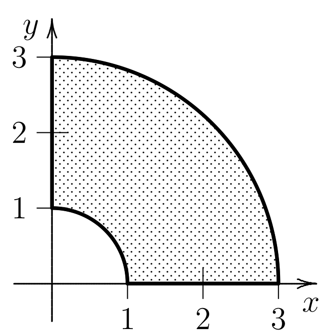


(b) In polar coordinates, the region is $-\pi/2 \le \theta \le \pi/2$, $0 \le r \le 1$. This is simple, so we should use polar coordinates.

(c) In polar coordinates, the integral becomes

$$\int_{-\pi/2}^{\pi/2}\int_0^1 r\cos\theta\ r\ dr\ d\theta = \int_{-\pi/2}^{\pi/2}\int_0^1 r^2\cos\theta\ dr\ d\theta = \int_{-\pi/2}^{\pi/2} \frac{1}{3}\cos\theta\ d\theta = \frac{2}{3}.$$

3 (a) The region of integration is one quarter of an annulus (the region between two circles):



(b) In polar coordinates, the region is $0 \le \theta \le \pi/2$, $1 \le r \le 3$. This is simple, so we should use polar coordinates.

(c) In polar coordinates, the integral becomes

$$\int_0^{\pi/2} \int_1^3 (r\cos\theta + r\sin\theta)^2 \; r \; dr \; d\theta = \int_0^{\pi/2} \int_1^3 r(r\cos\theta + r\sin\theta)^2 \; dr \; d\theta.$$

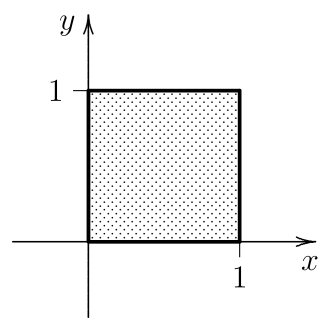
Multiplying out the square and using the identities $\sin^2\theta + \cos^2\theta = 1$ and $2\sin\theta\cos\theta = \sin 2\theta$, the integrand becomes

$$r(r\cos\theta + r\sin\theta)^2 = r^3\cos^2\theta + r^3\sin^2\theta + 2r^3\sin\theta\cos\theta = r^3(1+\sin 2\theta).$$

Thus the integral is

$$\int_0^{\pi/2} \int_1^3 r^3(1+\sin 2\theta) \; dr \; d\theta = \int_0^{\pi/2} 20(1+\sin 2\theta) \; d\theta = 10\pi + 20.$$

4 (a) The region of integration is a square:

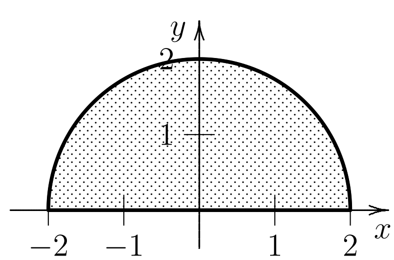


(b) This region is not easy to describe in polar coordinates! However, it's easy to describe in rectangular coordinates because it is just a square.

(c) In rectangular coordinates, the integral is

$$\int_0^1 \int_0^1 \left(x^2 + y^2\right) \; dx \; dy = \int_0^1 \left(\frac{1}{3} + y^2\right) \; dy = \frac{2}{3}.$$

5 (a) The region of integration is a half circle:



(b) In polar coordinates, the region is $0 \le \theta \le \pi$, $0 \le r \le 2$. This is simple, so we should use polar coordinates.

(c) In polar coordinates, the integral becomes

$$\int_0^{\pi}\int_0^2 e^{r^2}\, r\, dr\, d\theta = \int_0^{\pi}\int_0^2 re^{r^2}\, dr\, d\theta.$$

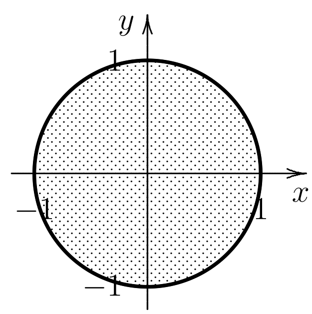
To evaluate the inner integral, set $u = r^2$. Then $du = 2rdr$ and u goes from 0 to 4 as r goes from 0 to 2, so

$$\int_0^2 re^{r^2}\, dr = \frac{1}{2}\int_0^4 e^u\, du = \frac{e^4-1}{2}.$$

Thus the double integral is

$$\int_0^{\pi} \frac{e^4-1}{2}\, d\theta = \pi\frac{e^4-1}{2}.$$

6 (a) The region of integration is the unit circle:



(b) In polar coordinates, the region is $0 \le \theta \le 2\pi$, $0 \le r \le 1$. This is simple, so we should use polar coordinates.

(c) In polar coordinates, the integral becomes

$$\int_0^{2\pi}\int_0^1 \sqrt{1-r^2}\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^1 r\sqrt{1-r^2}\, dr\, d\theta.$$

To evaluate the inner integral, set $u = 1-r^2$. Then $du = -2rdr$ and u goes from 1 to 0 as r goes from 0 to 1, so

$$\int_0^1 r\sqrt{1-r^2}\, dr = -\frac{1}{2}\int_1^0 \sqrt{u}\, du = \frac{1}{3}.$$

Thus the double integral is

$$\int_0^{2\pi} \frac{1}{3}\, d\theta = \frac{2\pi}{3}.$$

Notice that geometrically, this integral is just calculating the volume of the upper hemisphere of $x^2+y^2+z^2 = 1$.

7 First, we must find the region to integrate over; this will just be the interior of the intersection of the paraboloid with the xy-plane (see Figure 1). That intersection is given by setting $z = 0$ to get $9-x^2-y^2 = 0$, or $x^2+y^2 = 9$. Thus we want to integrate over the disk $x^2+y^2 \le 9$. The

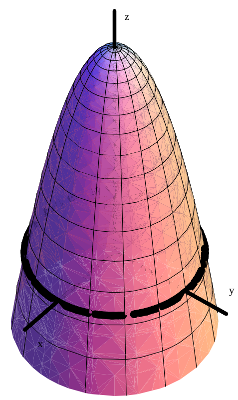


Figure 1: Surface for Problem 7

function to integrate is the height of the paraboloid above the xy-plane, which is $z = 9-x^2-y^2$. Thus we want to calculate the integral

$$\iint_R \left(9 - x^2 - y^2\right)\, dA$$

for R the disk $x^2 + y^2 \le 9$. In polar coordinates, this integral is

$$\int_0^{2\pi}\int_0^3 (9-r^2)\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^3 \left(9r - r^3\right)\, dr d\theta = \int_0^{2\pi}\left(\frac{81}{2} - \frac{81}{4}\right)\, d\theta = \frac{81\pi}{2}.$$

8 First, we must find the region to integrate over; this will just be the interior of the intersection of the sphere with the xy-plane (see picture). That intersection is given by setting $z = 0$ to get $x^2+y^2+1 = 4$, or $x^2+y^2 = 3$. Thus we want to integrate over the disk $x^2+y^2 \le 3$. The function to integrate is the height of the sphere above the xy-plane, which is $z = \sqrt{4-x^2-y^2} - 1$. Thus we want to calculate the integral

$$\iint_R \left(\sqrt{4-x^2-y^2} - 1\right)\, dA$$

for R the disk $x^2 + y^2 \le 3$. In polar coordinates, this integral is

$$\int_0^{2\pi}\int_0^{\sqrt{3}} \left(\sqrt{4-r^2} - 1\right)\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^{\sqrt{3}} r\sqrt{4-r^2}\, dr\, d\theta - \int_0^{2\pi}\int_0^{\sqrt{3}} r\, dr\, d\theta.$$

The second term is just

$$\int_0^{2\pi}\int_0^{\sqrt{3}} r\, dr\, d\theta = \int_0^{2\pi} \frac{3}{2}\, d\theta = 3\pi.$$

For the inner integral in the first term, set $u = 4 - r^2$. Then $du = -2rdr$ and u goes from 4 to 1 as r goes from 0 to $\sqrt{3}$, so

$$\int_0^{\sqrt{3}} r\sqrt{4-r^2}\, dr = -\frac{1}{2}\int_4^1 \sqrt{u}\, du = \frac{7}{3}.$$

Thus the first term is

$$\int_0^{2\pi} \frac{7}{3}\, d\theta = \frac{14\pi}{3}.$$

Putting it all together, the volume is

$$\int_0^{2\pi}\int_0^{\sqrt{3}} r\sqrt{4-r^2}\, dr\, d\theta - \int_0^{2\pi}\int_0^{\sqrt{3}} r\, dr\, d\theta = \frac{14\pi}{3} - 3\pi = \frac{5\pi}{3}.$$

Surface Area

1 In this problem, we'll find the surface area of a sphere of radius a. We think we know that the answer should be $4\pi a^2$, but now we've defined the area of a parameterized surface to be

$$A = \iint_R |\mathbf{r}_u \times \mathbf{r}_v| \; dA, \tag{$*$}$$

so we should be able to make sure.

(a) Use spherical coordinates to write down a parameterization of the sphere of radius a. Recall that we can use spherical coordinates to parameterize the sphere of radius a as

$$\mathbf{r}(\phi, \theta) = \langle a \sin(\phi)\cos(\theta), a\sin(\phi)\sin(\theta), a\cos(\phi)\rangle.$$

What values of ϕ and θ are needed to parameterize the entire sphere? (This will tell you the region R over which we will integrate.)

(b) Now find $\mathbf{r}_\phi$ and $\mathbf{r}_\theta$ using the parameterization from part (a).

(c) Compute $|\mathbf{r}_\phi \times \mathbf{r}_\theta|$. You should get $a^2\sin(\phi)$.

(d) Find the surface area of the sphere using the formula ($*$).

2 One particular parameterization that we might take is for the graph of a function $z = f(x, y)$. We can then replace (u, v) with (x, y), so we get the parameterization

$$\mathbf{r}(x, y) = \langle x, y, f(x, y)\rangle .$$

(a) Use the above expression for $\mathbf{r}(x, y)$ to compute $\mathbf{r}_x$, $\mathbf{r}_y$, and $|\mathbf{r}_x \times \mathbf{r}_y|$.

(b) Use your answer to part (a) and equation ($*$) to find that the surface area of the graph of the function $f(x, y)$ is

$$A = \iint_R \sqrt{1 + f_x^2 + f_y^2}\, dA = \iint_R \sqrt{1 + |\nabla f|^2}\, dA. \tag{$**$}$$

(c) Use equation ($**$) to find the surface area of the paraboloid $z = x^2 + y^2$ that lies over the disk $x^2 + y^2 \leq 9$ of radius 3.

3 Another particularly straightforward set of examples is surfaces of revolution. If we revolve the graph a function $f(x)$ $(a \leq x \leq b)$ around the x-axis, we get a surface that may be parameterized by

$$\mathbf{r}(x, \theta) = \langle x, f(x)\cos(\theta), f(x)\sin(\theta)\rangle .$$

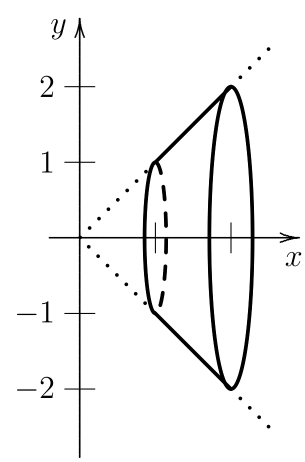
(a) Compute $\mathbf{r}_x$ and $\mathbf{r}_\theta$.

(b) Show that $|\mathbf{r}_x \times \mathbf{r}_\theta| = f(x)\sqrt{1 + (f'(x))^2}$.

(c) Use your answer to part (b) to deduce that the area of this surface of revolution is given by

$$A = 2\pi \int_a bf(x)\sqrt{1 + (f'(x))^2}\, dx. \qquad (\dagger)$$

4 Use equation (†) to compute the area of the *frustrum* of a cone obtained by revolving the line $y = x$ (between $x = 1$ and $x = 2$) around the x-axis. (Note: this problem is really easy using the above equation.)



5 Let's re-do Problem 1, now thinking of the sphere as a surface of revolution. Compute the surface area of a sphere of radius a by revolving the curve $y = \sqrt{a^2 - x^2}$ $(-a \leq x \leq a)$ around the x-axis.

Triple Integrals

1 Evaluate the following triple integrals as iterated integrals.

(a) $\iiint_E xy\, dV$, where $E = [0,1] \times [0,2] \times [0,3]$.

(b) $\iiint_E xy^3 z^2\, dV$, where $E = [-1,1] \times [-3,3] \times [0,3]$.

(c) $\iiint_E y^2 z \cos(xyz)\, dV$, where $E = [0,\pi] \times [0,1] \times [0,2]$.

Hint: Try using different orders of integration.

2 For each of the following regions E, write the triple integral $\iiint_E f(x,y,z)\, dV$ as an iterated integral. There may be up to six different ways to do this, depending on whether you write it with $dx\, dy\, dz$ or $dz\, dy\, dx$ or $dx\, dz\, dy$ or...

(a) The tetrahedron bounded by the planes $x + y + z = 1$, $x = 0$, $y = 0$, and $z = 0$.

(b) The (solid) sphere $x^2 + y^2 + z^2 = a^2$.

(c) The region between the paraboloid $x = 1 - y^2 - z^2$ and the yz-plane.

(d) The region bounded by the surface $z = 3xy + 1$ and the planes $z = 0$, $x = 0$, $x = 1$, $y = 0$, and $y = x$.

(e) The region bounded by the cylinder $x^2 + z^2 = 1$ and the planes $y = 0$ and $y + z = 2$.

(f) The pyramid whose base is the square $[-1, 1] \times [-1, 1]$ in the xy-plane and whose vertex is the point $(0, 0, 1)$.

3 Evaluate the following integrals.

(a) $\iiint_E 1 \, dV$, where E is the region in Problem 2(a).

(b) $\iiint_E z \, dV$, where E is the region in Problem 2(d).

4 Find the volume of the pyramid described in Problem 2(f).

5 Consider a brick in the region $[0, 1] \times [0, 2] \times [0, 1]$ whose density at a point (x, y, z) is $\rho(x, y, z) = 2 + xy - 2z$. Find the mass of the brick.

Triple Integrals – Solutions

1 (a) The integral is

$$\int_0^1 \int_0^2 \int_0^3 xy\, dz\, dy\, dx = \int_0^1 \int_0^2 3xy\, dy\, dx$$
$$= \int_0^1 3x \left(\frac{2^2}{2} - 0\right) dx$$
$$= \int_0^1 6x\, dx = 3.$$

(b) The integral is

$$\int_0^3 \int_{-3}^3 \int_{-1}^1 xy^3z^2\, dx\, dy\, dz = \int_0^3 \int_{-3}^3 y^3z^2 \left(\frac{1^2}{2} - \frac{(-1)^2}{2}\right) dy\, dz = 0.$$

Note that we can put the three integrals in whatever order we want, and putting the integral with respect to x first makes the computation easier.

(c) If we write the integral as

$$\int_0^\pi \int_0^1 \int_0^2 y^2z\cos(xyz)\, dz\, dy\, dx,$$

we have to use integration by parts and the integral is lots of work. However, if we write it as

$$\int_0^1 \int_0^2 \int_0^\pi y^2z\cos(xyz)\, dx\, dz\, dy,$$

it is a lot easier. In the inner integral, we can note that

$$\frac{\partial}{\partial x} y\sin(xyz) = y^2z\cos(xyz)$$

so we get

$$\int_0^1 \int_0^2 \int_0^\pi y^2z\cos(xyz)\, dx\, dz\, dy = \int_0^1 \int_0^2 (y\sin(\pi yz) - y\sin(0))\, dz\, dy$$
$$= \int_0^1 \int_0^2 y\sin(\pi yz)\, dz\, dy.$$

We then similarly have

$$\frac{\partial}{\partial z}\cos(xyz) = -y\sin(xyz)$$

so

$$\int_0^1 \int_0^2 y\sin(\pi yz)\, dz\, dy = \frac{-1}{\pi}\int_0^1 (\cos(2\pi y) - \cos(0))\, dy$$
$$= \frac{-1}{\pi}\int_0^1 (\cos(2\pi y) - 1)\, dy = \frac{1}{\pi}.$$

2 We only give one possible way to express each integral; there are others that are equally correct.

(a) $\iiint_E f(x,y,z)\,dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} f(x,y,z)\,dz\,dy\,dx.$

(b) $\iiint_E f(x,y,z)\,dV = \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} f(x,y,z)\,dz\,dy\,dx.$

(c) This region is defined by the inequality $0 \le x \le 1 - y^2 - z^2$. For $1 - y^2 - z^2$ to be nonnegative, we also need (y,z) to lie on the disk D bounded by $y^2 + z^2 = 1$. Thus we get

$$\begin{aligned}\iiint_E f(x,y,z)\,dV &= \iint_D \int_0^{1-y^2-z^2} f(x,y,z)\,dx\,dA \\ &= \int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_0^{1-y^2-z^2} f(x,y,z)\,dx\,dz\,dy.\end{aligned}$$

(d) This region is defined by the inequalities $0 \le z \le 3xy + 1$, $0 \le x \le 1$, and $0 \le y \le x$, so we get

$$\iiint_E f(x,y,z)\,dV = \int_0^1 \int_0^x \int_0^{3xy+1} f(x,y,z)\,dz\,dy\,dx.$$

(e) Being between the planes $y = 0$ and $y + z = 2$ says that $0 \le y \le 2 - z$, and being inside the cylinder says that (x,z) is in the disk D bounded by $x^2 + z^2 = 1$. Thus we get

$$\iiint_E f(x,y,z)\,dV = \iint_D \int_0^{2-z} f(x,y,z)\,y\,dA = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^{2-z} f(x,y,z)\,dy\,dz\,dx.$$

(f) In the pyramid, z ranges from 0 at the base to 1 at the top. The cross-section of the pyramid given by a fixed value of z is the square $[-1+z, 1-z] \times [-1+z, 1-z]$. Thus we get

$$\iiint_E f(x,y,z)\,dV = \int_0^1 \int_{-1+z}^{1-z} \int_{-1+z}^{1-z} f(x,y,z)\,dx\,dy\,dz.$$

3 (a) The integral is

$$\begin{aligned}\int_0^1 \int_0^{1-x} \int_0^{1-x-y} 1\,dz\,dy\,dx &= \int_0^1 \int_0^{1-x} (1-x-y)\,dy\,dx \\ &= \int_0^1 \left((1-x)^2 - \frac{(1-x)^2}{2}\right) dx \\ &= \int_0^1 \frac{(1-x)^2}{2}\,dx = \frac{1}{6}.\end{aligned}$$

(b) The integral is

$$\int_0^1 \int_0^x \int_0^{3xy+1} z \, dz \, dy \, dx = \int_0^1 \int_0^x \frac{(3xy+1)^2}{2} \, dy \, dx$$
$$= \frac{1}{2} \int_0^1 \int_0^x (9x^2y^2 + 6xy + 1) \, dy \, dx$$
$$= \frac{1}{2} \int_0^1 (3x^5 + 3x^3 + x) \, dx = \frac{7}{8}.$$

4 The volume of a region E is given by the integral $\iiint_E 1 \, dV$. In this case, that integral is

$$\int_0^1 \int_{-1+z}^{1-z} \int_{-1+z}^{1-z} 1 \, dx \, dy \, dz = \int_0^1 \int_{-1+z}^{1-z} (2-2z) \, dy \, dz$$
$$= \int_0^1 (2-2z)^2 \, dz = \frac{4}{3}.$$

5 The mass is given by integrating the density over the region, so the mass is

$$\int_0^1 \int_0^2 \int_0^1 (2 + xy - 2z) \, dz \, dy \, dx = \int_0^1 \int_0^2 (2 + xy - 1) \, dy \, dx$$
$$= \int_0^1 (4 + 2x - 2) \, dy \, dx$$
$$= 4 + 1 - 2 = 3.$$

Cylindrical & Spherical Coordinates

When we change from Cartesian coordinates (x, y, z) to cylindrical coordinates (r, θ, z) or spherical coordinates (ρ, θ, ϕ), integrals transform according to the rule

$$dV = dx\,dy\,dz = r\,dr\,d\theta\,dz = \rho^2 \sin\phi\,d\rho\,d\theta\,d\phi.$$

1 Using cylindrical coordinates, evaluate the integral $\iiint_E \sqrt{x^2+y^2}\,dV$, where E is the solid in the first octant inside the cylinder $x^2 + y^2 = 16$ and below the plane $z = 3$.

