

Abstract

Each section covers enough material so that students are exposed to the necessary terminology and examples to solve the homework. Homework is synchronized for all sections. Tuesday/Thursday sections assign the first homework set during the first lecture and two thirds of a weekly homework set during the second class. Section numbers refer to Stewart's Concept and Context, the 4'th edition.

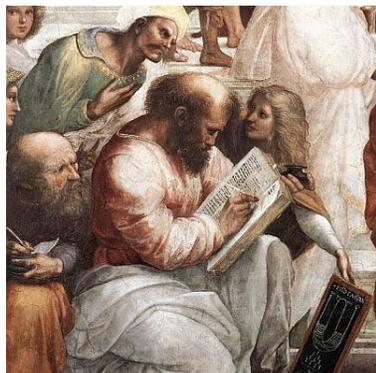
0. Week: Sectioning and intro lecture (9/4/2013)

A half an hour plenary introduction lecture of September 4 and the website gives students details about the syllabus. Sections therefore can refer to section-specific things, homework policies as well as getting to know your students.

1. Week: (9/09/2013 - 9/13/2013)**1. Lecture: Space, coordinates, distance (9.1)**

René Descartes

After breaking the ice, reviewing the syllabus and administrative things, we use **coordinates** like $P = (3, 4, 5)$ to describe points P in space. As promoted by **René Descartes** in the 16'th century, geometry can be described algebraically by using a **coordinate system**. The **distance** between two points $P(x, y, z)$ and $Q = (a, b, c)$ is defined as $d(P, Q) = \sqrt{(x - a)^2 + (y - b)^2 + (z - c)^2}$. This definition is motivated by **Pythagoras theorem**. We will prove the later. To become more familiar with space, we look at geometric objects in the plane and in space. We will focus on **cylinders** and **spheres** and learn how to find the **center** and **radius** of a sphere given as a quadratic expression in x, y, z . This is the **completion of the square**. The CA distributes a time questionnaire.

2. Lecture: Vectors, dot product (9.2-9.3)

Pythagoras

Two points P, Q define a **vector** \vec{PQ} which connects the initial point P with the end point Q . Vectors can be attached everywhere in space, but are identified if they have the same length and direction. Vectors can describe **velocities**, **forces** or **color** or **data**. We introduce **addition**, **subtraction** and **scaling** both graphically as well as algebraically. Unit vectors $\vec{v}/|\vec{v}|$ are also called directions. The **dot product** $\vec{v} \times \vec{w}$ results in a scalar. It allows to compute **length**, **angles** and **projections**. Using Cauchy-Schwarz, we can define the angle α between two vectors by $\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos \alpha$. One can use it to prove the cos formula in trigonometry and Pythagoras as a special case.

3. Lecture: The cross product and planes (9.4)



Rowan Hamilton

The **cross product** $\vec{v} \times \vec{w}$ of two vectors $\vec{v} = \langle a, b, c \rangle$ and $\vec{w} = \langle p, q, r \rangle$ is defined as $\langle br - cq, cp - ar, aq - bp \rangle$. It is perpendicular to $\langle a, b, c \rangle$ and $\langle p, q, r \rangle$. In two dimensions, the cross product is a scalar $\langle a, b \rangle \times \langle p, q \rangle = aq - bp$. The product is useful to compute **areas** of parallelograms, the **distance** between a point and a line, or to **construct** a plane through three points or to **intersect** two planes. We prove a formula $|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}|\sin(\alpha)$ which allows us to define the **area** of the parallelepiped spanned by \vec{v} and \vec{w} . The **triple scalar product** $(\vec{u} \times \vec{v}) \cdot \vec{w}$ is a scalar and defines the **signed volume** of the parallelepiped spanned by \vec{u}, \vec{v} and \vec{w} . Its sign of the triple scalar product informs about the **orientation** of the coordinate frame given by \vec{u}, \vec{v} and \vec{w} . The triple scalar product is zero if and only if the three vectors are contained in a common plane.

