10/2/2013 FIRST HOURLY PRACTICE VI **Math 21a, Fall 2013**

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- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3, we need to see **details** of your computation.
- All functions can be differentiated arbitrarily often unless otherwise specified.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) TF questions (20 points)

Mark for each of the 20 questions the correct letter. No justifications are needed.

1) **T** F — For a moving frame $(\vec{T}, \vec{N}, \vec{B})$, (remember that $\vec{T}$ is the unit tangent vector, $\vec{N}$ is the normal vector and $\vec{B}$ is the binormal vector), one always has $\vec{B} \cdot (\vec{T} \times \vec{N}) = 1$.

Solution:
The three vectors have length 1 and are perpendicular to each other.

2) **T** F — For any three points P, Q, R in space, $\vec{PQ} \times \vec{PR} = \vec{QP} \times \vec{RP}$

Solution:
The two vectors are switching sign on the right hand side.

3) T **F** — The triangle defined by the three points $(-1, 0, 2), (-4, 2, 1), (1, -1, 2)$ has a right angle.

Solution:
It would be inefficient to compute the angles. Better is to look at the squares of the lengths of the triangle which are $14, 5$ and 35. If there was a right angle, Pythagoras would apply.

4) T **F** — The function $f(x, y, z) = x^2 + y^2 + z^2/\sin(x^2 + y^2 + z^2)$ is continuous everywhere in space.

Solution:
The problem is not $(0, 0, 0)$. The function is continuous there becomes 1 as one could see in spherical coordinates $f(\rho) = \rho/\sin(\rho)$ is continuous at 0 (use Hopital's rule). Note however that there are other values like on the sphere $\rho = \pi$, where the function is not continuous. The function blows up there.

5) T **F** — $\vec{u} \times \vec{u} = 0$ implies $\vec{u} = \vec{0}$.

Solution:
The left hand side is always true. To see that it is false, take $\vec{u} = \langle 1, 0, 0 \rangle$. It is not the zero vector, but still $\vec{u} \times \vec{u} = 0$.

6) T F The level curves $f(x, y) = 1$ and $f(x, y) = 2$ of a smooth function f never intersect.

Solution:
If they would intersect in a point (x, y), then f would take two values 1 and 2, at the point which is not possible.

7) T F For any vector $\vec{v}$, we have $\text{proj}_{\vec{i}}(\text{proj}_{\vec{j}}(\vec{v})) = \vec{0}$.

Solution:
The vector $proj_{\vec{j}}(\vec{v}))$ is parallel to $\vec{j}$ which is perpendicular to $\vec{i}$ and the projection onto $\vec{i}$ is therefore the zero vector.

8) T F $(\vec{j} \times \vec{i}) \times \vec{i} = \vec{k} \times (\vec{i} \times \vec{k})$

Solution:
The left vector is parallel to j, the right vector is parallel to i.

9) T F If a parametrized curve $\vec{r}(t)$ lies in a plane and the velocity $\vec{r}\,'(t)$ is never zero, then the normal vector $\vec{N}(t)$ also lies in that plane.

Solution:
This is intuitively clear.

10) T F The angle between $\vec{r}\,'(t)$ and $\vec{r}\,''(t)$ is always 90 degrees.

Solution:
It is true for circles, but false in general. For example, on a line, the acceleration parallel to the velocity.

11) T F If $\vec{v}, \vec{w}$ are two nonzero vectors, then the projection vector $proj_{\vec{w}}(\vec{v})$ can be longer than $\vec{v}$.

Solution:
The projection vector has length $|\vec{v} \cdot \vec{w}|/|\vec{w}|$ which has length smaller or equal to $\vec{v}$ (use the cos formula).

12) T F A line intersects an ellipsoid in at most 2 distinct points.

Solution:
One can see this geometrically. Here is an argument: we know it for a sphere. When streching the picture with the sphere and line the number of intersections does not change.

13) T F For any vectors $\vec{v}$ and $\vec{w}$, the formula $(\vec{v} - \vec{w}) \cdot \vec{P}_{\vec{w}}(\vec{v}) = 0$ holds.

Solution:
Take $\vec{v} = \vec{i}$ and $\vec{w} = \vec{j}$.

14) T F Let S be a plane normal to the vector $\vec{n}$, and let P and Q be points not on S. If $\vec{n} \cdot \overrightarrow{PQ} = 0$, then P and Q lie on the same side of S.