2 Sketch the solid whose volume is given by the integral $\int_0^{\pi/2}\int_0^2\int_0^{9-r^2} r\,dz\,dr\,d\theta$, and evaluate the integral.

3 Use spherical coordinates to evaluate $\iiint_E z\,dV$, where E lies between the spheres $x^2+y^2+z^2 = 1$ and $x^2 + y^2 + z^2 = 4$ in the quarter-space where $y \le 0$ and $z \ge 0$.

4 Use spherical coordinates to set up a triple integral expressing the volume of the "ice-cream cone," which is the solid lying above the cone $\phi = \pi/4$ and below the sphere $\rho = \cos\phi$. Evaluate it.

5 Sketch the region of integration for

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} xy\, dz\, dy\, dx,$$

and evaluate the integral by changing to spherical coordinates.

6 Make an appropriate change of coordinates to evaluate the integral $\iiint_E (x^2+y^2)\, dV$, where E is the part of the sphere $x^2+y^2+z^2=1$ above the xy-plane.

Cylindrical & Spherical Coords. – Answers and Solutions

1 The solid E may be described as

$$\begin{aligned} E &= \left\{(x,y,z)\ :\ x \geq 0,\ y \geq 0,\ x^2+y^2 \leq 16, 0 \leq z \leq 3\right\} \\ &= \left\{(r,\theta,z)\ :\ 0 \leq \theta \leq \tfrac{\pi}{2},\ 0 \leq r \leq 4, 0 \leq z \leq 3\right\}. \end{aligned}$$

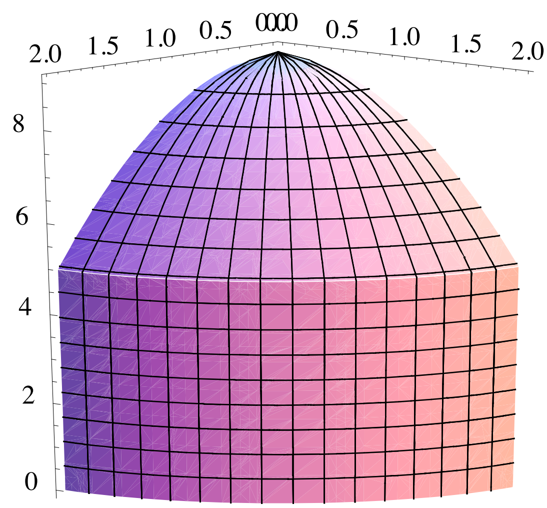
Thus it makes sense to evaluate this in cylindrical coordinates:

$$\iiint_E \sqrt{x^2+y^2}\,dV = \int_0^3 \int_0^{\pi/2} \int_0^4 r\ r\ dr\ d\theta\ dz = 3 \cdot \frac{\pi}{2} \cdot \frac{1}{3}(4)^3 = 32\pi.$$

2 This solid can be described as

$$\left\{(r,\theta,z)\ :\ 0 \leq z \leq 9-r^2,\ 0 \leq \theta \leq \tfrac{\pi}{2},\ 0 \leq r \leq 2\right\}.$$

This is that part of the cylinder of radius 2 (it is centered along the z-axis and has equation $x^2+y^2=2^2$) that lies in the first octant and underneath the elliptic paraboloid $z=9-x^2-y^2$. Here's a very simple Mathematica sketch of this solid:



3 In spherical coordinates, $z = \rho\cos(\phi)$. The region over which we're integrating can be described by the inequalities $0 \leq \phi \leq \frac{\pi}{2}$ (from $z \geq 0$), $\pi \leq \theta \leq 2\pi$ (from $y \leq 0$), and $1 \leq \rho \leq 2$ (from $1 \leq x^2+y^2+z^2 \leq 4$). Thus our integral is

$$\iiint_E z\,dV = \int_0^{\pi/2} \int_\pi^{2\pi} \int_1^2 \rho\cos(\phi)\cdot\rho^2\sin(\phi)d\rho\ d\theta\ d\phi = \frac{15\pi}{8}.$$

4 This region is

$$\left\{(\rho,\theta,\phi)\ :\ 0 \leq \rho \leq \cos(\phi),\ 0 \leq \phi \leq \tfrac{\pi}{4},\ 0 \leq \theta \leq 2\pi\right\}.$$

Thus the volume of the "ice-cream cone" is expressed by the iterated integral

$$V = \int_0^{2\pi} \int_0^{\pi/4} \int_0^{\cos(\phi)} \rho^2\sin(\phi)\ d\rho\ d\phi\ d\theta.$$

We were not asked to evaluate this, but it isn't difficult:

$$V = \int_0^{2\pi} \int_0^{\pi/4} \frac{1}{3}\cos^3(\phi)\sin(\phi)\ d\phi\ d\theta = \int_0^{2\pi} \frac{1}{3}\cdot\frac{1}{4}\left(1^4 - \left(\frac{1}{\sqrt{2}}\right)^4\right) d\theta = \frac{\pi}{8}.$$

5 This is the region

$$\left\{(x,y,z)\ :\ \sqrt{x^2+y^2}\le z\le\sqrt{2-x^2-y^2},\ 0\le y\le\sqrt{1-x^2},\ 0\le x\le 1\right\}.$$

The x and y restrictions mean we're integrating over the quarter of the unit circle in the first quadrant. The restrictions on z mean we're integrating the volume between the cone $z^2=x^2+y^2$ and the sphere $x^2+y^2+z^2=2$. In spherical coordinates, the cone is $\phi=\frac{\pi}{4}$ and the sphere is $\rho=\sqrt{2}$. Thus this is the region

$$\left\{(\rho,\phi,\theta)\ :\ 0\le\theta\le\tfrac{\pi}{2},\ 0\le\phi\le\tfrac{\pi}{4},\ 0\le\rho\le\sqrt{2}\right\}.$$

Thus (since $xy=\rho\sin(\phi)\cos(\theta)\cdot\rho\sin(\phi)\sin(\theta)$),

$$\begin{aligned}\int_0^1\int_0^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} xy\,dz\,dy\,dx &= \int_0^{\pi/4}\int_0^{\pi/2}\int_0^{\sqrt{2}}\rho^2\sin^2(\phi)\sin(\theta)\cos(\theta)\ \rho^2\sin(\phi)\ d\rho\ d\theta\ d\phi\\ &= \int_0^{\pi/4}\int_0^{\pi/2}\int_0^{\sqrt{2}}\rho^4\sin^3(\phi)\sin(\theta)\cos(\theta)\ d\rho\ d\theta\ d\phi.\end{aligned}$$

These iterated integrals are each independent of the others, so this quantity is

$$\left(\int_0^{\pi/4}\sin^3(\phi)\ d\phi\right)\left(\int_0^{\pi/2}\sin(\theta)\cos(\theta)\right)\left(\int_0^{\sqrt{2}}\rho^4\ d\rho\right)=\left(2-\frac{5}{2\sqrt{2}}\right)\cdot\frac{1}{2}\cdot\frac{4\sqrt{2}}{5}=\frac{1}{15}\left(4\sqrt{2}-5\right).$$

The second and third of these integrals are simple, and the first is not difficult using the substitution $u=\cos(\phi)$ and the relation $\sin^2(\phi)=1-\cos^2(\phi)$. We omit any further details.

6 We can use either spherical or cylindrical coordinates. Both have their appeal – the region (part of a sphere) calls out for spherical coordinates and the integrand ($r^2=x^2+y^2$) is asking for cylindrical. We'll do both.

In spherical coordinates, $x^2+y^2=\rho^2\sin^2(\phi)$ and the region E is simply

$$\left\{(\rho,\theta,\phi)\ :\ 0\le\rho\le 1,\ 0\le\theta\le 2\pi,\ 0\le\phi\le\tfrac{\pi}{2}\right\}.$$

Thus

$$\begin{aligned}\iiint_E(x^2+y^2)\,dV &= \int_0^{\pi/2}\int_0^{2\pi}\int_0^1\rho^2\sin^2(\phi)\cdot\rho^2\sin(\phi)\ d\rho\ d\theta\ d\phi\\ &= \int_0^{\pi/2}\int_0^{2\pi}\int_0^1\rho^4\sin^3(\phi)\ d\rho\ d\theta\ d\phi.\end{aligned}$$

The only difficulty here is the integral of $\sin^3(\phi)$, but we computed this in problem 5. Hence the answer we get is

$$\begin{aligned}\iiint_E(x^2+y^2)\,dV &= \frac{1}{5}\int_0^{\pi/2}\int_0^{2\pi}\sin^3(\phi)\ d\theta\ d\phi\\ &= \frac{2\pi}{5}\int_0^{\pi/2}\sin^3(\phi)\ d\phi\\ &= \frac{4\pi}{15}.\end{aligned}$$

In cylindrical coordinates, $x^2 + y^2 = r^2$ and the region E is

$$\left\{(r, \theta, z) \;:\; 0 \le z \le \sqrt{1 - r^2},\; 0 \le r \le 1,\; 0 \le \theta \le 2\pi\right\}.$$

(I'm angling to integrate z first, so I've written z in terms of r. One could integrate r first instead; in this case one would have $0 \le r \le \sqrt{1 - z^2}$ and $0 \le z \le 1$). Thus

$$\iiint_E (x^2 + y^2)\, dV = \int_0^{2\pi} \int_0^1 \int_0^{\sqrt{1-z^2}} r^2 \cdot r \; dr \; dz \; d\theta.$$

These integrals are not at all complicated, and we get the same answer $\frac{4\pi}{15}$ as before.

Math 21a

Vector Fields

Spring, 2009

1 Match the following vector fields to the pictures, below. Explain your reasoning.

(Notice that in some of the pictures all of the vectors have been uniformly scaled so that the picture is more clear. Also notice that there are eight vector fields but only six pictures. There's probably a reason behind this.)

Here are the possible vector fields:

(a) $\mathbf{F}(x, y) = \langle 1, x \rangle$

(b) $\mathbf{F}(x, y) = \langle -y, x \rangle$

(c) $\mathbf{F}(x, y) = \langle y, x \rangle$

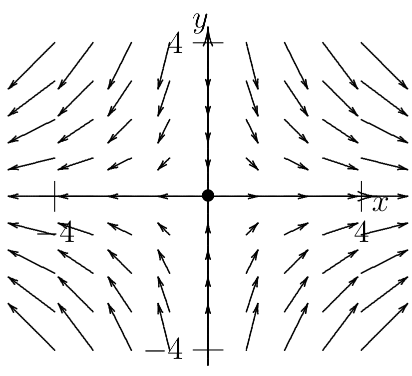
(d) $\mathbf{F}(x, y) = \langle 2x, -2y \rangle$

(e) ∇f, where $f(x, y) = x^2 + y^2$

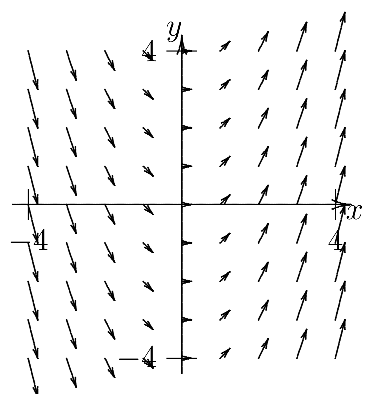
(f) ∇f, where $f(x, y) = \sqrt{x^2 + y^2}$

(g) ∇f, where $f(x, y) = xy$

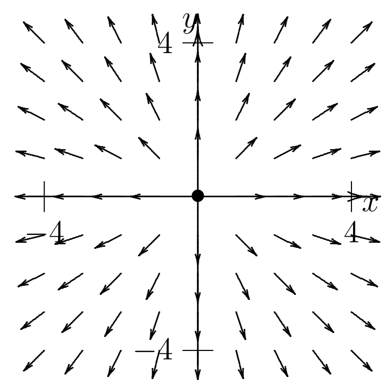
(h) ∇f, where $f(x, y) = x^2 - y^2$



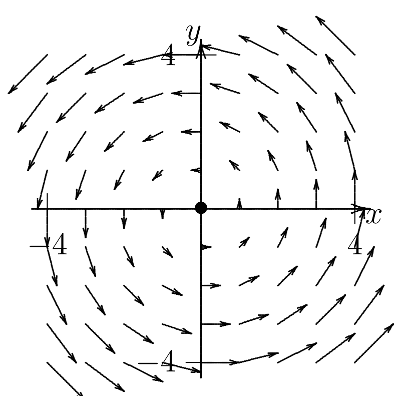
Field (I)



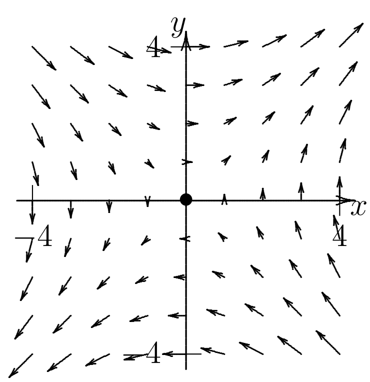
Field (II)



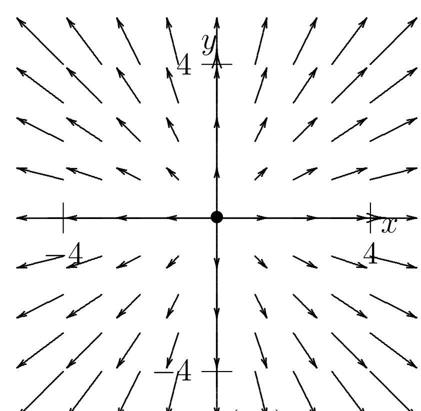
Field (III)



Field (IV)



Field (V)



Field (VI)

2 Recall that the gradient of a function is a vector normal to the level curve of this function. Explain how this confirms your identification of the pictures for vector fields (e) through (h), above.

Vector Fields – Answers and Solutions

1 (I) This is vector fields (d) and (h). (First notice that these two vector fields are the same!) We can see this by noticing that the vectors should point down when $y > 0$ and up when $y < 0$, and field (I) is the only one that does this.

(II) This is vector field (a). Notice that this vector field always has a positive rightward component, which is true only of Field (II).

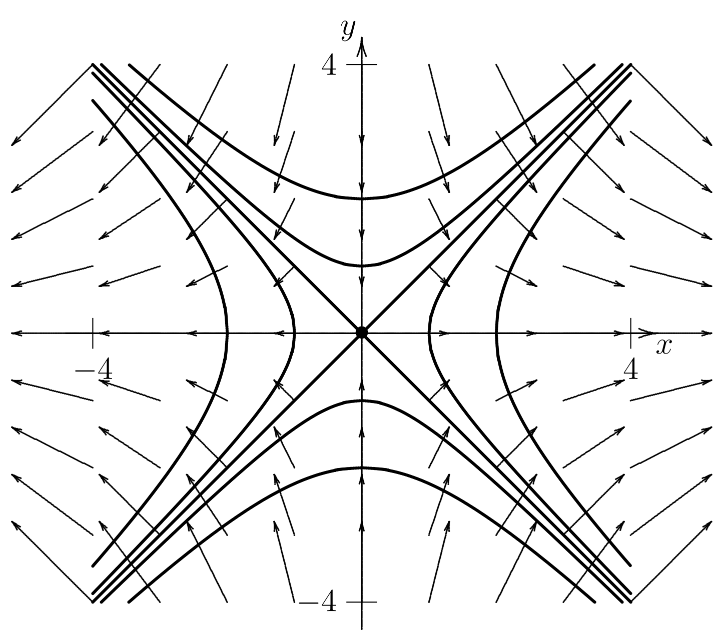
(III) This is vector field (f). Both (e) and (f) are vector fields that point radially outward, so they are Fields (III) and (VI). But which is which? Notice that the vector field in (e) is $\nabla f = \langle 2x, 2y \rangle$, which has length $2r = 2\sqrt{x^2 + y^2}$. On the other hand, the vector field in (f) is $\nabla f = \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right\rangle = \left\langle \frac{x}{r}, \frac{y}{r} \right\rangle$, which has length 1. Thus (e) is Field (VI), the field with the vectors that increase in length as the distance from the origin increases, while (f) is Field (III), the field with vectors all the same magnitude.

(IV) This is vector field (b). Look, for example, at the vectors on the axes. On the x-axis, the vector field is $\mathbf{F}(x, 0) = \langle 0, x \rangle$, a vector that points vertically up (if $x > 0$) or down (if $x < 0$). This narrows our choices to Fields (IV) or (V). On the y-axis, the vector field is $\mathbf{F}(0, y) = \langle -y, 0 \rangle$, a vector that points to the left (if $y > 0$) or to the right (if $y < 0$). This eliminates Field (V) and confirms Field (IV).