2. Week: (9/16/2013 - 9/20/2013)

4. Lecture: Lines distances (9.5)



Arthur Cayley

The problem to find a plane through three different points P, Q , and R allows to review dot and cross product. We introduce lines by parametrization $\vec{r}(t) = P + t\vec{v}$. If $\vec{v} = \langle a, b, c \rangle$ and $P = (x_0, y_0, z_0)$, then $(x - x_0)/a = (y - y_0)/b = (z - z_0)/c$ is the **symmetric equation** of a line. It can be interpreted as the intersection of two planes. As an application of dot and cross products, we look at **distance formulas**, the distance from a point to a plane, the distance from a point to a line or the distance between two lines. These allows to practice all the different products: dot product, cross product and triple scalar product.

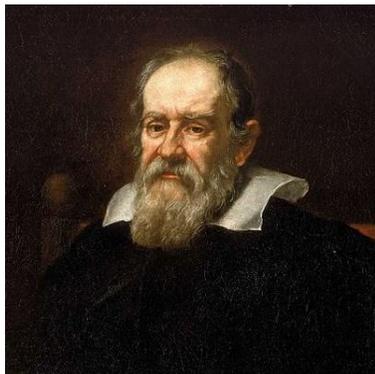
5. Lecture: Functions, graphs, quadrics (9.6)



Claudius Ptolemy

The **graph** of a function $f(x, y)$ of two variables is defined as the set of points (x, y, z) for which $g(x, y, z) = z - f(x, y) = 0$. We look at examples and match some graphs with functions $f(x, y)$. **Generalized traces** $f(x, y) = c$ form **level curves** of f and help to visualize surfaces. Drawing several level curves on the same plane builds a **contour map**. **Conic sections** like **ellipses**, **parabola** and **hyperbola** play an important role when looking at surfaces of the form $g(x, y, z) = 0$. We review the **sphere** and the **plane**. If $g(x, y, z)$ is a function which only involves linear and quadratic terms, then the level surface is called a **quadric**. Important quadrics are **spheres**, **ellipsoids**, **cones**, **paraboloids**, **cylinders** as well as **hyperboloids**.

6. Lecture: Parametrized curves (10.1, 10.2)

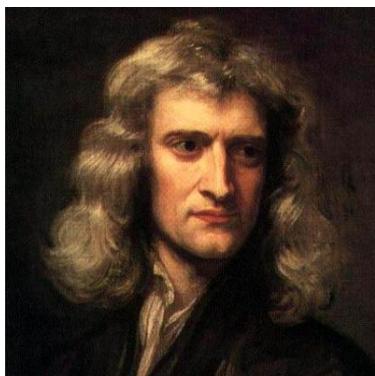


Johannes Kepler

We define curves by **parameterization** $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$, where the **parameter** t is in some **parameter interval** $I = [a, b]$. A parametrization has more information than the curve itself. It is a dynamical and constructive description which also tells, how the curve is traced if t is interpreted a **time variable**. Other examples of curves are **grid curves** on parametrized surfaces obtained by fixing one of the two uv parameters. Differentiation of a parametrization $\vec{r}(t)$ leads to the **velocity** $\vec{r}'(t)$, a vector which is tangent to the curve at $\vec{r}(t)$. A second differentiation with respect to t gives the **acceleration** vector $\vec{r}''(t)$. The **speed** $|\vec{r}'(t)|$ is a scalar. We also learn how to get from $\vec{r}''(t)$ and $\vec{r}'(0)$ and $\vec{r}(0)$ the position $\vec{r}(t)$ by integration. A special case is the **free fall**, where the acceleration vector is constant.

3. Week: (9/23/2013 - 9/27/2013)

7. Lecture: Arc length and curvature (10.3, 10.4)



Isaac Newton

The **arc length** of a curve is defined as a limiting length of polygons and leads to the **arc length** integral $\int_a^b |\vec{r}'(t)| dt$. A re-parametrization of a curve does not change the arc length. The **curvature** $\kappa(t)$ of a curve measures how much a curve is bent. Both acceleration and curvature involve second derivatives, but curvature is an intrinsic quantity which does not depend on parameterizations. One "feels" the acceleration but "sees" the curvature $\kappa(t) = |\vec{T}''(t)|/|\vec{T}'(t)| = |\vec{r}'(t) \times \vec{r}''(t)|/|\vec{r}'(t)|^3$ where $\vec{T}(t) = \vec{r}'(t)/|\vec{r}'(t)|$ is the **unit normal vector** \vec{T} . Together with **normal vector** \vec{N} and **bi-normal vector** \vec{B} these three vectors form an orthonormal frame. It is called the **Frenet frame**.