Solution:
The condition $\vec{n} \cdot \overrightarrow{PQ} = 0$ implies that the vector $\vec{PQ}$ is parallel to the plane.

15) T F The vectors $\langle 2, 2, 1 \rangle$ and $\langle 1, 1, -4 \rangle$ are perpendicular.

Solution:
The dot product vanishes

16) T F $\|\vec{v} \times \vec{w}\| = \|v\| \|w\| \cos(\alpha)$, where α is the angle between $\vec{v}$ and $\vec{w}$.

Solution:
It is sin not cos.

17) | T | **F** | The vector $\vec{i} \times (\vec{j} \times \vec{k})$ has length 1.

Solution:
It is the zero vector.

18) | **T** | F | The distance between the z-axis and the line $x - 1 = y = 0$ is 1.

Solution:
You can see that geometrically.

19) | **T** | F | There is a quadric surface which both hyperbola and parabola appear as traces. Traces are intersections of the surface with the coordinate planes $x = 0$, $y = 0$, or $z = 0$.

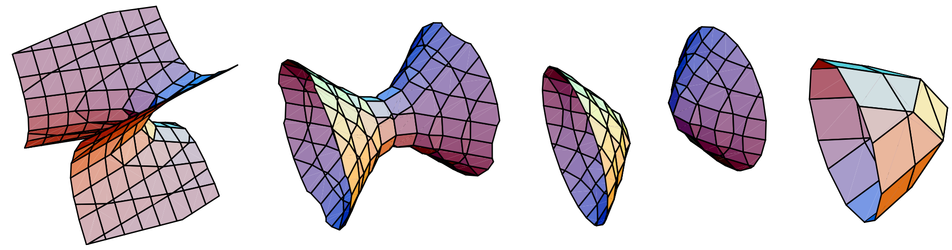
Solution:
The hyperbolic paraboloid has that.

20) | T | **F** | The equation $x^2 + y^2 - z^2 = -1$ defines a one-sheeted hyperboloid.

Solution:
$f(x, y, z) = x^2 + y^2 - z^2 = 1$ is a one-sheeted hyperboloid

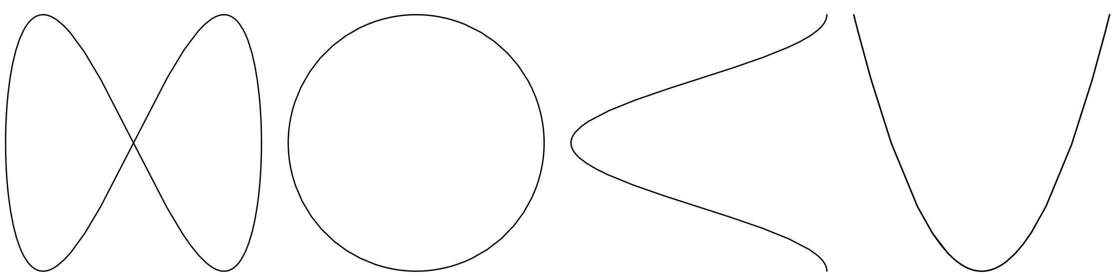
Problem 2) (10 points)

Match the equation with the pictures. No justifications are necessary in this problem.



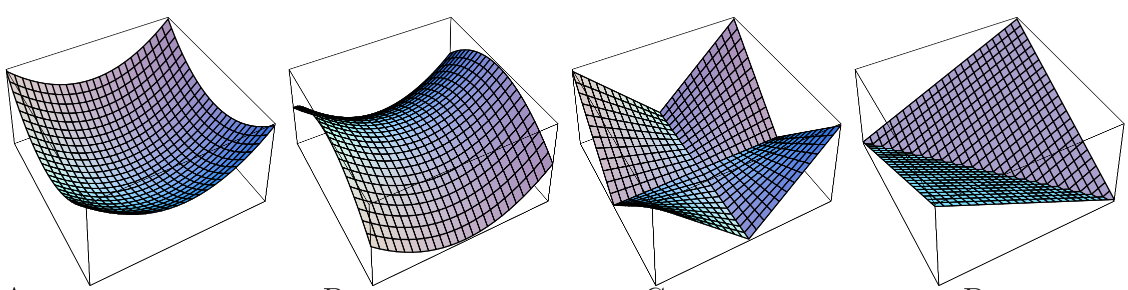
I II III IV

Enter I,II,III,IV here	Equation
	$x + y^2 - z^2 - 1 = 0$
	$-x^2 + y^2 + z^2 - 1 = 0$
	$-x^2 + y^2 + z^2 + 1 = 0$
	$-x + y^2 + z^2 + 1 = 0$