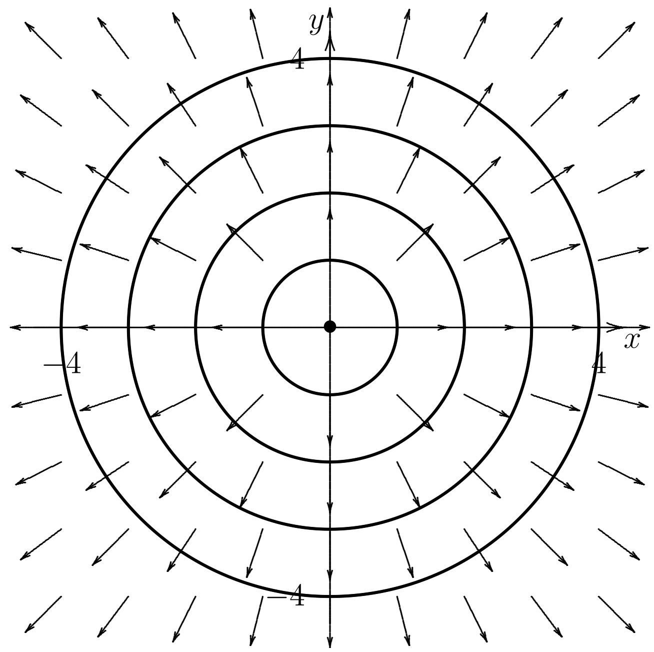
(V) This is vector field (c) and (g), by an analysis that is very similar to the one in Field (IV). (Notice that (c) and (g) are the same!)

(VI) This is vector field (e). See (III) for the explanation.

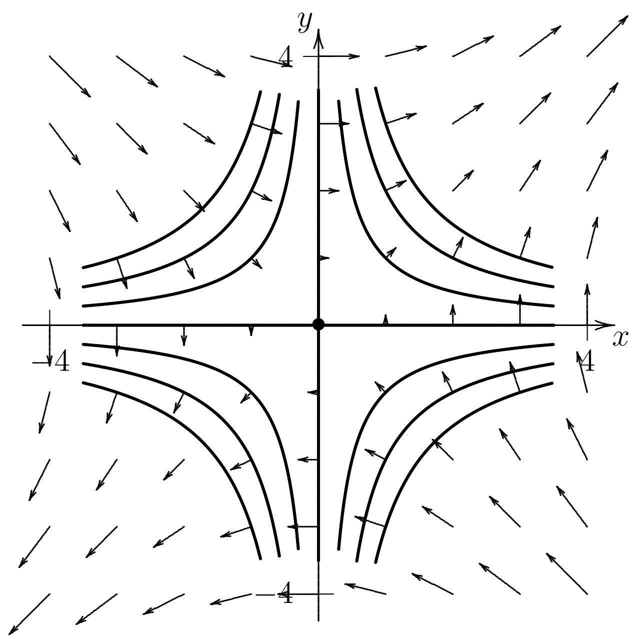
2 Four of the vector fields are (explicitly) gradient fields (we'll be able to tell later that vector fields (a) and (b) are *not* gradient fields). Since ∇f is perpendicular to the level curves of f, we should be able to see this in the vector field. Here I've re-drawn the four vector fields in question with some level curves drawn in as well – note the perpendicularity!



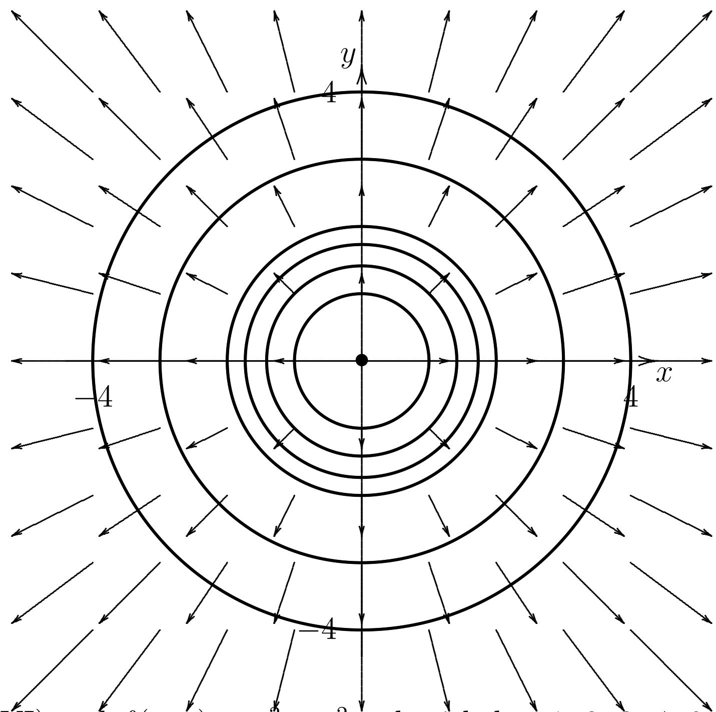
Field (I) and $f(x, y) = x^2 - y^2 = k$ for $k = 0, \pm 1, \pm 2$



Field (III) and $f(x, y) = \sqrt{x^2 + y^2} = k$ with $k = 1, 2, 3, 4$



Field (V) with $f(x, y) = xy = k$ for $k = 0, \pm1, \pm2, \pm3$



Field (VI) and $f(x, y) = x^2 + y^2 = k$ with $k = 1$, 2, 3, 4, 9, and 16

Green's Theorem

Green's Theorem: Suppose C is a positively oriented, piecewise-smooth, simple closed curve in the plane that bounds a region D. If P and Q have continuous derivatives (in an open set containing the region D), then

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,dA.$$

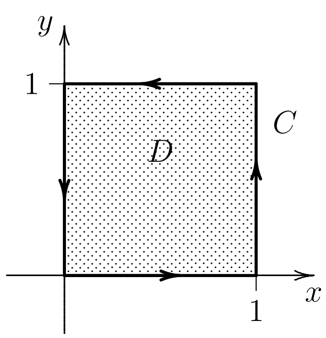
Sometimes the line integral is written $\oint_C P\,dx + Q\,dy$ to emphasize that the curve is closed.

For each of the following regions D, associated boundary curves C, and line integrals...

(a) Compute the given line integral directly by parameterizing the path C.

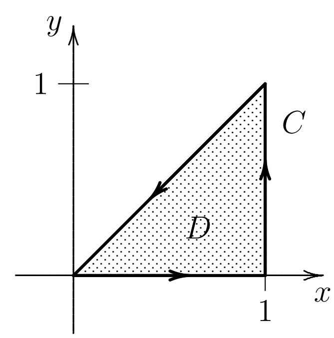
(b) Compute the given line integral by applying Green's theorem and computing a double integral.

1



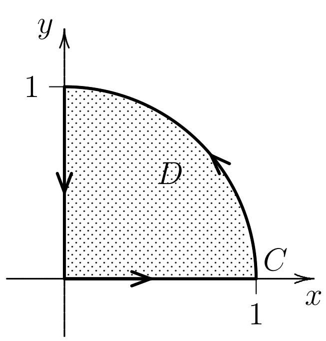
$$\int_C xy\,dx + (x^2 - y^2)\,dy$$

2



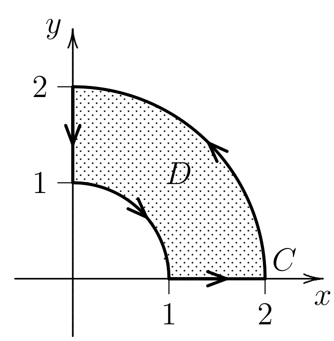
$$\int_C x^3\,dx - xy^2\,dy$$

3



$$\int_C x\,dy - y\,dx$$

4



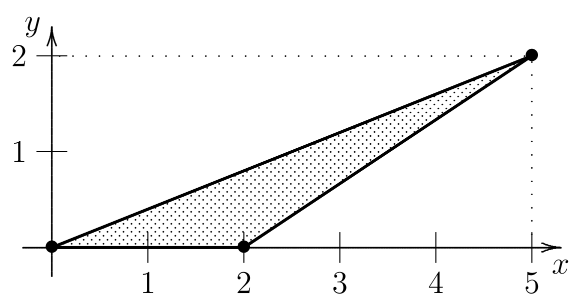
$$\int_C xy^2\,dx - x^2y\,dy$$

5 How does this relate to our work on Friday? If the vector field $\mathbf{F} = \langle P, Q\rangle$ is conservative (that is, if $\mathbf{F} = \nabla f$ for some f), what does Green's theorem say about the line integral and the double integral?

6 Some trickery: Recall that $\iint_D 1\, dA$ is the area of the region D. So if we choose P and Q so that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$, then $\int_C P\, dx + Q\, dy$ is also the area of D.

(a) One such choice is $P = -y$ and $Q = 0$. Write down two other choices of the vector field $\mathbf{F} = \langle P, Q\rangle$ so that $\int_C P\, dx + Q\, dy$ is the area of D.

(b) Use one of your line integrals from part (a) to find the area of the triangle with corners at $(0, 0)$, $(2, 0)$, and $(5, 2)$:



7 In this problem we'll calculate

$$\oint_C \frac{x\, dy - y\, dx}{x^2 + y^2}$$

where C is *any* positively oriented simple closed curve that encloses the origin, as follows:

(a) Let C_1 and C_2 be two different such simple closed curves that don't intersect. Let D be the region with boundary $C_1 \cup (-C_2)$. Show that

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = 0.$$

(b) Since

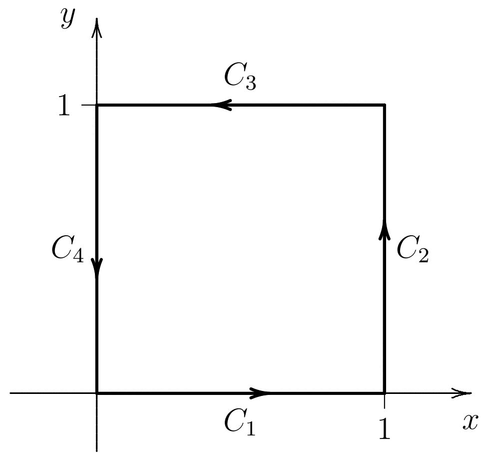
$$\int_C P\, dx+Q\, dy = \int_{C_1} P\, dx+Q\, dy+\int_{-C_2} P\, dx+Q\, dy = \int_{C_1} P\, dx+Q\, dy-\int_{C_2} P\, dx+Q\, dy,$$

use Green's theorem to show that the two line integrals (over C_1 and C_2) must agree.

(c) Choose a particularly nice parameterized curve (say, a simple circle) to compute the given integral.

Green's Theorem – Answers and Solutions

1 (a) I'll write this line integral as a sum of four line integrals, each of which I'll parameterize and compute separately:



We perform the parameterizations without comment:

C_1: $\mathbf{r}(t) = \langle x, y\rangle = \langle t, 0\rangle$ (for $0 \le t \le 1$), so $dx = dt$ and $dy = 0$. Thus

$$\int_{C_1} xy\ dx + \left(x^2 - y^2\right)\ dy = \int_0^1 t \cdot 0\ dt + \left(t^2 - 0^2\right)\ 0 = \int_0^1 0 = 0.$$

C_2: $\mathbf{r}(t) = \langle x, y\rangle = \langle 1, t\rangle$ (for $0 \le t \le 1$), so $dx = 0$ and $dy = dt$. Thus

$$\int_{C_2} xy\ dx + \left(x^2 - y^2\right)\ dy = \int_0^1 1 \cdot t\ 0 + \left(1^2 - t^2\right)\ dt = \int_0^1 (1 - t^2)\ dt = \frac{2}{3}.$$

C_3: $\mathbf{r}(t) = \langle x, y\rangle = \langle 1 - t, 1\rangle$ (for $0 \le t \le 1$), so $dx = -dt$ and $dy = 0$. Thus

$$\int_{C_3} xy\ dx + \left(x^2 - y^2\right)\ dy = \int_0^1 (1 - t) \cdot 1 \cdot -dt + \left((1-t)^2 - 1^2\right)\ 0 = \int_0^1 (t - 1)\ dt = -\frac{1}{2}$$

C_4: $\mathbf{r}(t) = \langle x, y\rangle = \langle 0, 1 - t\rangle$ (for $0 \le t \le 1$), so $dx = 0$ and $dy = -dt$. Thus

$$\int_{C_4} xy\ dx + \left(x^2 - y^2\right)\ dy = \int_0^1 0 \cdot (1 - t)\ 0 + \left(0^2 - (1-t)^2\right) \cdot -dt = \int_0^1 (1 - t)^2\ dt = \frac{1}{3}.$$

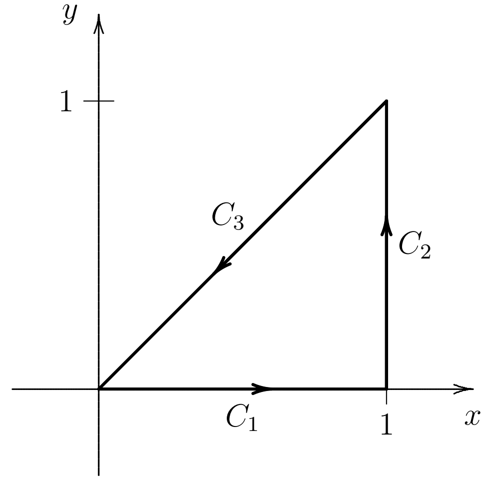
Putting this all together, we get

$$\int_C xy\ dx + \left(x^2 - y^2\right)\ dy = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4} = 0 + \frac{2}{3} - \frac{1}{2} + \frac{1}{3} = \frac{1}{2}.$$

(b) Here $P = xy$ and $Q = x^2 - y^2$, so the integrand of the double integral is $Q_x - P_y = 2x - x = x$. The region of integration is a square, so the limits of integration are simple:

$$\int_C xy\ dx + (x^2 - y^2)\ dy = \iint_D x\ dA = \int_0^1 \int_0^1 x\ dy\ dx = \frac{1}{2}.$$

[2] (a) Again we'll break the line integral into simple line segments:



Again we parameterize and integrate each piece more or less without commentary:
C_1: $\mathbf{r}(t) = \langle x, y \rangle = \langle t, 0 \rangle$ for $0 \le t \le 1$, so $dx = dt$ and $dy = 0$. Thus

$$\int_{C_1} x^3\,dx - xy^2\,dy = \int_0^1 t^3\,dt - t(0)^2\,0 = \int_0^1 t^3\,dt = \frac{1}{4}.$$

C_2: $\mathbf{r}(t) = \langle x, y \rangle = \langle 1, t \rangle$ for $0 \le t \le 1$, so $dx = 0$ and $dy = dt$. Thus

$$\int_{C_2} x^3\,dx - xy^2\,dy = \int_0^1 1^3\,0 - 1(t)^2\,dt = \int_0^1 -t^2\,dt = -\frac{1}{3}.$$

C_3: $\mathbf{r}(t) = \langle x, y \rangle = \langle 1, 1 \rangle + t\,(\langle 0, 0 \rangle - \langle 1, 1 \rangle) = \langle 1-t, 1-t \rangle$ for $0 \le t \le 1$, so $dx = -dt$ and $dy = -dt$. Thus

$$\int_{C_3} x^3\,dx - xy^2\,dy = \int_0^1 (1-t)^3 \cdot -dt - (1-t)(1-t)^2 \cdot -dt = \int_0^1 0 = 0.$$

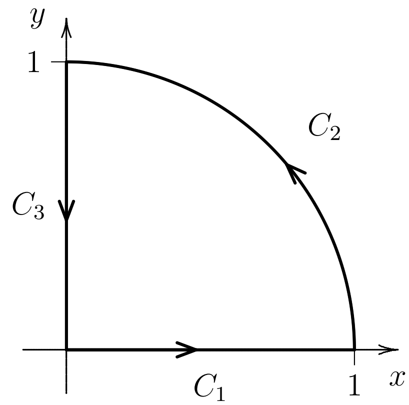
Putting this all together, we get

$$\int_C x^3\,dx - xy^2\,dy = \int_{C_1} + \int_{C_2} + \int_{C_3} = \frac{1}{4} - \frac{1}{3} + 0 = -\frac{1}{12}.$$

(b) Here $P = x^3$ and $Q = -xy^2$, so the integrand of the double integral is $Q_x - P_y = -y^2 - 0 = -y^2$. The region of integration is a triangle, so the limits of integration are reasonably straightforward:

$$\int_C x^3\,dx - xy^2\,dy = \iint_D -y^2\,dA = \int_0^1 \int_0^x -y^2\,dy\,dx = \int_0^1 -\frac{1}{3}x^3\,dx = -\frac{1}{12}.$$

3 (a) We'll break the line integral into two line segments and one arc:



Now we parameterize each part of the path and integrate, more or less without commentary:

C_1: $\mathbf{r}(t) = \langle t, 0\rangle$ $(0 \le t \le 1)$, so $dx = dt$, $dy = 0$, and $x\ dy - y\ dx = t \cdot 0 - 0\ dt = 0$. Thus $\int_{C_1} x\ dy - y\ dx = 0$.

C_2: $\mathbf{r}(t) = \langle \cos(t), \sin(t)\rangle$ $(0 \le t \le \frac{\pi}{2})$, so $\langle dx, dy\rangle = \langle -\sin(t), \cos(t)\rangle\ dt$ and thus $x\ dy - y\ dx = \big(\cos^2(t) + \sin^2(t)\big)\ dt = dt$. Thus $\int_{C_2} x\ dy - y\ dx = \int_0^{\pi/2} dt = \frac{\pi}{2}$.

C_3: $\mathbf{r}(t) = \langle 0, 1-t\rangle$ $(0 \le t \le 1)$, so $dx = 0$, $dy = -dt$, and $x\ dy - y\ dx = 0(-dt) - (1 - t)(0) = 0$. Thus $\int_{C_3} x\ dy - y\ dx = 0$.