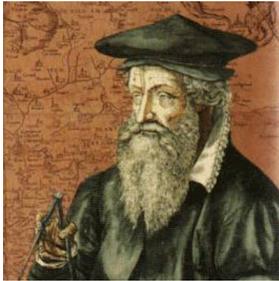
8. Lecture: Spherical coordinates (9.7)



Leonhard Euler

First we review **polar coordinates** (r, θ) in the plane, where $r \geq 0$ and $0 \leq \theta < 2\pi$. We stress that the radius r is always nonnegative and that the formula $\theta = \arctan(y/x)$ is ambiguous and only determines θ up to π . Next we introduce cylindrical coordinates, which is just adding a third component z to polar coordinates in the plane. Finally, we cover more Greek alphabet by introducing Euler angles θ, ϕ to describe points in space using **spherical coordinates**. To avoid confusion, it is good to use ρ as the distance to the origin and r as the distance to the z -axis. Then $x = r \cos(\theta) = \rho \sin(\phi) \cos(\theta)$, $y = r \sin(\theta) = \rho \sin(\phi) \sin(\theta)$, $z = \rho \cos(\phi)$ translates between the coordinate systems.

9. Lecture: Parametric surfaces (10.5)



Gerardus Mercator

Surfaces can be described in two fundamental ways: implicitly or parametrically. Implicit descriptions are $g(x, y, z) = 0$ or $x^2 + y^2 + z^2 - 1 = 0$, parametric descriptions are $r(u, v) = (x(u, v), y(u, v), z(u, v))$ like $\vec{r}(\theta, \phi) = \langle \rho \cos(\theta) \sin(\phi), \rho \sin(\theta) \sin(\phi), \rho \cos(\phi) \rangle$. In many cases, it is possible to switch from a parametric description to an implicit and back. Examples are the plane, spheres, graphs of functions of two variables or **surfaces of revolution**. It is important to have an example for each of these 4 basic classes of surfaces. Using a computer, one can **visualize** complicated surfaces. In computer applications, the parameterization $r(u, v)$ is called the "**uv-map**". More surfaces will be explored in the Mathematica project.

4. Week: (9/30/2013 - 10/4/2013)

10. Lecture: Review



Al Khashi

Review for first hourly on October 2, 2013. The first hourly covers the material from the first 3 weeks. Possible review strategies are answering questions, explaining again some key issues, discussion of the practice exam or to go through some True/False questions.

11. Lecture: Functions (11.1)



Augustin-Louis Cauchy

For functions of one variable, continuity can fail with jump discontinuities, infinities or singular oscillations. In higher dimensions, this is more interesting. Rather than giving an $\epsilon - \delta$ definition for continuity, we look at one or two examples which illustrates how continuity can fail in two dimensions. For most examples, the key is polar coordinates like to see whether $f(x, y) = (x^3 - y)/(x^2 + y^2)$ is continuous at $(0, 0)$. We also look at contour maps of functions and see the relation with continuity. For example, if contours to different levels intersect, the function can not be continuous.

12. Lecture: Partial derivatives (11.2- 11.3)



Alexis Clairot

Finally, we introduce **partial derivatives** $f_x = \partial_x f = \frac{\partial f}{\partial x}$. We also use the index notation f_x allowing to avoid writing many partial signs. To explain Clairot's theorem, one can ask something like to compute f_{xxyx} for $f(x, y) = x^2 \sin(\sin(y)) + x \tan(y)$.