1 2 3 4

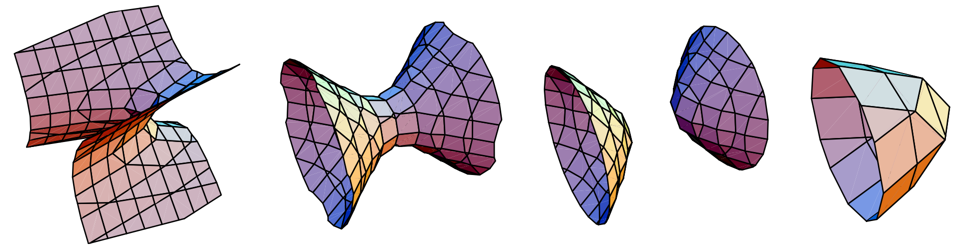
Enter 1,2,3,4 here	Equation
	$\langle \cos(t), \sin(t) \rangle$
	$\langle \cos(t), t \rangle$
	$\langle \cos(t), \cos^2(t) \rangle$
	$\langle \cos(t), \sin(2t) \rangle$



A B C D

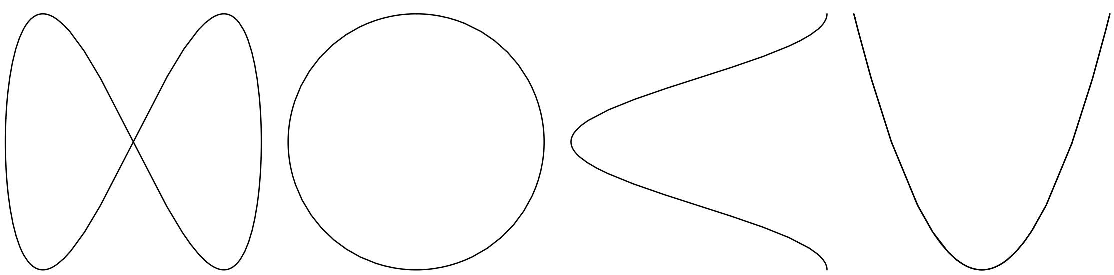
Enter A,B,C,D here	Equation
	$f(x, y) = x^2 - y^2$
	$f(x, y) = \|x + y\|$
	$f(x, y) = x^2 + y^2$
	$f(x, y) = \|xy\|$

Solution:



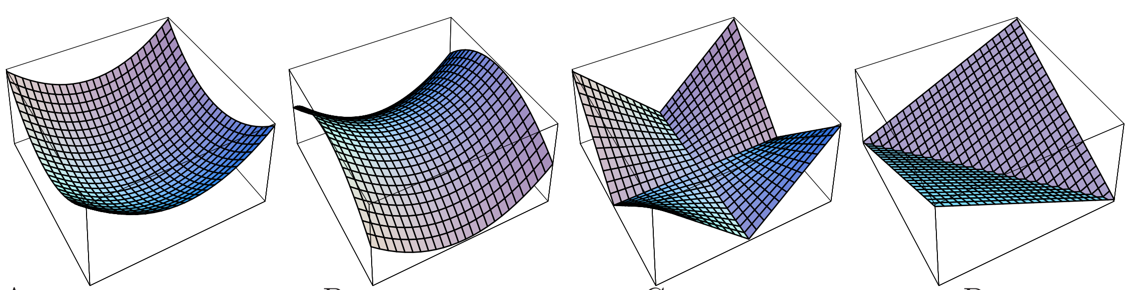
I II III IV

Enter I,II,III,IV here	Equation
I	$x + y^2 - z^2 - 1 = 0$
II	$-x^2 + y^2 + z^2 - 1 = 0$
III	$-x^2 + y^2 + z^2 + 1 = 0$
IV	$-x + y^2 + z^2 + 1 = 0$