Putting this all together, we get

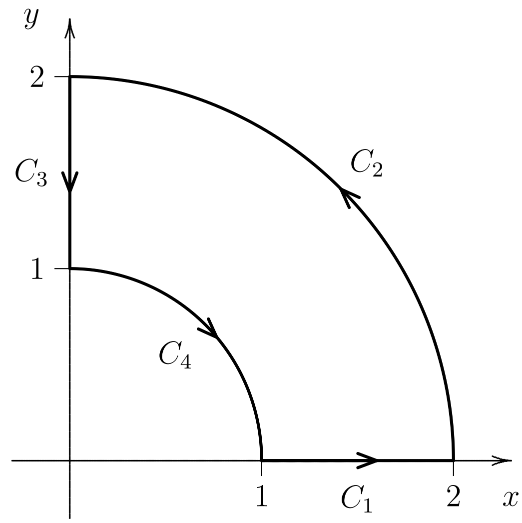
$$\int_C x\ dy - y\ dx = \int_{C_1} x\ dy - y\ dx + \int_{C_2} x\ dy - y\ dx + \int_{C_3} x\ dy - y\ dx = 0 + \frac{\pi}{2} + 0 = \frac{\pi}{2}.$$

(b) Here $P = -y$ and $Q = x$, so $Q_x - P_y = 1 - (-1) = 2$. Thus Green's theorem implies that

$$\int_C x\ dy - y\ dx = \iint_D 2\ dA = 2 \cdot \text{Area}(D) = \frac{\pi}{2},$$

as before.

4 (a) As usual, we'll break the line integral into simple line segments and arcs:



Again we parameterize and integrate each piece more or less without commentary:
C_1: $\mathbf{r}(t) = \langle x, y\rangle = \langle t, 0\rangle$ for $1 \le t \le 2$, so $dx = dt$ and $dy = 0$. Thus

$$\int_{C_1} xy^2\, dx - x^2y\, dy = \int_1^2 t(0)^2\, dt - t^2 \cdot 0\; 0 = 0.$$

C_2: $\mathbf{r}(t) = \langle x, y\rangle = \langle 2\cos(t), 2\sin(t)\rangle$ for $0 \le t \le \frac{\pi}{2}$, so $dx = -2\sin(t)\, dt$ and $dy = 2\cos(t)\, dt$. Thus

$$\begin{aligned}\int_{C_2} xy^2\, dx - x^2y\, dy &= \int_0^{\pi/2} 2\cos(t)\,(2\sin(t))^2 \cdot -2\sin(t)\, dt - (2\cos(t))^2\, 2\sin(t) \cdot 2\cos(t)\, dt \\ &= -16\int_0^{\pi/2} \left(\sin^3(t)\cos(t) + \cos^3(t)\sin(t)\right)\, dt \\ &= -16\int_0^{\pi/2} \sin(t)\cos(t)\, dt \qquad (\text{since } \sin^2(t) + \cos^2(t) = 1) \\ &= -8.\end{aligned}$$

C_3: $\mathbf{r}(t) = \langle x, y\rangle = \langle 0, 2\rangle + t\,(\langle 0, 2\rangle - \langle 0, 1\rangle) = \langle 0, 2 - t\rangle$ for $0 \le t \le 1$, so $dx = 0$ and $dy = -dt$. Thus

$$\int_{C_3} xy^2\, dx - x^2y\, dy = \int_0^1 0(1-t)^2\, 0 - 0^2(2-t) \cdot -dt = \int_0^1 0 = 0.$$

$-C_4$: (We'll parameterize $-C_4$ instead.) $\mathbf{r}(t) = \langle x, y\rangle = \langle \cos(t), \sin(t)\rangle$ for $0 \le t \le \frac{\pi}{2}$, so $dx = -\sin(t)\, dt$ and $dy = \cos(t)\, dt$. Thus

$$\begin{aligned}\int_{-C_4} xy^2\, dx - x^2y\, dy &= \int_0^{\pi/2} \cos(t)\,(\sin(t))^2 \cdot -\sin(t)\, dt - (\cos(t))^2 \sin(t) \cdot \cos(t)\, dt \\ &= -\int_0^{\pi/2} \left(\sin^3(t)\cos(t) + \cos^3(t)\sin(t)\right)\, dt \\ &= -\int_0^{\pi/2} \sin(t)\cos(t)\, dt \qquad (\text{since } \sin^2(t) + \cos^2(t) = 1) \\ &= -\frac{1}{2}.\end{aligned}$$

Putting this all together, we get

$$\int_C xy^2\,dx - x^2y\,dy = \int_{C_1} + \int_{C_2} + \int_{C_3} - \int_{-C_4} = 0 + (-8) + 0 - \left(-\frac{1}{2}\right) = -\frac{15}{2}.$$

(b) Here $P = xy^2$ and $Q = -x^2y$, so the integrand of the double integral is $Q_x - P_y = -2xy - 2xy = -4xy$. The region of integration is suited to polar coordinates, in which the limits of integration are simple:

$$\begin{aligned}
\int_C xy^2\,dx - x^2y\,dy &= \iint_D -4xy\,dA \\
&= \int_0^{\pi/2}\int_1^2 -4r^2\sin(\theta)\cos(\theta)\cdot r\,dr\,d\theta \\
&= -4\int_0^{\pi/2}\int_1^2 r^3\sin(\theta)\cos(\theta)\,dr\,d\theta \\
&= -4\int_0^{\pi/2} \frac{1}{4}r^4\bigg|_1^2 \sin(\theta)\cos(\theta)\,d\theta \\
&= -15\int_0^{\pi/2} \sin(\theta)\cos(\theta)\,d\theta \\
&= -15\cdot\frac{1}{2}\sin^2(\theta)\bigg|_0^{\pi/2} \\
&= -\frac{15}{2}.
\end{aligned}$$

5 If $\mathbf{F} = \langle P, Q\rangle$ is conservative, then we saw last time that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$. This means that Green's theorem says that

$$\int_C P\,dx + Q\,dy = \iint_D 0\,dA = 0.$$

Thus the line integral $\int_C \mathbf{F}\cdot d\mathbf{r} = 0$ for any simple closed curve C, provided the hypotheses of Green's theorem are satisfied. (In particular, P and Q must have continuous derivatives in an open set containing D, the region bounded by C. This is why Problem 7 isn't a contradiction, despite the fact that the vector field in that problem is $\mathbf{F} = \nabla\left(\arctan(\frac{y}{x})\right)$. Neither P nor Q, or even their derivatives are not continuous at the origin, which is part of the region bounded by each curve C.)

6 (a) There are lots of choices for P and Q that will result in $Q_x - P_y = 1$. For example, if $P = ay$ and $Q = bx$, then $Q_x - P_Y = b - a$, so there are many choices. Three typical ones are $(a, b) = (-1, 0)$ (the given $\langle P, Q\rangle = \langle -y, 0\rangle$ of the problem), $(a, b) = (0, 1)$ (so $\langle P, Q\rangle = \langle 0, x\rangle$) and $(a, b) = \left(\frac{1}{2}, \frac{1}{2}\right)$. These three examples give us the following line integrals that are equal to the area of D:

$$\int_C -y\,dx \qquad \text{and} \qquad \int_C x\,dy \qquad \text{and} \qquad \frac{1}{2}\int_C x\,dy - y\,dx.$$

While these are the traditional integrals, one could contrive much more complicated examples. For example, if

$$P(x, y) = y(f'(x) + 1) + g(x) + h(y)$$
$$Q(x, y) = f(x) + h'(y)x + k(y)$$

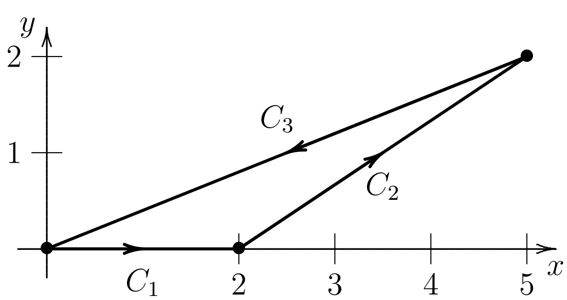
(where $f(x)$, $g(x)$, $h(y)$ and $k(y)$ are any continuously twice-differentiable functions), then

$$\frac{\partial P}{\partial y} = f'(x) + 1 + 0 + h'(y) \qquad \text{and} \qquad \frac{\partial Q}{\partial x} = f'(x) + h'(y) + 0,$$

so $P_y - Q_x = 1$. Thus

$$\oint_C \Big(y(f'(x) + 1) + g(x) + h(y)\Big)\,dx + \Big(f(x) + h'(y)x + k(y)\Big)\,dy = \text{Area}(D).$$

(b) Let's use one of the simpler integrals – say, $\int_C x\,dy$ – to find the area of this triangle. We'll label the three sides and compute the three line integrals, as usual, without much commentary:



C_1: $\mathbf{r}(t) = \langle x, y\rangle = \langle t, 0\rangle$ (for $0 \le t \le 2$), so $dx = dt$ and $dy = 0$. Thus

$$\int_{C_1} x\,dy = \int_0^2 t \cdot 0 = 0.$$

C_2: $\mathbf{r}(t) = \langle x, y\rangle = \langle 2, 0\rangle + t\langle 3, 2\rangle = \langle 2 + 3t, 2t\rangle$ (for $0 \le t \le 1$), so $dx = 3\,dt$ and $dy = 2\,dt$. Thus

$$\int_{C_2} x\,dy = \int_0^1 (2 + 3t) \cdot 2\,dt = 7.$$

C_3: $\mathbf{r}(t) = \langle x, y\rangle = \langle 5, 2\rangle + t\langle -5, -2\rangle = \langle 5 - 5t, 2 - 2t\rangle$ (for $0 \le t \le 1$), so $dx = -5\,dt$ and $dy = -2\,dt$. Thus

$$\int_{C_3} x\,dy = \int_0^1 (5 - 5t) \cdot -2\,dt = -5.$$

Putting this all together, we get

$$\text{Area}(D) = \int_C y\,dx = \int_{C_1} y\,dx + \int_{C_2} y\,dx + \int_{C_3} y\,dx = 0 + 7 - 5 = 2.$$

7 (a) Here $P = -\frac{y}{x^2+y^2}$ and $Q = \frac{x}{x^2+y^2}$, so

$$\frac{\partial P}{\partial y} = -\frac{1 \cdot (x^2+y^2) - y \cdot 2y}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

andz

$$\frac{\partial Q}{\partial x} = \frac{1 \cdot (x^2+y^2) - x \cdot 2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}.$$

Thus $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$, so

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\, dA = 0.$$

(b) We've already seen that the double integral in Green's theorem must be zero, so we must have

$$\int_{C_1} P\, dx + Q\, dy - \int_{C_2} P\, dx + Q\, dy = 0$$

or that the two integrals are equal.

This now implies that the integrals are the same for *every* such simple closed curves, even pairs that intersect. If C_1 and C_2 intersect, then choose a third curve C_3 (for example a really large or really small circle) that intersects neither curve. The above argument shows that the integrals over C_1 and C_2 both equal the integral over C_3, so all three must be the same!

(c) Let C be the unit circle, parameterized by $\mathbf{r}(t) = \langle x, y\rangle = \langle \cos(t), \sin(t)\rangle$ (for $0 \le t \le 2\pi$). Then $dx = -\sin(t)\, dt$ and $dy = \cos(t)\, dt$, so

$$\int_C \frac{x\, dy - y\, dx}{x^2+y^2} = \int_0^{2\pi} \frac{\cos^2(t)\, dt + \sin^2(t)\, dt}{1}$$

$$= \int_0^{2\pi} dt = 2\pi.$$

Thus $\oint_C \frac{x\, dy - y\, dx}{x^2+y^2} = 2\pi$ for *any* positively oriented simple closed curve that encloses the origin.

Comment: The interesting thing about this example is that $\mathbf{F}$ is a gradient field:

$$\mathbf{F} = \nabla\left(\arctan\left(y/x\right)\right) = \left\langle \frac{-y/x^2}{1+(y/x)^2}, \frac{1/x}{1+(y/x)^2}\right\rangle = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right\rangle.$$

But the domain D of $\mathbf{F}$ is the entire plane *except* for the origin. Thus D is open but not simply connected (it has a hole!), so being a gradient field is not the same thing as integrals being independent of path on D. (This is worked out in detail in the book as well; see example 13.4.5 on page 937).

Curl and Divergence

1 Define the operator ∇ (pronounced "del") by

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}.$$

Notice that the gradient ∇f (or also grad f) is just ∇ applied to f.

(a) We define the *divergence* of a vector field $\mathbf{F}$, written div $\mathbf{F}$ or $\nabla \cdot \mathbf{F}$, as the dot product of del with $\mathbf{F}$. So if $\mathbf{F} = \langle P, Q, R \rangle$, then

$$\operatorname{div}\mathbf{F} = \nabla \cdot \mathbf{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle P, Q, R \rangle = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

Notice that div $\mathbf{F}$ is a scalar.

Find div $\mathbf{F}$ for each of the following vector fields:

(i) $\mathbf{F} = \langle xy, yz, xz \rangle$

(ii) $\mathbf{F} = \langle yz, xz, xy \rangle$

(iii) $\mathbf{F} = \left\langle \frac{x}{r}, \frac{y}{r}, \frac{z}{r} \right\rangle$ where $r = \sqrt{x^2 + y^2 + z^2}$

(iv) $\mathbf{F} = \operatorname{grad} f$, where f is a function with continuous second derivatives

(b) We define the *curl* of a vector field $\mathbf{F}$, written curl $\mathbf{F}$ or $\nabla \times \mathbf{F}$, as the cross product of del with $\mathbf{F}$. So if $\mathbf{F} = \langle P, Q, R \rangle$, then

$$\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \times \langle P, Q, R \rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)\mathbf{k}.$$

Notice that curl $\mathbf{F}$ is a vector.

Find curl $\mathbf{F}$ for each of the following vector fields:

(i) $\mathbf{F} = \langle xy, yz, xz \rangle$

(ii) $\mathbf{F} = \langle yz, xz, xy \rangle$

(iii) $\mathbf{F} = \left\langle \frac{x}{r}, \frac{y}{r}, \frac{z}{r} \right\rangle$ where $r = \sqrt{x^2 + y^2 + z^2}$

(iv) $\mathbf{F} = \operatorname{grad} f$, where f is a function with continuous second derivatives

2 Check the appropriate box ("Vector", "Scalar", or "Nonsense") for each quantity.

Quantity	**Vector**	**Scalar**	**Nonsense**	**Quantity**	**Vector**	**Scalar**	**Nonsense**
$\text{curl}(\nabla f)$	☐	☐	☐	$\text{curl}(\text{curl}\, f)$	☐	☐	☐
$\nabla \cdot (\nabla \times \mathbf{F})$	☐	☐	☐	$\text{grad}(\text{div}\, f)$	☐	☐	☐
$\text{div}(\text{curl}\, f)$	☐	☐	☐	$\text{div}(\text{grad}\, \mathbf{F})$	☐	☐	☐
$\text{curl}(\text{curl}\, \mathbf{F})$	☐	☐	☐	$\nabla \cdot (\nabla f)$	☐	☐	☐
$\nabla(\nabla \cdot \mathbf{F})$	☐	☐	☐	$\text{div}(\text{div}\, \mathbf{F})$	☐	☐	☐

3 (a) Suppose we have a vector $\langle P, Q\rangle$ that we extend to a vector in space: $\mathbf{F} = \langle P(x,y), Q(x,y), 0\rangle$. Find $\text{curl}\, \mathbf{F}$.

(b) When is the vector field $\mathbf{F}$ in part (a) conservative? Use the curl in your answer.

(c) If $\mathbf{F} = \nabla f = \langle f_x, f_y, f_z\rangle$ is conservative, then is $\text{curl}\, \mathbf{F} = \mathbf{0}$? (See Problem 1(b)(iv).)

In fact, we can identify conservative vector fields with the curl:

> **Theorem:** Let $\mathbf{F}$ be a vector field defined on all of $\mathbf{R}^3$. If the component functions of $\mathbf{F}$ all have continuous derivatives and $\text{curl}\, \mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is a conservative vector field.

4 Show that, if $\mathbf{F}$ is a vector field on $\mathbf{R}^3$ with components that have continuous second-order derivatives, then $\text{div}\,\text{curl}\, \mathbf{F} = 0$. (This is less useful for us right now than $\text{curl}\,\text{grad}\, f = \mathbf{0}$, but it's not a difficult computation.)

5 We can re-write Green's theorem in vector form (we get the formula on the left, below). The formula on the right can be thought of as a version of Green's theorem that uses the normal component rather than the tangential component:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\operatorname{curl} \mathbf{F}) \cdot \mathbf{k} \, dA \qquad \text{and} \qquad \oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \iint_D \operatorname{div} \mathbf{F}(x, y) \, dA.$$

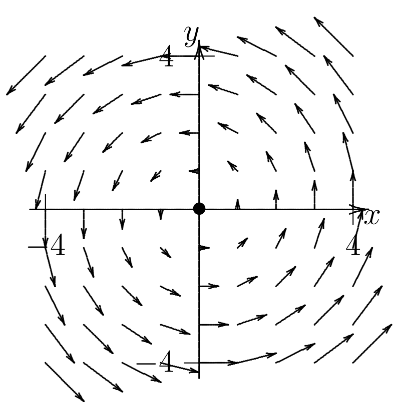
(If $\mathbf{r} = \langle x(t), y(t) \rangle$, then $\mathbf{n} = \frac{\langle y'(t), -x'(t) \rangle}{|\langle y'(t), -x'(t) \rangle|}$ is the outward unit normal.) For each of the following vector fields $\mathbf{F}$ and paths C, compute the integrals above using these vector forms of Green's theorem:

(a) $\mathbf{F} = \langle -y, x \rangle$ and C is the unit circle, positively oriented.

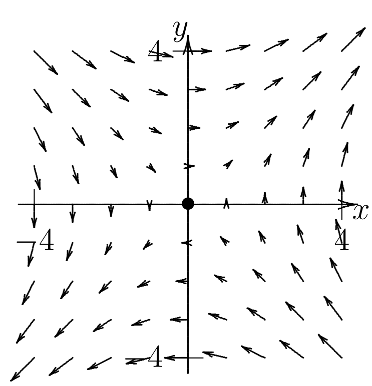
(b) $\mathbf{F} = \langle 2x, 2y \rangle$ and C is the unit square (with corners at $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$), positively oriented.

(While we'll never really look at the normal version of Green's theorem again, one of our big integral theorems next week can be thought of as a higher-dimensional version of it.)