5. Week: (10/7/2013 - 10/11/2013)

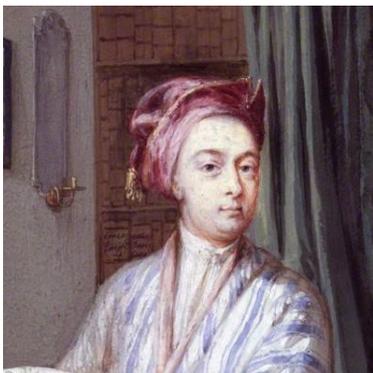
13. Lecture: Partial differential equations (11.3)



Johannes Burgers

To practice differentiation and to get a glimpse of how calculus is used in science, we check whether functions are solutions to **partial differential equations**, abbreviated PDE. More precisely, we look at the transport equation $f_x(t, x) = f_t(t, x)$, the wave equation $f_{tt}(t, x) = f_{xx}(t, x)$ and the heat equation $f_t(t, x) = f_{xx}(t, x)$. There will be a handout on differential equations. Partial differential equations are important not only because they describe the fabric of the structure we are made of like the Einstein equations in cosmology, or Maxwell equations in electromagnetism. They also are very useful for us: Navier Stokes equations help in weather prediction and turbine design, the Black-Scholes equation in pricing financial options, the shallow water equation to model Tsunamis, the nonlinear wave equation to design optical fibers or the diffusion-migration equation to discover oil.

14. Lecture: Linear approximation (11.4)



Brook Taylor

Linearization is an important concept in science because many physical laws are linearization of more complicated laws. Linearization is also useful to estimate quantities. After a review of linearization of functions of one variables, we introduce the **linearization** of a function $f(x, y)$ of two variables at a point (p, q) . It is defined as the function $L(x, y) = f(p, q) + f_x(p, q)(x - p) + f_y(p, q)(y - q)$. The **tangent line** $ax + by = d$ at a point (p, q) is a level curve of L and $a = f_x, b = f_y$. Linearization works similarly in three dimensions, where it allows to compute the **tangent plane** $ax + by + cz = d$. The key is the **gradient** $\nabla f = \langle f_x, f_y, f_z \rangle$.

15. Lecture: Chain rule, implicit deriv. (11.5)



Gottfried Wilhelm Leibniz

The **chain rule** $d/dt f(g(t)) = f'(g(t))g'(t)$ in one dimension is generalized to higher dimensions. It becomes $d/dt f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}'(t)$, where $\nabla f = \langle f_x, f_y, f_z \rangle$ is the gradient. Written out, this formula is $d/dt f(x(t), y(t), z(t)) = f_x(x(t), y(t), z(t))x'(t) + f_y(x(t), y(t), z(t))y'(t) + f_z(x(t), y(t), z(t))z'(t)$. All other chain rule versions can be derived from this. A nice application of the chain rule is **implicit differentiation**: if $f(x, y, z) = 0$ defines a surface which looks locally like $z = g(x, y)$ and because $f_x + f_z z' = 0$ we can compute the partial derivatives $g_x = -f_x/f_z$ and $g_y = -f_y/f_z$ of g without knowing g .

6. Week: (10/14/2013 - 10/18/2013)

Columbus day, no class

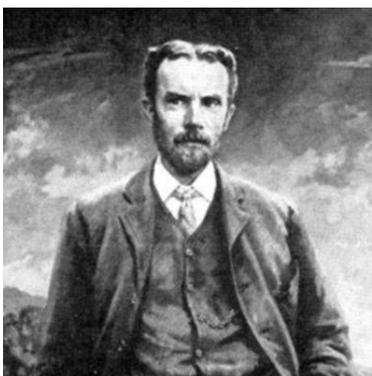
16. Lecture: Tangent lines and planes (11.6)



Pierre-Simon Laplace

The **gradient** of a function is useful for the geometry of a surface $g(x, y, z) = c$ because it is perpendicular to this level surface at that point. One can see this both using linearization or by using the chain rule for a curve $\vec{r}(t)$ on the surface $f(\vec{r}(t)) = 0$. A special case is the plane $g(x, y, z) = ax + by + cz = d$, where $\nabla g = \langle a, b, c \rangle$. The gradient helps to find tangent planes and tangent lines.