1 2 3 4

Enter 1,2,3,4 here	Equation
2	$\langle \cos(t), \sin(t) \rangle$
3	$\langle \cos(t), t \rangle$
4	$\langle \cos(t), \cos^2(t) \rangle$
1	$\langle \cos(t), \sin(2t) \rangle$



A B C D

Enter A,B,C,D here	Equation
B	$f(x,y) = x^2 - y^2$
D	$f(x,y) = \|x + y\|$
A	$f(x,y) = x^2 + y^2$
C	$f(x,y) = \|xy\|$

Problem 3) (10 points)

Imagine the planet Earth as the unit sphere in 3D space centered at the origin. An asteroid is approaching from the point $P = (0, 4, 3)$ along the path

$$\vec{r}(t) = \langle (4 - t)\sin(t), (4 - t)\cos(t), 3 - t \rangle \ .$$

a) When and where will it first hit the Earth?

b) What velocity will it have at the impact?

Solution:
a) The distance to the origin $|\vec{r}(t)| = \sqrt{(4-t)^2 + (3-t)^2} = \sqrt{25 + 2t^2 - 14t}$ is equal 1 for $t = 3$ or $t = 4$.
b) The velocity is $\vec{r}\,'(t) = \langle (4-t)\cos(t) - \sin(t), -\cos(t) - (4-t)\sin(t), -1 \rangle$. The velocity at time $t = 3$ is $\langle \cos(3) - \sin(3), -\sin(3) - \cos(3), -1 \rangle$. (The speed at time $t = 3$ is $\sqrt{3}$.)

Problem 4) (10 points)

Find the distance between the cylinder $x^2 + y^2 = 1$ and the line

$$L : \frac{x+2}{4} = \frac{y-1}{3} = \frac{z}{2}.$$

Solution:
We first compute the distance between the z axes and the line L. The z axes can be parametrized as

$$\vec{r}(t) = P + t\vec{v} = \langle 0, 0, 0 \rangle + t\langle 0, 0, 1 \rangle$$

The line L can be parametrized as

$$\vec{r}(t) = Q + t\vec{w} = \langle -2, 1, 0 \rangle + t\langle 4, 3, 2 \rangle$$

The distance is the length of the projection of $\vec{PQ} = \langle -2, 1, 0 \rangle$ onto the normal vector $\vec{n} = \vec{v} \times \vec{w} = \langle -3, 4, 0 \rangle$. This is

$$d = \frac{|\langle -2, 1, 0 \rangle \cdot \langle -3, 4, 0 \rangle|}{|\langle -3, 4, 0 \rangle|} = 10/5 = 2 \ .$$

The distance between the line L and the cylinder is by 1 smaller. The answer is $\boxed{1}$.

Problem 5) (10 points)

a) Find a parametrization $\vec{r}(t)$ of the line which is the intersection of the two planes

$$4x + 6y - z = 1$$

and

$$4x + z = 0 \; .$$

b) Find the point on the line which is closest to the origin.

Solution:
a) In order to find the line of intersection, we have to find a point Q in the intersection as well as the direction of intersection. We get a point in the intersection by setting one variable zero. Lets take $x = 0$. Then $6y - z = 1, z = 0$ so that $Q = (0, 1/6, 0)$.
The cross product of the normal vectors between two vectors is perpendicular to the normal vectors of the plane. The vector $\vec{v} = \langle 4, 6, -1 \rangle$ is perpendicular to the plane $4x + 6y - z = 1$. The vector $\vec{w} = \langle 4, 0, 1 \rangle$ is perpendicular to the plane $4x + z = 0$. The vector $\vec{u} = \vec{v} \times \vec{w} = \langle 6, -8, -24 \rangle$ is a vector in the direction of the line. b) The vectors $\vec{r}(t)$ and $\vec{u}$ must be perpendicular, that is the dot product between $\vec{r}(t) = \langle 6t, 1/6 - 8t, -24t \rangle$ and $\vec{u} = \langle 6, -8, -24 \rangle$ is zero. This gives $t = 1/507$. The closest point is $\vec{r}(t) = (0, 1/6, 0) + (6, -8, -24)/507$.

Problem 6) (10 points)

Consider the parameterized curve

$$\vec{r}(t) = \langle e^t + e^{-t}, 2\cos(t), 2\sin(t) \rangle \; .$$

Find the arc length of this curve from $t = 0$ to $t = 4$.