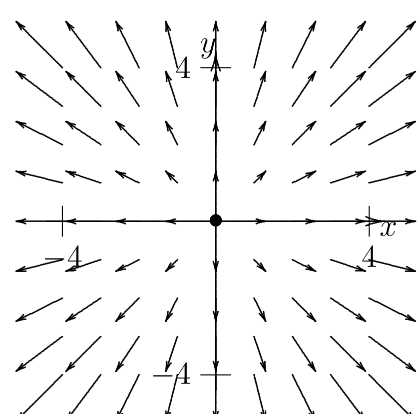
6 Here are sketches of a few vector fields $\mathbf{F} = \langle P(x,y), Q(x,y), 0 \rangle$ (they're drawn in the plane, but they're defined in all of space). Can you tell which one has zero curl? Zero divergence?



$\mathbf{F} = \langle -y, x, 0 \rangle$



$\mathbf{F} = \langle y, x, 0 \rangle$



$\mathbf{F} = \langle 2x, 2y, 0 \rangle$

Curl and Divergence – Answers and Solutions

1 (a) (i) $\operatorname{div}\mathbf{F} = x + y + z$

(ii) $\operatorname{div}\mathbf{F} = 0$

(iii) $\operatorname{div}\mathbf{F} = \frac{y^2+z^2}{r^3} + \frac{x^2+z^2}{r^3} + \frac{x^2+y^2}{r^3} = \frac{2(x^2+y^2+z^2)}{r^3} = \frac{2}{r}$

(iv) $\operatorname{div}\mathbf{F} = \operatorname{div}(\operatorname{grad} f) = f_{xx} + f_{yy} + f_{zz}$. This is the Laplace operator applied to f.

(b) (i) $\operatorname{curl}\mathbf{F} = \langle -y, -z, -x\rangle$

(ii) $\operatorname{curl}\mathbf{F} = \mathbf{0}$

(iii) $\operatorname{curl}\mathbf{F} = \mathbf{0}$

(iv) $\operatorname{curl}\mathbf{F} = \operatorname{curl}(\operatorname{grad} f) = \mathbf{0}$.

Two quick comments:

- Notice that the vector from part (ii) is actually an example of this, since $\mathbf{F} = \operatorname{grad}(xyz) = \langle yz, xz, xy\rangle$. The vector for part (iii) is $\mathbf{F} = \operatorname{grad}(\sqrt{x^2+y^2+z^2})$, so it is another example as well.
- We'll see later on in this worksheet that $\operatorname{curl}\mathbf{F} = \mathbf{0}$ precisely when $\mathbf{F} = \operatorname{grad} f$ (when we make some continuity assumptions about the derivatives of the components of $\mathbf{F}$.

2 Here are some answers, although half are blank as they match questions from the homework:

Quantity	**Vector**	**Scalar**	**Nonsense**	**Quantity**	**Vector**	**Scalar**	**Nonsense**
$\operatorname{curl}(\nabla f)$	☒	☐	☐	$\operatorname{curl}(\operatorname{curl} f)$	☐	☐	☒
$\nabla\cdot(\nabla\times\mathbf{F})$	☐	☒	☐	$\operatorname{grad}(\operatorname{div} f)$	☐	☐	☐
$\operatorname{div}(\operatorname{curl} f)$	☐	☐	☒	$\operatorname{div}(\operatorname{grad}\mathbf{F})$	☐	☐	☒
$\operatorname{curl}(\operatorname{curl}\mathbf{F})$	☐	☐	☐	$\nabla\cdot(\nabla f)$	☐	☐	☐
$\nabla(\nabla\cdot\mathbf{F})$	☐	☐	☐	$\operatorname{div}(\operatorname{div}\mathbf{F})$	☐	☐	☐

3 (a) $\operatorname{curl}\mathbf{F} = \left\langle 0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right\rangle = \langle 0, 0, Q_x - P_y\rangle$

(b) A planar vector field $\mathbf{F} = \langle P, Q\rangle$ is conservative when $Q_x - P_y = 0$. This means there's a function $f(x,y)$ with $P = f_x$ and $Q = f_y$. We also get $f_z = 0$ since f is a function of only x and y, so $\mathbf{F} = \langle P(x,y), Q(x,y), 0\rangle$ is conservative under this same condition. From part (a), this means that $\mathbf{F}$ is conservative when $\operatorname{curl}\mathbf{F} = \mathbf{0}$.

(c) Yep, we've already seen that $\operatorname{curl}\operatorname{grad} f = \mathbf{0}$.

4 Here's a quick detailed computation:

$$\begin{aligned}
\operatorname{div}\operatorname{curl}\mathbf{F} &= \operatorname{div}\left(\nabla\times\langle P, Q, R\rangle\right) \\
&= \nabla\cdot\left[\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}\right] \\
&= \frac{\partial}{\partial x}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \\
&= \left(\frac{\partial^2 R}{\partial x\,\partial y} - \frac{\partial^2 Q}{\partial x\,\partial z}\right) + \left(\frac{\partial^2 P}{\partial y\,\partial z} - \frac{\partial^2 R}{\partial y\,\partial x}\right) + \left(\frac{\partial^2 Q}{\partial z\,\partial x} - \frac{\partial^2 P}{\partial z\,\partial y}\right) \\
&= (R_{yx} - Q_{zx}) + (P_{zy} - R_{xy}) + (Q_{xz} - P_{yz}).
\end{aligned}$$

5 (a) We do this both ways, of course. First let's parameterize C by $\mathbf{r}(t) = \langle x, y\rangle = \langle \cos(t), \sin(t)\rangle$, so $dx = -\sin(t)\ dt$ and $dy = \cos(t)\ dt$. Thus

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \langle -\sin(t), \cos(t)\rangle \cdot \langle -\sin(t), \cos(t)\rangle\ dt = \int_0^{2\pi} 1\ dt = 2\pi.$$

The double integral is also straightforward: writing $\mathbf{F} = \langle -y, x, 0\rangle$ as a vector field in space, we get $\operatorname{curl}\mathbf{F} = \langle 0,0,2\rangle$. Thus, since $\mathbf{k} = \langle 0,0,1\rangle$, we get

$$\iint_D (\operatorname{curl}\mathbf{F})\cdot\mathbf{k}\ dA = \iint_D 2\ dA = 2\cdot\text{Area}(D) = 2\pi.$$

The normal component version requires us to have a unit outward-pointing normal $\mathbf{n}$. For the unit circle, the unit normal at the point (x, y) is simply $\mathbf{n} = \langle x, y\rangle$. (One way to see this is to parameterize the circle as $\mathbf{r}(t) = \langle x(t), y(t)\rangle = \langle \cos(t), \sin(t)\rangle$, so by our formula $\mathbf{n}$ is the unit vector in the direction $\langle y'(t), -x'(t)\rangle = \langle \cos(t), \sin(t)\rangle$. But this is a unit vector already, and in fact it is the vector $\langle x, y\rangle$.) Thus $\mathbf{F}\cdot\mathbf{n} = \langle -y, x\rangle \cdot \langle x, y\rangle = 0$, so the line integral is zero. Note also that $\operatorname{div}\mathbf{F} = \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x) = 0$, so the double integral is also zero.

(b) Again we do this both ways. We'll parameterize C in four parts (all with $0 \le t \le 1$):

$$C_1: \ \mathbf{r}(t) = \langle x, y\rangle = \langle t, 0\rangle \qquad C_2: \ \mathbf{r}(t) = \langle x, y\rangle = \langle 1, t\rangle$$
$$C_3: \ \mathbf{r}(t) = \langle x, y\rangle = \langle 1-t, 1\rangle \qquad C_4: \ \mathbf{r}(t) = \langle x, y\rangle = \langle 0, 1-t\rangle$$

From this we get four integrals:

$$\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_{C_1} \langle 2x, 2y\rangle\cdot\langle dx, dy\rangle = \int_0^1 \langle 2t, 0\rangle\cdot\langle dt, 0\rangle = \int_0^1 2t\ dt = 1$$
$$\int_{C_2} \mathbf{F}\cdot d\mathbf{r} = \int_{C_2} \langle 2x, 2y\rangle\cdot\langle dx, dy\rangle = \int_0^1 \langle 2, 2t\rangle\cdot\langle 0, dt\rangle = \int_0^1 2t\ dt = 1$$
$$\int_{C_3} \mathbf{F}\cdot d\mathbf{r} = \int_{C_3} \langle 2x, 2y\rangle\cdot\langle dx, dy\rangle = \int_0^1 \langle 2(1-t), 2\rangle\cdot\langle -dt, 0\rangle = \int_0^1 2(t-1)\ dt = -1$$
$$\int_{C_4} \mathbf{F}\cdot d\mathbf{r} = \int_{C_4} \langle 2x, 2y\rangle\cdot\langle dx, dy\rangle = \int_0^1 \langle 0, 2(1-t)\rangle\cdot\langle 0, -dt\rangle = \int_0^1 2(t-1)\ dt = -1.$$

Thus

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4} = 1 + 1 - 1 - 1 = 0.$$

The double integral is considerably easier: writing $\mathbf{F} = \langle 2x, 2y, 0\rangle$ as a vector field in space, we compute $\operatorname{curl}\mathbf{F} = \mathbf{0}$, so the integrand in the double integral is zero. Thus the double integral is zero as well.

The normal component version requires us to have a unit outward-pointing normal $\mathbf{n}$. These are simple to find for each component (using the above parameterization):

$$C_1: \ \mathbf{n} = -\mathbf{j} = \langle 0, -1\rangle \qquad C_2: \ \mathbf{n} = \mathbf{i} = \langle 1, 0\rangle,$$
$$C_3: \ \mathbf{n} = \mathbf{j} = \langle 0, 1\rangle, \qquad C_4: \ \mathbf{n} = -\mathbf{i} = \langle -1, 0\rangle.$$

Thus the integrals are all (note that $ds = dt$ in each case)

$$\int_{C_1} \mathbf{F}\cdot\mathbf{n}\,ds = \int_0^1 \langle 2t, 0\rangle \cdot \langle 0,-1\rangle\,dt = 0$$

$$\int_{C_2} \mathbf{F}\cdot\mathbf{n}\,ds = \int_0^1 \langle 2, 2t\rangle \cdot \langle 1,0\rangle\,dt = \int_0^1 2\,dt = 2$$

$$\int_{C_3} \mathbf{F}\cdot\mathbf{n}\,ds = \int_0^1 \langle 2(1-t), 2\rangle \cdot \langle 0,1\rangle\,dt = \int_0^1 2\,dt = 2$$

$$\int_{C_4} \mathbf{F}\cdot\mathbf{n}\,ds = \int_0^1 \langle 0, 2(1-t)\rangle \cdot \langle -1,0\rangle = 0.$$

Thus

$$\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4} = 0+2+2+0 = 4.$$

The double integral is again much simpler: we compute $\operatorname{div}\mathbf{F} = 2+2 = 4$, so the double integral is just 4 times the area of the square. Thus the double integral has value 4 as well.

6 The idea here is that we can do this two ways: first, we can compute the curl and divergence of the given vector fields:

(a) $\operatorname{div}\mathbf{F} = 0$
$\operatorname{curl}\mathbf{F} = \langle 0,0,2\rangle$

(b) $\operatorname{div}\mathbf{F} = 0$
$\operatorname{curl}\mathbf{F} = \mathbf{0}$

(c) $\operatorname{div}\mathbf{F} = 4$
$\operatorname{curl}\mathbf{F} = \mathbf{0}$

Thus we see that the first vector field is the only one with a non-zero curl, and that the last vector field is similarly the only one with a non-zero divergence.

Here's a way to see this from the graph. We'll use the two Green's theorems from the previous problem:

$$\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \iint_D (\operatorname{curl}\mathbf{F})\cdot\mathbf{k}\,dA$$

$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \iint_D \operatorname{div}\mathbf{F}\,dA.$$

(Here we've written the first one slightly differently, but it's the same. Since $d\mathbf{r} = \mathbf{r}'(t)\,dt$ and $\mathbf{T} = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$ and $ds = |\mathbf{r}'(t)|\,dt$, we get $d\mathbf{r} = \mathbf{T}\,ds$.)

Let's start with the curl and the first of Green's theorems. Let's integrate around a small circle C centered at the origin. We'll choose it so small that $\operatorname{curl}\mathbf{F}$ is essentially constant on the tiny little disk D bounded by C. By this argument, the double integral is roughly

$$\iint_D (\operatorname{curl}\mathbf{F})\cdot\mathbf{k}\,dA \approx \iint_D (\operatorname{curl}\mathbf{F}(0,0,0))\cdot\mathbf{k}\,dA = \Big((\operatorname{curl}\mathbf{F}(0,0,0))\cdot\mathbf{k}\Big)\cdot\operatorname{Area}(D).$$

On the other hand, the line integral is the integral of $\mathbf{F}\cdot\mathbf{T}$. This is the tangential component of $\mathbf{F}$, and in particular

$\mathbf{F}\cdot\mathbf{T} > 0$ means $\mathbf{F}$ and $\mathbf{T}$ are mostly in the same direction,

$\mathbf{F}\cdot\mathbf{T} < 0$ means $\mathbf{F}$ and $\mathbf{T}$ are mostly in opposing directions, and

$\mathbf{F}\cdot\mathbf{T} = 0$ means $\mathbf{F}$ and $\mathbf{T}$ are perpendicular.

In the case of the first sketch, it's clear that $\mathbf{F}$ is mostly parallel to the tangent vector to the small circle C, so $\mathbf{F} \cdot \mathbf{T} > 0$. Thus $\oint_C \mathbf{F} \cdot \mathbf{T}\, ds > 0$, so by Green's theorem,

$$(\operatorname{curl} \mathbf{F}(0,0,0)) \cdot \mathbf{k} > 0.$$

The moral: the curl of $\mathbf{F}$ is non-zero means that there is some kind of rotation in the vector field. We'll see much more of this later.

We could do something similar with the divergence. Let's cut straight to the chase: the (outward-pointing) normal $\mathbf{n}$ produces a positive divergence at the origin. This gives us what we call a *source* (when $\operatorname{div} \mathbf{F} > 0$) at the origin; if the vector fields were pointing in we'd get a sink (and $\operatorname{div} \mathbf{F} < 0$).

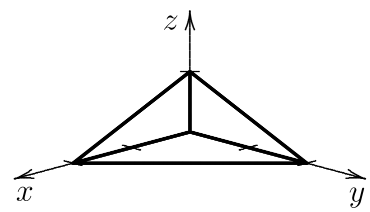
Flux Integrals

A surface integral is

$$\iint_S f(x,y,z)\,dS = \iint_D f(\mathbf{r}(u,v))\ |\mathbf{r}_u \times \mathbf{r}_v|\ du\,dv,$$

where f is a function defined on the parametric surface $\mathbf{r}(u,v)$.

1 Evaluate the surface integral



$$\iint_S (1+z)\ dS,$$

where S is that part of the plane $x+y+2z=2$ in the first octant.

Suppose $\mathbf{F}$ is a continuous vector field on an oriented surface S with unit normal vector $\mathbf{n}$. The surface integral of $\mathbf{F}$ over S is

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_D \mathbf{F}\cdot(\mathbf{r}_u \times \mathbf{r}_v)\ du\,dv$$

for a parametrically defined surface.

2 Evaluate the surface integral $\iint_S \mathbf{F}\cdot d\mathbf{S}$, where $\mathbf{F} = y\mathbf{i} - x\mathbf{j} + z\mathbf{k}$ and S is the part of the sphere $x^2+y^2+z^2=4$ in the first octant with inward orientation.

3 Evaluate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = \langle x, y, 2z \rangle$ and S is the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the unit square $[0,1] \times [0,1]$ with the *downward* orientation.

4 Evaluate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + (2x + 2y)\mathbf{k}$ and S is the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the unit disk (centered at the origin) with upward orientation.

5 Evaluate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = \langle -z, x, y \rangle$ and S is the full unit hemisphere (including the base!) on and above the xy-plane (so $x^2 + y^2 + z^2 = 1$ plus a disk) with the outward orientation.

Flux Integrals – Answers and Solutions

1 Here we use the parameterization $\mathbf{r}(x,y) = \langle x, y, 1 - \frac{1}{2}x - \frac{1}{2}y\rangle$. From this we find that

$$\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \end{vmatrix} = \left\langle \tfrac{1}{2},\ \tfrac{1}{2},\ 1 \right\rangle.$$

Therefore

$$\iint_S (1+z)\ dS = \iint_D \left[1 + \left(1 - \frac{1}{2}x - \frac{1}{2}y\right)\right]\ dA.$$

The region D in our parameter space (the xy-plane) need to cover our surface is the triangle

$$D = \{(x,y)\ :\ x \geq 0,\ y \geq 0,\ x + y \leq 2\},$$

so we can write our limits as follows:

$$\begin{aligned} \iint_S (1+z)\ dS &= \int_0^2 \int_0^{2-x} \left(2 - \frac{1}{2}x - \frac{1}{2}y\right)\ dy\ dx \\ &= \int_0^2 \left(\frac{1}{4}x^2 - 2x + 3\right)\ dx \\ &= \frac{1}{12} \cdot 2^3 - 2^2 + 3 \cdot 2 = \frac{8}{3}. \end{aligned}$$

2 This is based on Problem 23 from Section 13.6 of the textbook.

One common parameterization of the sphere of radius a is simply using spherical coordinates (with $\rho = a$):

$$\mathbf{r}(\phi,\theta) = \big\langle a\sin(\phi)\cos(\theta), a\sin(\phi)\cos(\theta), a\cos(\phi)\big\rangle.$$

One could then compute $\mathbf{r}_\phi \times \mathbf{r}_\theta$, but it is easier to just remember that we've done this before and the answer is:

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = a^2\sin(\phi)\big\langle \sin(\phi)\cos(\theta), \sin(\phi)\cos(\theta), \cos(\phi)\big\rangle.$$

(See, for example, page 869 of the text.) The coefficient in front is simply the coefficient $\rho^2\sin(\phi)$ from the spherical volume element (with $\rho = a$) while the vector is simply the unit vector in the direction $\langle x, y, z\rangle$ (the radial vector, which is perpendicular to the tangent plane to the sphere).

Notice that the orientation is specified to be the *inward* normal, so we actually want $\mathbf{r}_\theta \times \mathbf{r}_\phi = -\mathbf{r}_\phi \times \mathbf{r}_\theta$.