17. Lecture: Directional derivative (11.6)



Oliver Heaviside

We introduce the **directional derivative** $D_{\vec{v}}(f)$ as $D_{\vec{v}}f = \nabla f \cdot \vec{v}$ for unit vectors \vec{v} . Partial derivatives are special directional derivatives. The direction of the normal vector gives a nonnegative partial derivative. Moving into the direction of the normal vector, increases f because $D_{\nabla f/|\nabla f|}f = |\nabla f|$. In other words, the gradient vector points in the direction of **steepest ascent**.

7. Week: (10/21/2013-10/25/2013)

18. Lecture: Extrema, 2. derivative test (11.7)



Pierre de Fermat

To **extremize** functions $f(x, y)$ of two variables, we first identify **critical points**, where the gradient $\nabla f(x, y)$ vanishes. The nature of these critical points (p, q) can often be established using the **second derivative test**. Let (p, q) be a critical point and let $D = f_{xx}f_{yy} - f_{xy}^2$ denote the **discriminant** of f at this critical point. There are three fundamentally different cases: **local maxima**, **local minima** as well as **saddle points**. If $D < 0$, then (p, q) is a saddle point, if $D > 0$ and $f_{xx} < 0$ then we have a local maximum, if $D > 0$ and $f_{xx} > 0$ then we have a local minimum. If $D = 0$, we can not determine the nature of the critical point from the second derivatives alone. We also look already at **global maxima**, places where the function is larger or equal than the value at any other point.

19. Lecture: Extrema with constraints (11.8)



Joseph Louis Lagrange

We extremize a function $f(x, y)$ in the presence of a **constraint** $g(x, y) = 0$. A necessary condition for a critical point is that the gradients of f and g are parallel. This leads to a system of equations which are called the **Lagrange equations**. They are a system of nonlinear equations $\nabla f = \lambda \nabla g, g = 0$. Extrema solve this equation of $\nabla g = 0$. When extremizing functions on a domain bounded by a curve $g(x, y) = 0$, we have to solve two problems: find the extrema in the interior and the extrema on the boundary. The second problem is a Lagrange problem. With the same method we can also extremize functions $f(x, y, z)$ of three variables, under the constraint $g(x, y) = 0, h(x, y) = 0$. In two or three dimensions, extrema could also be obtained without Lagrange from $\nabla f \times \nabla g = \vec{0}$.

20. Lecture: Global extrema (11.8)



Bernard Bolzano

A point (x_0, y_0) is called a global maximum of f in a domain G if $f(x_0, y_0) \geq f(x, y)$ for all $(x, y) \in G$. When extremizing functions on a domain bounded by a curve $g(x, y) = 0$, we have to solve two problems: find the extrema in the interior and the extrema on the boundary. This is an opportunity to practice both extremization problems again and give more examples.

8. Week: (10/28/2013-11/1/2013)

21. Lecture: Double integrals (12.1,12.2)



Guido Fubini

Integration in two dimensions is first done on **rectangles**, then on regions G bound by graphs of functions. Depending on whether curves $y = c(x), y = d(x)$ or curves $x = a(y), y = b(y)$ are the boundaries, we call the region **Type I** or **Type II**. Similar than in one dimension, there is a **Riemann sum approximation** of the integral. This allows us to derive results like **Fubini's theorem** on a rectangular region or the change of the order of integration which often enables the integration. The double integral $\int \int_G f(x, y) \, dx \, dy$ is the signed volume under the graph of the function of two variables. Double integrals define **area** if $f(x, y) = 1$. By changing of order of integration in regions which are both type I and type II, we sometimes can integrate otherwise impossible integrals.

22. Lecture: Polar integration (12.4)



James Gregory

Some regions can be described better in **polar coordinates** (r, θ) , where r is the distance to the origin and θ is the angle with the positive x -axes. Examples of regions which can be treated like that are **polar region** is $r \leq g(\theta)$ which trace flower-like shapes in the plane. Polar integration involves describing the region in polar coordinates, rewrite the function f in polar coordinates and rewrite $dx \, dy$ as $r \, dr \, d\theta$.