Solution:
The velocity is $\vec{r}\,'(t) = \langle e^t - e^{-t}, -2\sin(t), 2\cos(t) \rangle$. The speed is $\sqrt{2 + e^{-2t}, +e^{2t}} = (e^t + e^{-t})$. The integral

$$L = \int_0^4 (e^t + e^{-t}) \; dt$$

gives $e^4 - e^{-4} = 2\sinh(4)$.

Problem 7) (10 points)

The set of points P for which the distance from P to $A = (1,2,3)$ is equal to the distance from P to $B = (5,8,5)$ forms a plane S.

a) Find the equation $ax + by + cz = d$ of the plane S.

b) Find the distance from A to S.

Solution:
a) The key insight is that the point $Q = (A+B)/2 = (3,5,4)$ is in the middle of the two points. The plane has to pass through this point. The normal vector is parallel to $\vec{n} = \langle 4,6,2 \rangle$. The equation of the plane is

$$4x + 6y + 2z = 50 .$$

b) The distance from A to S is half the distance from A to B which is $|\vec{AB}|/2 = |\langle 4,6,2 \rangle|/2 = \sqrt{56}/2$.

Problem 8) (10 points)

The Swiss tennis player Roger Federer hits the ball at the point $\vec{r}(0) = (0,0,3)$. The initial velocity is $\vec{r}\,'(0) = \langle 100, 10, 13 \rangle$. The tennis ball experiences a constant acceleration $\vec{r}\,''(t) = \langle 2, 0, -32 \rangle$ which is due to the combined force of gravity and a constant wind in the x direction.

a) Where does the tennis ball hit the ground $z = 0$?

b) What is the z-component = (projection onto z vector) $proj_{\vec{k}}(\vec{r}\,'(t))$ of the ball velocity at the impact?

Solution:
a) This is a typical free fall problem. After integrating twice the equation $\vec{r}\,''(t) = \langle 2, 0, -32\rangle$, we get

$$\vec{r}(t) = (0,0,3) + t(100,10,13) + t^2(1,0,-16)$$

$$\vec{r}\,'(t) = (100,10,13) + 2t(1,0,-16)$$

We get an impact with the ground $z = 0$ at time $t = 1$. This is at the position $\vec{r}(1) = (101, 10, 0)$.
b) The velocity at time $t = 1$ is $(102, 10, -19)$. The projection onto the vector $\vec{k}$ is $(0, 0, -19)$. Note that this is a vector. The z-component of this vector is the third component of this vector which is -19.

Problem 9) (10 points)

a) (4 points) Parameterize the intersection of the ellipsoid

$$\frac{x^2}{4} + \frac{(y-5)^2}{4} + \frac{z^2}{9} = 2$$

with the plane $z = 3$.

b) (3 points) Parametrize the ellipsoid itself in the form

$$\vec{r}(\theta, \phi) = ... \,.$$

c) (3 points) What is the curvature of the curve at the point $(2, 5, 3)$?

Hint. While you can use the curvature formula $\kappa(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}$ you are also allowed to cite a fact which you know about the curvature.

Solution:
a) The parametrization is

$$\vec{r}(t) = \langle 2\cos(t), 5 + 2\sin(t), 3\rangle \,.$$

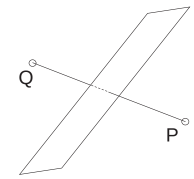
This is a circle of radius 2.
b) The parametrization is

$$\vec{r}(\theta, \phi) = \langle 2\sqrt{2}\cos(\theta)\sin(\phi), 5 + 2\sqrt{2}\sin(\theta)\sin(\phi), \sqrt{2}3\cos(\phi) \,.$$

c) The curvature is $1/2$ at all points.

Problem 10) (10 points)

Find an equation $ax + by + cz = d$ for the plane which has the property that $Q = (5, 4, 5)$ is the reflection of $P = (1, 2, 3)$ through that plane.



Solution:
The plane contains the point $(P + Q)/2 = (6, 6, 8)/2 = (3, 3, 4)$ which is the midpoint between P and Q. The direction of the normal vector to the plane is $\vec{n} = (Q - P) = (4, 2, 2)$. The equation is $4x + 2y + 2z = 12 + 6 + 8 = 26$ or $2x + y + z = 13$.