In any case, with this we proceed. In spherical coordinates (with $\rho = a = 2$ in this case), our vector field is

$$\mathbf{F} = \langle y, -x, z\rangle = \langle 2\sin(\phi)\sin(\theta), -2\sin(\phi)\cos(\theta), 2\cos(\phi)\rangle.$$

Thus

$$\mathbf{F} \cdot (\mathbf{r}_\theta \times \mathbf{r}_\phi)$$
$$= -4\sin(\phi)\big\langle 2\sin(\phi)\sin(\theta), -2\sin(\phi)\cos(\theta), 2\cos(\phi)\big\rangle \cdot \big\langle \sin(\phi)\cos(\theta), \sin(\phi)\cos(\theta), \cos(\phi)\big\rangle$$
$$= -8\sin(\phi)\cos^2(\phi).$$

We integrate this over the domain $\{(\phi, \theta) \ : \ 0 \le \phi \le \pi/2, \ 0 \le \theta \le \pi/2\}$, so we have

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \int_0^{\pi/2} \int_0^{\pi/2} -8\sin(\phi)\cos^2(\phi)\ d\theta\ d\phi$$
$$= -4\pi \int_0^{\pi/2} \sin(\phi)\cos^2(\phi)\ d\phi$$
$$= -4\pi \left[-\frac{1}{3}\cos^3(\phi) \right]_0^{\pi/2} = -\frac{4\pi}{3}.$$

Another approach is to use the parameterization by x and y

$$\mathbf{r}(x, y) = \big\langle x, y, \sqrt{4 - x^2 - y^2} \big\rangle$$

then integrating over the quarter circle in the xy-plane. Let's see how this goes:

$$\mathbf{r}_y \times \mathbf{r}_x = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & -\frac{y}{\sqrt{4-x^2-y^2}} \\ 1 & 0 & -\frac{x}{\sqrt{4-x^2-y^2}} \end{vmatrix} = \left\langle -\frac{x}{\sqrt{4 - x^2 - y^2}}, \ -\frac{y}{\sqrt{4 - x^2 - y^2}}, \ -1 \right\rangle.$$

(Notice we've used $\mathbf{r}_y \times \mathbf{r}_x$ to get the inward-pointing normal.) Thus

$$\mathbf{F} \cdot (\mathbf{r}_y \times \mathbf{r}_x) = \langle y, -x, z \rangle \cdot \left\langle -\frac{x}{\sqrt{4 - x^2 - y^2}}, \ -\frac{y}{\sqrt{4 - x^2 - y^2}}, \ -1 \right\rangle$$
$$= \left\langle y, -x, \sqrt{4 - x^2 - y^2} \right\rangle \cdot \left\langle -\frac{x}{\sqrt{4 - x^2 - y^2}}, \ -\frac{y}{\sqrt{4 - x^2 - y^2}}, \ -1 \right\rangle$$
$$= -\sqrt{4 - x^2 - y^2}$$

and so

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \int_0^2 \int_0^{\sqrt{4-x^2}} -\sqrt{4 - x^2 - y^2}\ dy\ dx$$
$$= -\int_0^2 \left[\frac{y}{2}\sqrt{4 - x^2 - y^2} + \frac{4 - x^2}{2}\sin^{-1}\left(\frac{y}{\sqrt{4 - x^2}}\right) \right]_0^{\sqrt{4-x^2}} dx$$
$$= -\frac{\pi}{4}\int_0^2 \left(4 - x^2\right)\ dx = -\frac{\pi}{4} \cdot \frac{16}{3} = -\frac{4\pi}{3},$$

as before.

3 A simple parameterization of this surface is $\mathbf{r}(x, y) = \langle x, y, 4 - x^2 - y^2 \rangle$. Thus

$$\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -2x \\ 0 & 1 & -2y \end{vmatrix} = \langle 2x, 2y, 1 \rangle .$$

(Note that this is *not* properly oriented. We're told to use the "downward orientation" but here the $\mathbf{k}$ component is positive. Thus we should be using $\mathbf{r}_y \times \mathbf{r}_x = -\mathbf{r}_x \times \mathbf{r}_y = \langle -2x, -2y, -1 \rangle$.) Thus

$$\begin{aligned} \iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_D \langle x, y, 2z \rangle \cdot \langle -2x, -2y, -1 \rangle \; dx \; dy \\ &= \iint_D \left(-2x^2 - 2y^2 - 2z \right) \; dx \; dy. \end{aligned}$$

Now we notice that (according to our parameterization) $z = 4 - x^2 - y^2$. The region D in our parameter space is a unit square, so we get

$$\begin{aligned} \iint_S \mathbf{F} \cdot d\mathbf{S} &= \int_0^1 \int_0^1 \left[-2x^2 - 2y^2 - 2(4 - x^2 - y^2) \right) \; dx \; dy \\ &= \int_0^1 \int_0^1 (-8) \; dx \; dy = -8 \text{ Area}(D) = -8. \end{aligned}$$

4 One way to do this is to use the parameterization $\mathbf{r}(x,y) = \langle x, y, 4-x^2-y^2\rangle$, so $\mathbf{r}_x = \langle 1, 0, -2x\rangle$, $\mathbf{r}_y = \langle 0, 1, -2y\rangle$, and so

$$\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -2x \\ 0 & 1 & -2y \end{vmatrix} = \langle 2x, 2y, 1\rangle.$$

Thus we can write

$$\mathbf{F} \cdot (\mathbf{r}_x \times \mathbf{r}_y) = \langle x, y, 2x+2y\rangle \cdot \langle 2x, 2y, 1\rangle = 2x^2 + 2y^2 + 2x + 2y.$$

Hence our flux is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \left(2x^2 + 2y^2 + 2x + 2y\right)\, dx\, dy,$$

where D is the unit disk. Thus we make the change to polar coordinates, where $2x^2+2y^2+2x+2y = 2r^2 + 2r\left(\cos(\theta) + \sin(\theta)\right)$ and $dx\,dy = r\,dr\,d\theta$. We get

$$\begin{aligned}
\iint_S \mathbf{F} \cdot d\mathbf{S} &= \int_0^{2\pi} \int_0^1 \Big[2r^2 + 2r\left(\cos(\theta) + \sin(\theta)\right)\Big] r\, dr\, d\theta \\
&= \int_0^{2\pi} \int_0^1 \Big[2r^3 + 2r^2\left(\cos(\theta) + \sin(\theta)\right)\Big]\, dr\, d\theta \\
&= \int_0^{2\pi} \left[\frac{1}{2} + \frac{2}{3}\left(\cos(\theta) + \sin(\theta)\right)\right] d\theta \\
&= \left[\frac{1}{2}\theta + \frac{2}{3}\left(\sin(\theta) - \cos(\theta)\right)\right]_0^{2\pi} = \pi.
\end{aligned}$$

Another approach is to use polar coordinates to parameterize the surface from the very beginning. That is, we could use the parameterization $\mathbf{r}(r,\theta) = \langle r\cos(\theta), r\sin(\theta), 4-r^2\rangle$, in which case $\mathbf{r}_r = \langle \cos(\theta), \sin(\theta), -2r\rangle$ and $\mathbf{r}_\theta = \langle -r\sin(\theta), r\cos(\theta), 0\rangle$, so

$$\mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos(\theta) & \sin(\theta) & -2r \\ -r\sin(\theta) & r\cos(\theta) & 0 \end{vmatrix} = \langle 2r^2\cos(\theta), 2r^2\sin(\theta), r\rangle.$$

In these coordinates our vector field is $\mathbf{F} = \langle r\cos(\theta), r\sin(\theta), 2r(\cos(\theta)+\sin(\theta))\rangle$, and therefore our flux is

$$\begin{aligned}
\iint_S \mathbf{F} \cdot d\mathbf{S} &= \int_0^{2\pi} \int_0^1 \langle r\cos(\theta), r\sin(\theta), 2r(\cos(\theta)+\sin(\theta))\rangle \cdot \langle 2r^2\cos(\theta), 2r^2\sin(\theta), r\rangle\, dr\, d\theta \\
&= \int_0^{2\pi} \int_0^1 \Big[2r^3 + 2r^2\left(\cos(\theta) + \sin(\theta)\right)\Big]\, dr\, d\theta,
\end{aligned}$$

which is identical to an integral computed above.

5 Here we need to compute two integrals:

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_{S_1} \mathbf{F}\cdot d\mathbf{S} + \iint_{S_2} \mathbf{F}\cdot d\mathbf{S},$$

where S_1 is the hemisphere (with outward-pointing normal) and S_2 is the unit disk in the xy-plane (with downward-pointing normal).

The integral over S_1 is very similar to the previous problem. We'll use the first parameterization of Problem 3, namely

$$\mathbf{r}(\phi,\theta) = \big\langle \sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi)\big\rangle.$$

This gives us an outward-pointing normal

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = \sin(\phi)\big\langle \sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi)\big\rangle,$$

so

$$\begin{aligned}
&\mathbf{F}\cdot(\mathbf{r}_\phi \times \mathbf{r}_\theta) \\
&= \big\langle -\cos(\phi), \sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta)\big\rangle \cdot \big\langle \sin^2(\phi)\cos(\theta), \sin^2(\phi)\sin(\theta), \sin(\phi)\cos(\phi)\big\rangle \\
&= -\sin^2(\phi)\cos(\phi)\cos(\theta) + \sin^3(\phi)\sin(\theta)\cos(\theta) + \sin^2(\phi)\cos(\phi)\sin(\theta).
\end{aligned}$$

Thus

$$\begin{aligned}
&\iint_{S_1} \mathbf{F}\cdot d\mathbf{S} \\
&= \int_0^{\pi/2}\int_0^{2\pi} \big(-\sin^2(\phi)\cos(\phi)\cos(\theta) + \sin^3(\phi)\sin(\theta)\cos(\theta) + \sin^2(\phi)\cos(\phi)\sin(\theta)\big)\, d\theta\, d\phi \\
&= \int_0^{\pi/2} \left(-\sin^2(\phi)\cos(\phi)\sin(\theta) - \sin^3(\phi)\cdot\frac{1}{2}\sin^2(\theta) - \sin^2(\phi)\cos(\phi)\cos(\theta)\right)\Bigg|_{\theta=0}^{2\pi} d\phi \\
&= 0.
\end{aligned}$$

The second integral is even simpler. Here S_2 is the unit disk in the xy-plane, with the unit normal $\mathbf{n} = -\mathbf{k}$ (straight down). Thus

$$\iint_{S_2} \mathbf{F}\cdot d\mathbf{S} = \iint_D \big\langle -z, x, y\big\rangle \cdot \big\langle 0,0,-1\big\rangle\, dA = \iint_D -y\, dA,$$

where D is the unit disk in the xy-plane. We use polar coordinates to compute this integral:

$$\begin{aligned}
\iint_{S_2} \mathbf{F}\cdot d\mathbf{S} = \iint_D -y\, dA &= \int_0^1\int_0^{2\pi} -r\sin(\theta)\cdot r\, d\theta\, dr \\
&= \int_0^1 r^2\cos(\theta)\Big|_0^{2\pi} dr = 0.
\end{aligned}$$

Thus the full integral is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} \mathbf{F} \cdot d\mathbf{S} + \iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = 0 + 0 = 0.$$

We'll see a simpler way of computing this integral on Wednesday when we learn about the Divergence Theorem.

Stokes' Theorem

Cast of Players:

S – an oriented, piecewise-smooth surface

C – a simple, closed, piecewise-smooth curve that bounds S

$\mathbf{F}$ – a vector field whose components have continuous derivatives in an open region of $\mathbf{R}^3$ containing S

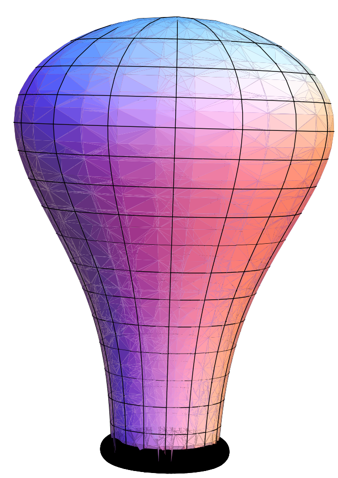
Stokes' Theorem:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$

1 Suppose C is the curve obtained by intersecting the plane $z = x$ and the cylinder $x^2 + y^2 = 1$, oriented counter-clockwise when viewed from above. Let S be the inside of this ellipse, oriented with the upward-pointing normal. If $\mathbf{F} = x\mathbf{i} + z\mathbf{j} + 2y\mathbf{k}$, verify Stokes' theorem by computing both $\oint_C \mathbf{F} \cdot d\mathbf{r}$ and $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$.

2 Suppose S is that part of the plane $x + y + z = 1$ in the first octant, oriented with the upward-pointing normal, and let C be its boundary, oriented counter-clockwise when viewed from above. If $\mathbf{F} = \langle x^2 - y^2, y^2 - z^2, z^2 - x^2 \rangle$, verify Stokes' theorem by computing both $\oint_C \mathbf{F} \cdot d\mathbf{r}$ and $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$.

3 Suppose S is a "light-bulb-shaped region" as follows. Imagine a light-bulb cut off at the base so that its boundary is the unit circle $x^2 + y^2 = 1$, oriented with the outward-pointing normal. (You can use either an old-fashioned light-bulb or a compact fluorescent if you're feeling green.) Suppose $\mathbf{F} = \left\langle e^{z^2-2z}x, \sin(xyz) + y + 1, e^{z^2}\sin(z^2) \right\rangle$. Compute the flux integral $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$ using Stokes' theorem.



4 Suppose $\mathbf{F} = \langle -y, x, z \rangle$ and S is the part of the sphere $x^2 + y^2 + z^2 = 25$ below the plane $z = 4$, oriented with the outward-pointing normal (so that the normal at $(5, 0, 0)$ is $\mathbf{i}$). Compute the flux integral $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$ using Stokes' theorem.

5 Suppose S_1 and S_2 are two oriented surfaces that share C as boundary. What can you say about $\iint_{S_1} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$ and $\iint_{S_2} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$?

6 Suppose S_1 and S_2 are two oriented surfaces that share C as boundary. Is it true that $\iint_{S_1} \mathbf{G} \cdot d\mathbf{S} = \iint_{S_2} \mathbf{G} \cdot d\mathbf{S}$ for any vector field $\mathbf{G}$? That is, can you *always* choose the easiest surface to work with when computing flux integrals over a surface with boundary?

7 Suppose S is a closed surface (that is, a surface without a boundary). Must $\iint_S \mathbf{F} \cdot d\mathbf{S} = 0$?

Stokes' Theorem – Answers and Solutions

1 There are two integrals to compute here, so we do them both.

The line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ The ellipse is a graph (using $z = x$) over the unit circle in the xy-plane. Thus we can parameterize it as $\mathbf{r}(t) = \langle \cos(t), \sin(t), \cos(t) \rangle$ for $0 \leq t \leq 2\pi$. Since $\mathbf{F} = \langle x, z, 2y \rangle$, we get

$$\begin{aligned} \mathbf{F}(\mathbf{r}(t)) &= \langle \cos(t), \cos(t), 2\sin(t) \rangle \\ d\mathbf{r} &= \langle -\sin(t), \cos(t), -\sin(t) \rangle \; dt \end{aligned}$$

and so

$$\begin{aligned} \mathbf{F}(\mathbf{r}(t)) \cdot d\mathbf{r} &= \left(-\sin(t)\cos(t) + \cos^2(t) - 2\sin^2(t)\right) \; dt \\ &= \left(-\sin(t)\cos(t) + 1 - 3\sin^2(t)\right) \; dt. \end{aligned}$$

Thus

$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \left(-\sin(t)\cos(t) + 1 - 3\sin^2(t)\right) \; dt \\ &= \left(-\frac{1}{2}\sin^2(t) + t - \frac{3}{2}t + \frac{3}{4}\sin(2t)\right)\Bigg|_0^{2\pi} \\ &= -\pi. \end{aligned}$$

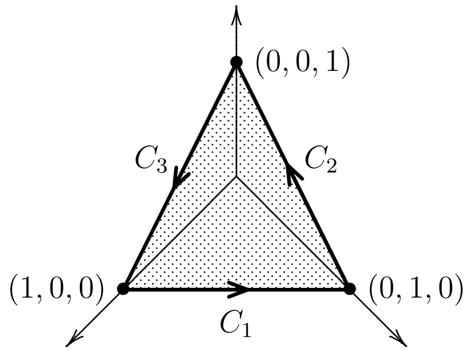
The flux integral $\iint\limits_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$ Again the elliptical disk is a graph (using $z = x$) over the unit disk in the xy-plane. Thus we can parameterize it as $\mathbf{r}(x, y) = \langle x, y, x \rangle$ for $x^2 + y^2 \leq 1$. Since $\mathbf{F} = \langle x, z, 2y \rangle$, we get $\operatorname{curl} \mathbf{F} = \langle 1, 0, 0 \rangle$ and $\mathbf{r}_x \times \mathbf{r}_y = \langle -1, 0, 1 \rangle$. Thus

$$\begin{aligned} \iint_C \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} &= \iint\limits_{\text{unit disk}} \langle 1, 0, 0 \rangle \cdot \langle -1, 0, 1 \rangle \; dx \; dy \\ &= -\iint\limits_{\text{unit disk}} 1 \; dx \; dy \\ &= -\pi, \end{aligned}$$

since the last integral is simply the area $\pi(1)^2 = \pi$ of the unit disk.

Note that the two integrals agree. Another victory for Stokes' theorem!

2 There are two integrals to compute here, so we do them both. Here's a picture of the surface and curve, so we're all on the same page:



The region S is the dotted triangle (with the upward normal coming straight toward the viewer) and the curve C is the union $C_1 \cup C_2 \cup C_3$.

The line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ This integral will really be the sum of three separate integrals, over each of C_1, C_2, and C_3. We begin with C_1. A simple parameterization of this line segment is

$$\begin{aligned}\mathbf{r}(t) &= \text{starting point} + t\,(\text{ ending point } - \text{ starting point })\\ &= \langle 1,0,0\rangle + t\,(\langle 0,1,0\rangle - \langle 0,1,0\rangle)\\ &= \langle 1-t,t,0\rangle.\end{aligned}$$

Thus $d\mathbf{r} = \langle -1,1,0\rangle\ dt$. In terms of this parameterization, the vector field $\mathbf{F} = \langle x^2-y^2, y^2-z^2, z^2-x^2\rangle$ becomes

$$\begin{aligned}\mathbf{F}(\mathbf{r}(t)) &= \left\langle (1-t)^2-t^2, t^2-0^2, 0^2-(1-t)^2\right\rangle\\ &= \left\langle 1-2t, t^2, -(1-t)^2\right\rangle\end{aligned}$$

Then

$$\begin{aligned}\mathbf{F}(\mathbf{r}(t)) \cdot d\mathbf{r} &= \left\langle 1-2t, t^2, -(1-t)^2\right\rangle \cdot \langle -1,1,0\rangle\ dt\\ &= \left(t^2+2t-1\right)\ dt.\end{aligned}$$

Thus

$$\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \left(t^2+2t-1\right)\ dt = \frac{1}{3}.$$

The curve C_2 is similar: it's parameterized by $\mathbf{r}(t) = \langle 0, 1-t, t\rangle$, so

$$\begin{aligned}\mathbf{F}(\mathbf{r}(t)) \cdot d\mathbf{r} &= \left\langle -(1-t)^2, 1-2t, t^2\right\rangle \cdot \langle 0,-1,1\rangle\ dt\\ &= \left(t^2+2t-1\right)\ dt.\end{aligned}$$

Thus

$$\int_{C_2} \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \left(t^2+2t-1\right)\ dt = \frac{1}{3},$$

and very similarly $\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \frac{1}{3}$ as well. Thus

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} + \int_{C_3} \mathbf{F} \cdot d\mathbf{r} = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1.$$

The flux integral $\iint\limits_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$ This triangular surface is a graph (using $z = 1 - x - y$) over the triangle $T = \{(x, y) \ : \ 0 \le y \le 1 - x, \ 0 \le x \le 1\}$ in the first quadrant of the xy-plane. Thus we can parameterize it as $\mathbf{r}(x, y) = \langle x, y, 1 - x - y \rangle$ for $(x, y) \in T$. Since $\mathbf{F} = \langle x^2 - y^2, y^2 - z^2, z^2 - x^2 \rangle$, we get

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 - y^2 & y^2 - z^2 & z^2 - x^2 \end{vmatrix} = \langle 2z, 2x, 2y \rangle .$$

A similar computation shows that and $\mathbf{r}_x \times \mathbf{r}_y = \langle 1, 1, 1 \rangle$ (and, since the $\mathbf{k}$ coefficient is positive, this is the upward-pointing normal). Thus

$$\begin{aligned} \iint_C \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} &= \iint\limits_T \langle 2(1 - x - y), 2x, 2y \rangle \cdot \langle 1, 1, 1 \rangle \, dA \\ &= \int_0^1 \int_0^{1-x} 2 \, dy \, dx \\ &= 1, \end{aligned}$$

as before. More success for Stokes' theorem!

3 The point of this problem is to find use Stokes' theorem to avoid computing the flux integral over S (whatever confusing surface that could be) and instead compute the line integral over the unit circle C in the xy-plane. We use the parameterization $\mathbf{r}(t) = \langle \cos(t), \sin(t), 0 \rangle$, so

$$\begin{aligned} \mathbf{F}(\mathbf{r}(t)) &= \langle \cos(t), \sin(t) + 1, 0 \rangle \\ d\mathbf{r} &= \langle -\sin(t), \cos(t), 0 \rangle \, dt \\ \mathbf{F}(\mathbf{r}(t)) \cdot d\mathbf{r} &= \cos(t) \, dt. \end{aligned}$$

Thus

$$\iint\limits_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \cos(t) \, dt = 0.$$

That was pretty easy.

4 Again we integrate the line integral over the boundary curve C rather than the flux integral over the (more complicated) surface S. The boundary curve is the circle $x^2 + y^2 + 4^2 = 25$ (or $x^2 + y^2 = 9$) in the plane $z = 4$, but a note of caution is in order. The natural parameterization

(or the one we usually think of) is $\mathbf{r}(t) = \langle 3\cos(t), 3\sin(t), 4\rangle$ actually parameterizes $-C$ (that is, C with the opposite orientation)! Why is that? Imagine a person walking this boundary with their head in the normal (outward) direction. The remaining part of the sphere is on their *right* if they're walking counter-clockwise. It should be on their left, so they should be walking clockwise.

We'll calculate $\oint_{-C} \mathbf{F} \cdot d\mathbf{r}$ anyway, since we like the parameterization. In terms of this parameterization,

$$\begin{aligned} \mathbf{F}(\mathbf{r}(t)) &= \langle -3\sin(t), 3\cos(t), 4\rangle \\ d\mathbf{r}(t) &= \langle -3\sin(t), 3\cos(t), 0\rangle\ dt \\ \mathbf{F}(\mathbf{r}(t)) \cdot d\mathbf{r}(t) &= 9\ dt. \end{aligned}$$

Thus

$$\int_{-C} \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} 9\ dt = 18\pi,$$

and so $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r} = -\int_{-C} \mathbf{F} \cdot d\mathbf{r} = -18\pi.$

5 If the boundaries of S_1 and S_2 are both C (with the same orientation!), then two applications of Stokes' theorem means that

$$\iint\limits_{S_1} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint\limits_{S_2} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S},$$

so these two flux integrals must be the same.

6 What the previous problem was getting at was that this is always true when $\mathbf{G} = \operatorname{curl} \mathbf{F}$ for some vector field $\mathbf{F}$. But, alas, it is not true in general.

Here's a "simple" example: Let $\mathbf{G} = \langle 0, 0, z\rangle$ and S_1 be the unit square in the xy-plane:

$$S_1 = \{(x, y, z) \ :\ 0 \le x \le 1,\ 0 \le y \le 1,\ z = 0\}.$$

Then we'll let S_2 be the rest of the boundary of the unit cube, oriented so $S_1 \cup S_2$ encloses this cube and we have the outward-pointing normal. It should be clear that the flux across S_1 is zero (since $\mathbf{G} = \mathbf{0}$ on this surface), but a computation shows that, in fact, $\iint_{S_2} \mathbf{G} \cdot d\mathbf{S} = 1$.

We'll see in the next section that what is claimed here is true when $\operatorname{div} \mathbf{G} = 0$. But, like $\operatorname{curl} \mathbf{F} = \mathbf{0}$ implying that $\mathbf{F} = \nabla f$, it turns out that (under suitable assumptions) $\operatorname{div} \mathbf{G} = 0$ implies $\mathbf{G} = \operatorname{curl} \mathbf{F}$. (This will also give us an easy way to compute our "simple" example.)

7 No, $\displaystyle\iint\limits_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = 0$ in this case, but not the given integral $\displaystyle\iint\limits_S \mathbf{F} \cdot d\mathbf{S}$.

If you're wondering how, say, the total net flow in or out of a closed surface can be something other than zero, then chances are you're too focused on water. A liquid like water is *incompressible*, so for water that flows into a region must be balanced by an equal amount that flows out (assuming that this region is totally submerged / full of water). But this is not true of all things that can flow; for example, electrical charge or temperature (two examples we had in homework due today) can have "sinks" or "sources" as we'll see Friday.

The Divergence Theorem

Cast of Players:

E – a simple solid region with boundary surface...

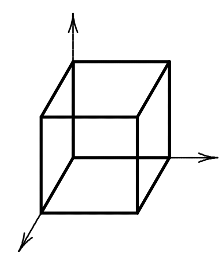
S – given the positive (outward) orientation, and

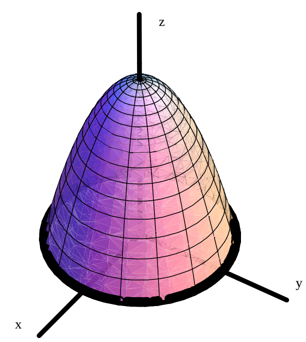
$\mathbf{F}$ – a vector field whose components have continuous derivatives in an open region of $\mathbf{R}^3$ containing E

The Divergence (or Gauss's) Theorem:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F} \, dV$$

1 Let E be the solid unit cube with opposing corners at the origin and $(1, 1, 1)$ and faces parallel to the coordinate planes. Let S be the boundary surface of E, oriented with the outward-pointing normal. If $\mathbf{F} = \langle 2xy, 3ye^z, x\sin(z)\rangle$, find $\iint_S \mathbf{F} \cdot d\mathbf{S}$ using the divergence theorem. (That is, find this flux integral by computing a triple integral instead.)





2 Let E be the solid bounded by the xy-plane and the paraboloid $z = 4 - x^2 - y^2$. Let S be the boundary of E (that is, piece of the paraboloid and a disk in the xy-plane), oriented with the outward-pointing normal. If $\mathbf{F} = \langle xz\sin(yz) + x^3, \cos(yz), 3zy^2 - e^{x^2+y^2}\rangle$, find $\iint_S \mathbf{F} \cdot d\mathbf{S}$ using the divergence theorem.

3 Here's a modification of the previous problem that adds a little clever trickery. Let's split up the surface S into a union $S = S_1 \cup S_2$ of the piece S_1 of the paraboloid and the flat disk S_2, with both pieces oriented as in the previous problem. If the vector field $\mathbf{F}$ is the same as in the previous problem, find $\iint_{S_1} \mathbf{F} \cdot d\mathbf{S}$, the flux of $\mathbf{F}$ through S_1.

4 A friend tells you that if S is a closed surface (that is, a surface without a boundary curve), then by Stokes' theorem we ought to have $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = 0$ for any appropriate $\mathbf{F}$ (since there is no boundary curve C). Is this true? Can you justify it using the Divergence theorem?

Hint: Is $\operatorname{div}(\operatorname{curl} \mathbf{F})$ something simple?

Hint 2: Recall that $\operatorname{div}(\operatorname{curl} \mathbf{F}) = \nabla \cdot (\nabla \times \mathbf{F})$. Is $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v})$ simple?

5 This problem should help you understand what the divergence means. Suppose P is a point in space, let E_a be the solid ball centered at P with radius a, and let S_a be the boundary of E_a with the outward-pointing normal. Then

$$\iint_{S_a} \mathbf{F}\cdot d\mathbf{S} = \iiint_{E_a} \operatorname{div}\mathbf{F}\, dV \approx \iiint_{E_a} \operatorname{div}\mathbf{F}(P)\, dV = \operatorname{div}\mathbf{F}(P)\ \operatorname{Vol}(E_a)$$

or

$$\operatorname{div}\mathbf{F}(P) \approx \frac{1}{\operatorname{Vol}(E_a)} \iint_{S_a} \mathbf{F}\cdot d\mathbf{S}.$$

(Notice that this approximation improves as the radius a shrinks to zero.)

(a) If $\operatorname{div}\mathbf{F} > 0$ at P, would you expect more flux into P or out of P?

(b) If $\operatorname{div}\mathbf{F} < 0$ at P, would you expect more flux into P or out of P?

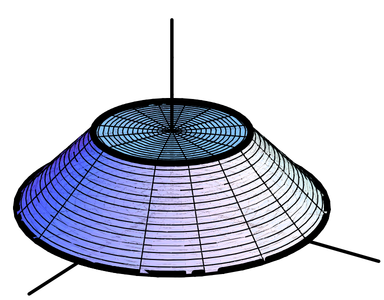
(c) What does it mean when $\operatorname{div}\mathbf{F} = 0$ at P?

6 Suppose E is the ball of radius a centered at the origin, so S the unit sphere with the outward-pointing normal.

(a) Suppose $\mathbf{F} = \langle x - 3y^2z, e^{xz} - y, 2z - \cos(xy)\rangle$. Find the flux of $\mathbf{F}$ across S using the divergence theorem. (Thus you should be computing only the triple integral over E.)

(b) Let's turn things around: suppose you wanted to compute the volume of E (and, for some reason, you didn't know that the volume was $\frac{4}{3}\pi a^3$). If you could find a vector field $\mathbf{F}$ so that $\operatorname{div}\mathbf{F} = 1$, then both integrals in the divergence theorem would find the volume of E. Try this with this E and $\mathbf{F} = \frac{1}{3}\langle x, y, z\rangle$.

7 Use the trick of the previous problem to find the volume of a truncated cone by computing a flux integral. Let E be the part of the solid $x^2+y^2 \le (2-z)^2$ with $0 \le z \le 1$. Use the vector field $\mathbf{F} = \frac{1}{2}\langle x, y, 0\rangle$ (or another, if you prefer). Notice that S, the boundary of E, typically needs to be broken into three pieces, so it would be ideal for $\mathbf{F}\cdot\mathbf{n}$ to be simple (zero, for example) on one or two of these surfaces.



The Divergence Theorem – Answers and Solutions

1 We don't want to do the tedious work of parameterizing six surfaces (the six faces of the cube) in order to compute the flux integral directly. Instead we'll use the divergence theorem. Notice that

$$\operatorname{div}\mathbf{F} = \frac{\partial}{\partial x}\Big(2xy\Big) + \frac{\partial}{\partial y}\Big(3ye^z\Big) + \frac{\partial}{\partial z}\Big(x\sin(z)\Big) = 2y + 3e^z + x\cos(z).$$

Thus the divergence theorem says that

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E \operatorname{div}\mathbf{F}\,dV &= \int_0^1\int_0^1\int_0^1 \left(2y + 3e^z + x\cos(z)\right)\,dx\,dy\,dz \\ &= \int_0^1\int_0^1 \left(2y + 3e^z + \frac{1}{2}\cos(z)\right)\,dy\,dz \\ &= \int_0^1 \left(1 + 3e^z + \frac{1}{2}\cos(z)\right)\,dz \\ &= 1 + 3\,(e-1) + \frac{1}{2}\sin(1).\end{aligned}$$

That looks like it would have been unpleasant to compute via six flux integrals.

2 This is even worse than the first problem. We couldn't possibly compute the flux integrals over the two pieces of the boundary (the paraboloid piece and the disk in the xy-plane) – the vector field $\mathbf{F}$ is simply too complicated. Instead, we'll use the divergence theorem. Notice that

$$\begin{aligned}\operatorname{div}\mathbf{F} &= \frac{\partial}{\partial x}\Big(xz\sin(yz) + x^3\Big) + \frac{\partial}{\partial y}\Big(\cos(yz)\Big) + \frac{\partial}{\partial z}\Big(3zy^2 - e^{x^2+y^2}\Big) \\ &= \Big(z\sin(yz) + 3x^2\Big) + \Big(-z\sin(yz)\Big) + \Big(3y^2\Big) = 3x^2 + 3y^2.\end{aligned}$$

Thus we have from the divergence theorem

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iiint_E \operatorname{div}\mathbf{F}\,dV \\ &= \iint_D \int_0^{4-x^2-y^2} \left(3x^2+3y^2\right)\,dz\,dA,\end{aligned}$$

where D is the disk $x^2+y^2 \le 4$ in the xy-plane. Thus we'll use polar coordinates for this double integral, or cylindrical coordinates for the triple integral:

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \int_0^{2\pi}\int_0^2\int_0^{4-r^2} \left(3r^2\right)\,r\,dz\,dr\,d\theta \\ &= \int_0^{2\pi}\int_0^2 \left(12r^3 - 3r^5\right)\,dr\,d\theta \\ &= \int_0^{2\pi} \left[3r^4 - \frac{1}{2}r^6\right]_0^2\,d\theta \\ &= \int_0^{2\pi} (48-32)\,d\theta = 32\pi.\end{aligned}$$

That wasn't bad at all.

3 The difficulty here is that S_1 is *not* the boundary surface of a simple solid region, so we can't blindly apply the divergence theorem. The standard trick is to add another piece S_2 to the boundary so that $S = S_1 \cup S_2$ bounds a region E. This was already done for us in the previous problem, and in fact we found that

$$32\pi = \iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} \mathbf{F} \cdot d\mathbf{S_1} + \iint_{S_2} \mathbf{F} \cdot d\mathbf{S}.$$

Thus to find the requested flux we can either integrate over S_1 (icky!) or S_2 (much nicer). We parameterize S_2 with polar coordinates, so $\mathbf{r}(r,\theta) = \langle r\cos(\theta), r\sin(\theta), 0\rangle$ (with $0 \le \theta \le 2\pi$ and $0 \le r \le 2$). Then

$$\mathbf{r}_\theta \times \mathbf{r}_r = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -r\sin(\theta) & r\cos(\theta) & 0 \\ \cos(\theta) & \sin(\theta) & 0 \end{vmatrix} = \langle 0, 0, -r\rangle.$$

Note that this vector points downward, with is the appropriate orientation for S_2 (and why we chose $\mathbf{r}_\theta \times \mathbf{r}_r$ rather than $\mathbf{r}_\theta \times \mathbf{r}_\theta$). Thus the flux through S_2 is

$$\begin{aligned} \iint_{S_2} \mathbf{F} \cdot d\mathbf{S} &= \int_0^{2\pi} \int_0^2 \mathbf{F}(\mathbf{r}(r,\theta)) \cdot \langle 0,0,-r\rangle \, dr \, d\theta \\ &= \int_0^{2\pi} \int_0^2 \langle r^3\cos^3(\theta), 1, -e^{r^2}\rangle \cdot \langle 0,0,-r\rangle \, dr \, d\theta \\ &= \int_0^{2\pi} \int_0^2 re^{r^2} \, dr \, d\theta = \int_0^{2\pi} \left. \frac{e^{r^2}}{2} \right|_0^2 d\theta \\ &= \int_0^{2\pi} \frac{e^4 - 1}{2} \, d\theta = 2\pi \, \frac{e^4-1}{2} = \pi\left(e^4 - 1\right). \end{aligned}$$

Thus the flux through S_2 is $32\pi - \pi\,(e^4 - 1) = \pi\,(33 - e^4)$.

4 This is true.

We can justify it with the divergence theorem by recalling that $\operatorname{div}(\operatorname{curl} \mathbf{F}) = 0$. We proved this in the Curl and Divergence worksheet and Stewart does it in that section as well. The point of the second hint is that $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0$, since $\mathbf{u} \times \mathbf{v}$ is perpendicular to $\mathbf{u}$, so you can remember that $\nabla \cdot (\nabla \times \mathbf{F}) = 0$ by analogy.

In any case, if S is a closed surface, then we can let E be the solid enclosed by S. Then the divergence theorem says that

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div}(\operatorname{curl} \mathbf{F}) \, dV = \iiint_E 0 \, dV = 0.$$

5 (a) If $\operatorname{div} \mathbf{F} > 0$ at P, then $\iint_{S_a} \mathbf{F} \cdot d\mathbf{S} > 0$ as well, at least when a is small enough. Thus we would expect flux out of P.

(b) If $\operatorname{div} \mathbf{F} < 0$ at P, a similar argument to part (a) says that we would expect more flux into P.

(c) When $\operatorname{div} \mathbf{F} = 0$ at P, we expect that $\iint_{S_a} \mathbf{F} \cdot d\mathbf{S} \approx 0$. This means that the same amount of flow in toward P as out from P.

6 (a) We're aiming to use the divergence theorem, so we compute the divergence of $\mathbf{F}$:

$$\begin{aligned}\operatorname{div}\mathbf{F} &= \frac{\partial}{\partial x}\left(x-3y^2z\right)+\frac{\partial}{\partial y}\left(e^{xz}-y\right)+\frac{\partial}{\partial z}\left(2z-\cos(xy)\right)\\ &= 1-1+2=2.\end{aligned}$$

Thus the divergence theorem implies that

$$\iint\limits_S \mathbf{F}\cdot d\mathbf{S} = \iiint\limits_E \operatorname{div}\mathbf{F}\,dV = \iiint\limits_E 2\,dV = 2\mathrm{Vol}(E) = \frac{8}{3}\pi a^3.$$

(b) What this problem is getting at is that if we have a vector field $\mathbf{F}$ with $\operatorname{div}\mathbf{F}=1$, then we can compute the volume of E via the flux over the boundary S:

$$\mathrm{Vol}(E) = \iiint\limits_E 1\,dV = \iiint\limits_E \operatorname{div}\mathbf{F}\,dV = \iint\limits_S \mathbf{F}\cdot d\mathbf{S}.$$

We'll compute this flux with the given $\mathbf{F}=\frac{1}{3}\langle x,y,z\rangle$.
We parameterize the sphere of radius a with

$$\mathbf{r}(\phi,\theta) = \langle a\sin(\phi)\cos(\theta), a\sin(\phi)\sin(\theta), a\cos(\phi)\rangle,$$

from which we find that

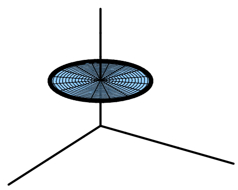
$$\begin{aligned}\mathbf{r}_\phi\times\mathbf{r}_\theta &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a\cos(\phi)\cos(\theta) & a\cos(\phi)\sin(\theta) & -a\sin(\phi)\\ -a\sin(\phi)\sin(\theta) & a\sin(\phi)\cos(\theta) & 0\end{vmatrix}\\ &= a^2\sin(\phi)\,\big\langle \sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi)\big\rangle\\ &= a\sin(\phi)\,\big\langle a\sin(\phi)\cos(\theta), a\sin(\phi)\sin(\theta), a\cos(\phi)\big\rangle\\ &= a\sin(\phi)\,\mathbf{r}(\phi,\theta)\end{aligned}$$

is the outward-pointing normal. Since $\mathbf{F}=\frac{1}{3}\mathbf{r}$ and $|\mathbf{r}|=a$, we get $\mathbf{F}\cdot(\mathbf{r}_\phi\times\mathbf{r}_\theta)=\frac{a\sin(\phi)}{3}|\mathbf{r}|^2=\frac{a^3\sin(\phi)}{3}$. Thus the flux is

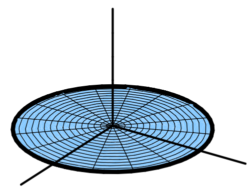
$$\iint\limits_S \mathbf{F}\cdot d\mathbf{S} = \int_0^{2\pi}\int_0^{\pi}\frac{a^3\sin(\phi)}{3}\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi}\frac{2a^3}{3}\,d\theta = \frac{4\pi a^3}{3}.$$

As expected, this flux is the volume of E.

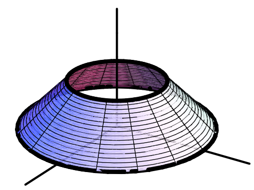
7 As in Problem 6(b), we will compute the volume by computing the flux. This time (as suggested) we'll use $\mathbf{F} = \frac{1}{2}\langle x, y, 0\rangle$, so $\operatorname{div}\mathbf{F} = 1$. We split S into three pieces, the top, bottom, and sides:



S_1, the top



S_2, the bottom



S_3, the sides

Notice that S is the union $S_1 \cup S_2 \cup S_3$, so

$$\operatorname{Vol}(E) = \iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_{S_1} \mathbf{F}\cdot d\mathbf{S} + \iint_{S_2} \mathbf{F}\cdot d\mathbf{S} + \iint_{S_3} \mathbf{F}\cdot d\mathbf{S}.$$

Two of these three new flux integrals are really simple. Notice that the normals for S_1 and S_2 are either $\mathbf{n} = \mathbf{k} = \langle 0, 0, 1\rangle$ or $\mathbf{n} = -\mathbf{k}$. Because of the clever way we chose $\mathbf{F}$, however, we thus get $\mathbf{F}\cdot d\mathbf{S} = 0$! So the flux integrals over S_1 and S_2 are zero.

On S_3 we need to actually compute the flux integrals. This involves parameterizing the surface, given to us as $x^2 + y^2 = (2-z)^2$. Thus, for fixed z, this is a circle of radius $2 - z$. This means we have the parameterization

$$\mathbf{r}(z, \theta) = \langle (2-z)\cos(\theta), (2-z)\sin(\theta), z\rangle,$$

and so

$$\mathbf{r}_\theta \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -(2-z)\sin(\theta) & (2-z)\cos(\theta) & 0 \\ -\cos(\theta) & -\sin(\theta) & 1 \end{vmatrix} = \big\langle (2-z)\cos(\theta), (2-z)\sin(\theta), (2-z)\big\rangle$$

is the appropriately oriented normal. Since $\mathbf{F}(\mathbf{r}(z,\theta)) = \frac{1}{2}\langle (2-z)\cos(\theta), (2-z)\sin(\theta), 0\rangle$, we get the flux integral

$$\iint_{S_3} \mathbf{F}\cdot d\mathbf{S} = \int_0^{2\pi}\int_0^1 \frac{1}{2}(2-z)^2\,dz\,d\theta = \int_0^{2\pi}\left[-\frac{1}{6}(2-z)^3\right]_0^1 d\theta = \int_0^{2\pi} \frac{7}{6}\,d\theta = \frac{7\pi}{3}.$$

Finally, we have found the volume of E: $0 + 0 + \frac{7\pi}{3} = \frac{7\pi}{3}$. (Notice that this agrees with the usual formula for the volume of a cone: $V = \frac{1}{3}\pi r^2 h$. The "full" cone has $r = h = 2$, while the removed cone has $r = h = 1$, so the truncated cone has volume $\frac{1}{3}\pi 2^3 - \frac{1}{3}\pi 1^3 = \frac{7\pi}{3}$.)

Integral Theorems Review

1 For these problems, find $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}$ and C are as given.

(a) $\mathbf{F} = \langle x, y, z \rangle$ and C is parameterized by $\mathbf{r}(t) = \langle t, t, t \rangle$ $(0 \le t \le 1)$

(b) $\mathbf{F} = \langle x, y, z \rangle$ and C is parameterized by $\mathbf{r}(t) = \langle t, \sqrt{t}, \sqrt[3]{t} \rangle$ $(0 \le t \le 1)$

(c) $\mathbf{F} = \langle 2xy, x^2 + z, y + 2z \rangle$ and C is parameterized by

$$\mathbf{r}(t) = \langle t^2 - t,\ \sin(\pi t),\ \cos^2(\pi t) \rangle \qquad (0 \le t \le 1)$$

(d) $\mathbf{F} = \left\langle -\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$ and C is the unit circle in the xy-plane, oriented counter-clockwise.

2 For these problems, find $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}$ and S are as given.

(a) $\mathbf{F} = \operatorname{curl}\langle 2y, z - 2x, yz \rangle$ and S is the hemisphere of radius 1, centered at the origin and above the xy-plane, oriented with the upward-pointing normal.

(b) $\mathbf{F} = \operatorname{curl}\langle 2y, z - 2x, yz \rangle$ and S is the solid disk of radius 1 in the xy-plane, centered at the origin, oriented with the upward-pointing normal.

(c) $\mathbf{F} = \operatorname{curl}\langle 2y, z - 2x, yz \rangle$ and $S = S_{(a)} \cup \left(-S_{(b)}\right)$ is the union of the two surfaces from parts (a) and (b), oriented with the outward-pointing normals.

(d) $\mathbf{F} = \langle x^2 y - 3x,\ -xy^2 + 2\cos(y)z,\ \sin(y)z^2 \rangle$ and $S = S_{(a)} \cup \left(-S_{(b)}\right)$ is the same as in part (c).

3 For these problems, find $\iiint_E \operatorname{div} \mathbf{F}\, dV$, where $\mathbf{F}$ and E are as given.

(a) $\mathbf{F} = \operatorname{curl} \mathbf{G}$ (where $\mathbf{G}$ is any appropriately smooth vector field) and E is any simple solid

(b) $\mathbf{F} = \langle x, y^2, -2yz \rangle$ and E is the solid ball of radius a centered at the origin

(c) $\mathbf{F} = \langle x^2, 2yz, x^2 - z^2 \rangle$ and E is the unit cube with corners at $(0,0,0)$ and $(1,1,1)$

Integral Theorems Review – Answers and Solutions

1 (a) To compute this we "just do it". Using $\mathbf{r}(t) = \langle t,t,t\rangle$, we get $\mathbf{F}(\mathbf{r}(t)) = \langle t,t,t\rangle$ and $d\mathbf{r} = \langle 1,1,1\rangle\ dt$, so

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \langle t,t,t\rangle \cdot \langle 1,1,1\rangle\ dt = \int_0^1 3t\ dt = \frac{3}{2}$$

(b) Here we could "just do it" again, but the parameterization is messier. It's simpler to notice that the vector field $\mathbf{F} = \langle x,y,z\rangle$ is conservative and thus independent of path. We see this by noticing that

$$\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \mathbf{0}$$

or simply that $\mathbf{F} = \nabla f$ where $f = \frac{1}{2}\left(x^2+y^2+z^2\right)$. Since the path in part (b) starts and ends at the same places as the path in part (a), we can simply integrate over that path. This is what we've done in part (a), so we get the same answer: $\frac{3}{2}$.

Another approach would be to use the fundamental theorem of line integrals:

$$\begin{aligned}\int_C \nabla f\cdot d\mathbf{r} &= f(\mathbf{r}(1)) - f(\mathbf{r}(0)) = f(1,1,1) - f(0,0,0) \\ &= \frac{1}{2}\left(1^2+1^2+1^2\right) - 0 = \frac{3}{2},\end{aligned}$$

as before.

(c) Here the key is that the given vector field is conservative. We can see this by computing $\operatorname{curl}\mathbf{F} = 0$ or writing $\mathbf{F} = \nabla f$, where $f = x^2y + yz + z^2$. Since C is a closed curve (one that starts and ends at the point $(0,0,1)$), the integral must be zero. We could also see this via the fundamental theorem of line integrals:

$$\int_C \nabla f\cdot d\mathbf{r} = f(\mathbf{r}(1)) - f(\mathbf{r}(0)) = f(0,0,1) - f(0,0,1) = 0.$$

It's also possible to see this using Stokes' theorem (where S is some (any!) oriented surface with C as its boundary):

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \iint_S \mathbf{0}\cdot d\mathbf{S} = 0.$$

Either approach is fine.

(d) This is a cautionary tale. Let's think of our vector as $\mathbf{F} = \langle P,Q\rangle$. Then it's straightforward to see that

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} = \frac{y^2-x^2}{(x^2+y^2)^2},$$

so we might think that we can proceed as in part (c). That is, we could say that $\mathbf{F}$ is conservative and thus $\int_C \mathbf{F}\cdot d\mathbf{r} = 0$. Or we might apply Green's theorem: let D be the unit disk (with C as it's boundary), so

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\ dA = \iint_D 0\ dA = 0.$$

Both these conclusions are **WRONG!** and in fact the line integral has value 2π.

What has gone wrong? We've tried to blindly apply theorems without checking the hypotheses. In the first case we tried to apply theorem 6 (on page 928) that says that $\mathbf{F}$ is conservative when $P_y = Q_x$. But we must have this equality on a simply connected region D. In our case, both P and Q are undefined at the origin $(0,0)$, so any region D containing the unit circle also contains a hole at the origin. Similarly, Green's theorem require that P and Q have continuous derivatives inside D, which again fails at the origin.

We can compute the actual value using a simple parameterization of C:

$$\mathbf{r}(t) = \langle x, y \rangle = \langle \cos(t), \sin(t) \rangle \qquad \text{and so} \qquad d\mathbf{r}(t) = \langle -\sin(t), \cos(t) \rangle \; dt.$$

Thus

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \left\langle -\frac{\sin(t)}{1}, \frac{\cos(t)}{1} \right\rangle \cdot \langle -\sin(t), \cos(t) \rangle \; dt \\
&= \int_0^{2\pi} \left(\sin^2(t) + \cos^2(t) \right) \; dt \\
&= \int_0^{2\pi} 1 \; dt = 2\pi,
\end{aligned}$$

as claimed.

2 (a) While we could compute this directly, it seems easier to use Stokes' theorem to compute a line integral instead. Here the boundary is simply the unit circle in the xy-plane, so we can parameterize C as

$$\mathbf{r}(t) = \langle x, y, z \rangle = \langle \cos(t), \sin(t), 0 \rangle.$$

Thus the vector we are integrating is

$$\langle 2y, z - 2x, yz \rangle = \langle 2\sin(t), -2\cos(t), 0 \rangle$$

while $d\mathbf{r} = \langle -\sin(t), \cos(t), 0 \rangle \; dt$. Thus

$$\begin{aligned}
\iint_S \operatorname{curl}\langle 2y, z - 2x, yz \rangle \cdot d\mathbf{S} &= \oint_C \langle 2y, z - 2x, yz \rangle \cdot d\mathbf{r} \\
&= \int_0^{2\pi} \langle 2\sin(t), -2\cos(t), 0 \rangle \cdot \langle -\sin(t), \cos(t), 0 \rangle \; dt \\
&= \int_0^{2\pi} \left(-2\sin^2(t) - 2\cos^2(t) \right) \; dt \\
&= \int_0^{2\pi} -2 \; dt = -2t \Big|_0^{2\pi} = -4\pi.
\end{aligned}$$

(b) As with part (a), we could compute this integral directly. Now, however, we have even more of an incentive to use Stokes' theorem and compute the line integral – we've already computed this line integral! That is, our surface has the same boundary C (the unit circle in the xy-plane) as the surface from part (a), so we can simply use our answer from there:

$$\iint_S \operatorname{curl}\langle 2y, z - 2x, yz \rangle \cdot d\mathbf{S} = \oint_C \langle 2y, z - 2x, yz \rangle \cdot d\mathbf{r} = -4\pi.$$

(c) Our surface is $S_{(a)} \cup S_{(-b)}$, the union of the surface from part (a) (with the same outward-pointing normal) and the surface from part (b) (with the downward-pointing normal, the opposite orientation from part (b)). Thus

$$\begin{aligned}
\iint_S \operatorname{curl}\langle 2y, z-2x, yz\rangle \cdot d\mathbf{S} &= \iint_{S_{(a)}} \operatorname{curl}\langle 2y, z-2x, yz\rangle \cdot d\mathbf{S} + \iint_{-S_{(b)}} \operatorname{curl}\langle 2y, z-2x, yz\rangle \cdot d\mathbf{S} \\
&= \iint_{S_{(a)}} \operatorname{curl}\langle 2y, z-2x, yz\rangle \cdot d\mathbf{S} - \iint_{S_{(b)}} \operatorname{curl}\langle 2y, z-2x, yz\rangle \cdot d\mathbf{S} \\
&= (-4\pi) - (-4\pi) = 0.
\end{aligned}$$

Notice that it doesn't matter what the values of the surface integrals from part (a) and part (b) were, it only matters that they were the same. Thus the given surface integral over the closed surface $S_{(a)} \cup -S_{(b)}$ will always be zero. This is because of two facts:

(i) Both $S_{(a)}$ and $S_{(b)}$ are oriented surfaces that have the curve C (as oriented) as their boundary, and

(ii) The flux integrals over $S_{(a)}$ and $S_{(b)}$ involve integrating a vector field that is a curl, and thus we can apply Stokes' theorem.

Another approach is to apply the Divergence theorem. That is, since we're integrating over a closed surface S that bounds a solid hemisphere (we'll call this E), then

$$\iint_S \operatorname{curl}\langle 2y, z-2x, yz\rangle \cdot d\mathbf{S} = \iiint_E \operatorname{div}\left(\operatorname{curl}\langle 2y, z-2x, yz\rangle\right)\, dV.$$

Now the key is the $\operatorname{div}(\operatorname{curl}\mathbf{G}) = 0$ for any vector field $\mathbf{G}$, so this integral is zero.

(d) This is very similar to part (c). Perhaps the simplest approach from (c) that we could use here is the last one, using the Divergence theorem. In this approach we again let E be the solid unit hemisphere that has S as its (oriented) boundary. Thus

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E (\operatorname{div}\mathbf{F})\, dV,$$

where $\mathbf{F} = \langle x^2y - 3x,\ -xy^2 + 2\cos(y)z,\ \sin(y)z^2\rangle$ so

$$\begin{aligned}
\operatorname{div}\mathbf{F} &= \frac{\partial}{\partial x}\left(x^2y - 3x\right) + \frac{\partial}{\partial y}\left(-xy^2 + 2\cos(y)z\right) + \frac{\partial}{\partial z}\left(\sin(y)z^2\right) \\
&= 2xy + (-2xy - 2\sin(y)z) + 2\sin(y)z = -3.
\end{aligned}$$

Thus our flux integral is (again!) zero.

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E (-3)\, dV = -3\mathrm{Vol}(E) = -3\left(\frac{2}{3}\pi\right) = -2\pi,$$

where we've used the fact that the volume of a hemisphere is $\frac{2}{3}\pi r^3$ (and here $r = 1$).

3 (a) The key here is that $\operatorname{div}(\operatorname{curl}\mathbf{G}) = 0$ (this is just an application of Clairaut's theorem on order of partial derivatives). Thus

$$\iiint_E \operatorname{div}\mathbf{F}\, dV = \iiint_E \operatorname{div}(\operatorname{curl}\mathbf{G})\; dV = \iiint_E 0\; dV = 0.$$

(b) Here

$$\operatorname{div}\mathbf{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(-2yz) = 1 + 2y - 2y = 1.$$

Thus

$$\iiint_E \operatorname{div}\mathbf{F}\, dV = \iiint_E 1\; dV = \operatorname{Vol}(E) = \frac{4}{3}\pi a^3,$$

since E is a ball of radius a.

(c) Here

$$\operatorname{div}\mathbf{F} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(2yz) + \frac{\partial}{\partial z}(x^2 - z^2) = 2x + 2z - 2z = 2x.$$

Thus, since $E = \{(x, y, z)\ :\ 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1\}$, we get

$$\iiint_E \operatorname{div}\mathbf{F}\, dV = \iiint_E 2x\; dV = \int_0^1 \int_0^1 \int_0^1 2x\; dx\; dy\; dz = 1.$$