23. Lecture: Surface area (12.6)



Bonaventura Cavalieri

We derive the formula $\int \int_R |r_u \times r_v| \, du \, dv$ and give examples like graphs, surfaces of revolution and especially the sphere. Similar as for arc length, it is easy to give examples, where the surface area can be computed in closed form, like for a paraboloid, part of a plane or a sphere.

9. Week: (11/4/2013-11/8/2013)

24. Lecture: Review for second midterm



Carl Jacobi

The second midterm covers the material from week 1-8. The material focuses on weeks 4-8.

25. Lecture: Triple integrals, (12.7)



Archimedes of Syracuse

Triple integrals measure volume, moment of inertia or the center of mass of a solid. First introduced for cuboids, then to more general regions like regions sandwiched between the graphs of two functions of two variables. Applications are computations of **mass** $\int \int \int_E \delta(x, y, z) dx dy dz$, **moment of inertia** $\int \int \int_E (x^2 + y^2 + z^2) dx dy dz$, **center of mass**, $\int \int \int \langle x, y, z \rangle dV$ the **expectation** $E[X] = \int \int \int X(x, y, z) dV / \int \int \int 1 dV$ of a random variable=function $X(x, y, z)$ on a region Ω .

26. Lecture: Spherical coordinates (12.8)

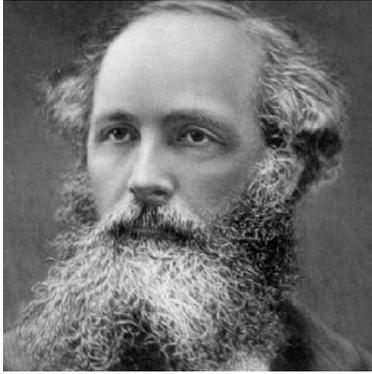


Bernhard Riemann

Some objects can be described better in **cylindrical coordinates** (r, θ, z) , polar coordinates for the x, y variables in space. Examples of such regions are parts of cylinders or solids of revolution. The important factor to include when changing to cylindrical coordinates is r . Other regions are integrated over better in **spherical coordinates** (ρ, ϕ, θ) . Example of such regions are parts of cones or spheres. The important factor to include when changing to spherical coordinates is $\rho^2 \sin(\phi)$. As an application we compute **moments of inertia** of some bodies.

10. Week: (11/11/2013-11/16/2013)

27. Lecture: Vector field (13.1)



James Maxwell

Vector fields occur as force fields or velocity fields or mechanics. We look at examples in 2 or 3 dimensions. We learn how to **match vector fields** with formulas and introduce **flow lines**, parametrized curves $\vec{r}(t)$ for which the vector $\vec{F}(\vec{r}(t))$ is parallel to $\vec{r}'(t)$ at all times.

28. Lecture: Vector field/line integral (13.1-13.2)



Michael Faraday

Given a parametrized curve $\vec{r}(t)$ and a vector field \vec{F} , we can define the **line integral** $\int_C \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt$ along a curve in the presence of a vector field. An important example is the case if \vec{F} is a **force field** and where line integral is **work**.

29. Lecture: Fund. thm of line integrals (13.3)



André-Marie Ampère

For **conservative vector fields** one can evaluate a line integral using the **fundamental theorem of line integrals**. Conservative fields are also called **path independence** or being a **gradient field** $\vec{F} = \nabla f$. It is equivalent to being **irrotational** $\text{curl}(F) = Q_x - P_y = 0$ if the topological condition of **simply connected** is satisfied: any closed curve can be contracted to a point. The plane with the unit disc cut out is not simply connected because the path $\langle 2 \cos(t), 2 \sin(t) \rangle$ can not be pulled together to a point. In two dimensions, the curl of a field $\text{curl}\langle P, Q \rangle = Q_x - P_y$ measures the **vorticity** of the field and if this is zero, the line integral along a simply connected region is zero.

11. Week, (11/18/2013-11/23/2013)

30. Lecture: Green theorem (13.4)



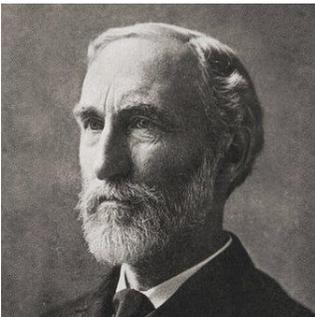
Mikhail Ostrogradsky

Greens theorem equates the line integral along a boundary curve C with a double integral of the curl inside the region G : $\int \int_G \text{curl}(\vec{F})(x, y) dx dy = \int_C \vec{F}(\vec{r}(t)) \vec{r}'(t) dt$. The theorem is useful to **compute areas**: take a field $\vec{F} = \langle 0, x \rangle$ which has constant curl 1. It also allows to compute complicated line integrals. Greens theorem implies that if $\text{curl}(F) = 0$ everywhere in the plane, then the field has the **closed loop property** and is therefore conservative. The **curl** of a vector field $\vec{F} = \langle P, Q, R \rangle$ in three dimensions is a new vector field which can be computed as $\nabla \times \vec{F}$. The three components of $\text{curl}(F)$ are the vorticity of the vector field in the x, y and z direction.

Thanksgiving, no class.

12. Week, (11/24/2013-11/29/2013)

31. Lecture: Div, curl, flux (13.5-13.6)



Josiah Gibbs

The **curl** of a vector field $\vec{F} = \langle P, Q, R \rangle$ in three dimensions is a new vector field which can be computed as $\nabla \times \vec{F}$. The three components give the vorticity of the vector field in the x, y and z direction. In two dimensions, it is a scalar field $\text{curl}(P, Q) = Q_x - P_y$. We also introduce the **divergence**, a scalar field, which is a scalar field. The divergence at a point measures "how much field is generated" at a point. Given a surface S and a fluid with has the velocity $\vec{F}(x, y, z)$ at the point (x, y, z) . the amount of fluid which passes through the membrane S in unit time is the **flux**. This integral is $\int \int_S \vec{F} \cdot (\vec{r}_u \times \vec{r}_v) dudv$. The angle between \vec{F} and the normal vector $\vec{n} = \vec{r}_u \times \vec{r}_v$ determines the sign of $d\vec{S} = \vec{F} \cdot \vec{n} dudv$.

32. Lecture: Stokes theorem (13.7)



George Gabriel Stokes

Stokes theorem equates the line integral along the boundary C of the surface S with the flux of the "curl" of the field through the surface: $\int_C \vec{F} \cdot d\vec{r} = \int \int_S \text{curl}(\vec{F}) \cdot d\vec{S}$. Matching orientations is important. The theorem has applications. It illustrates the Maxwell equations in electromagnetism and explains why the line integral of an irrotational field along a closed curve in space is zero if the region where \vec{F} is defined is simply connected. It is the flux of the curl of \vec{F} through the surface S bound by the curve C .

33 Lecture: Divergence theorem (13.8)

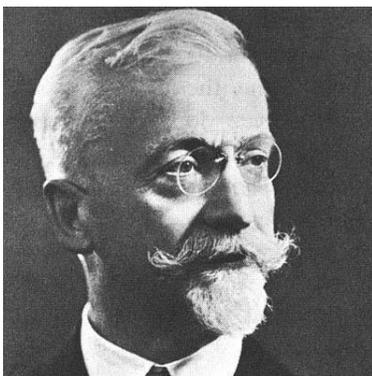


Carl Friedrich Gauss

The **divergence** of a vector field in the interior of a solid E is related to the flux of the vector field \vec{F} through the boundary of the surface using the **divergence theorem**. It relates the "local expansion rate" integrated over the solid $\int \int \int_E \text{div}(\vec{F}) \cdot dV$ of a vector field \vec{F} with the flux $\int \int_S \vec{F} \cdot d\vec{S}$ through the boundary surface S of E . It is useful for example to compute the gravitational field inside a solid like the sphere or a hollow sphere.

13. Week, (12/2/2013-12/3/2013)

34 Lecture: Overview over integral theorems



Elie Cartan

We give an overview over all integral theorems. It turns out that in d -dimensions, there are d theorems. We have here seen them in dimension 2 and 3:

