Vectors and the Dot Product

1. Are the following better described by vectors or scalars?

   (a) The cost of a Super Bowl ticket.

   (b) The wind at a particular point outside.

   (c) The number of students at Harvard.

   (d) The velocity of a car.

   (e) The speed of a car.

2. Bert and Ernie are trying to drag a large box on the ground. Bert pulls the box toward the north with a force of 30 N, while Ernie pulls the box toward the east with a force of 40 N. What is the resultant force on the box?

Definition. The dot product $\vec{v} \cdot \vec{w}$ of two vectors $\vec{v}$ and $\vec{w}$ is defined as follows.

- If $\vec{v}$ and $\vec{w}$ are two-dimensional vectors, say $\vec{v} = \langle v_1, v_2 \rangle$ and $\vec{w} = \langle w_1, w_2 \rangle$, then their dot product is $v_1 w_1 + v_2 w_2$.

- If $\vec{v}$ and $\vec{w}$ are three-dimensional vectors, say $\vec{v} = \langle v_1, v_2, v_3 \rangle$ and $\vec{w} = \langle w_1, w_2, w_3 \rangle$, then their dot product is $v_1 w_1 + v_2 w_2 + v_3 w_3$.

It is not possible to dot a two-dimensional vector with a three-dimensional vector!

3. (a) What is $\langle 1, 2 \rangle \cdot \langle 3, 4 \rangle$?

   (b) What is $\langle 1, 2, 3 \rangle \cdot \langle 4, -5, 6 \rangle$?

Here are some basic algebraic properties of the dot product. If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors of the same dimension and c is a scalar, then

1. $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$.
2. $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$.
3. $(c\vec{v}) \cdot \vec{w} = c(\vec{v} \cdot \vec{w}) = \vec{v} \cdot (c\vec{w})$.

4. True or false: if $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors of the same dimension, then $\vec{u} \cdot (\vec{v} \cdot \vec{w}) = (\vec{u} \cdot \vec{v}) \cdot \vec{w}$.

5. What is the relationship between $\vec{v} \cdot \vec{v}$ and $|\vec{v}|$?

6. Find the angle between $\langle 1, 2, 1 \rangle$ and $\langle 1, -1, 1 \rangle$.

7. Find the vector projection of $\langle 0, 0, 1 \rangle$ onto $\langle 1, 2, 3 \rangle$.

8. True or false: If $\vec{v}$ and $\vec{w}$ are parallel, then $|\vec{v} - \vec{w}| = |\vec{v}| - |\vec{w}|$.

9. If $\vec{v}$ and $\vec{w}$ are vectors with the property that $|\vec{v} + \vec{w}|^2 = |\vec{v}|^2 + |\vec{w}|^2$, which of the following must be true?

   (a) $\vec{v} = \vec{w}$.

   (b) $\vec{v} = \vec{0}$.

   (c) $\vec{v}$ is orthogonal to $\vec{w}$.

   (d) $\vec{v}$ is parallel to $\vec{w}$.

Vectors and the Dot Product

1. Are the following better described by vectors or scalars?

   (a) *The cost of a Super Bowl ticket.*

   Solution. Scalar — the cost is just a number.

   (b) *The wind at a particular point outside.*

   Solution. Vector — the wind has both a speed and a direction.

   (c) *The number of students at Harvard.*

   Solution. Scalar.

   (d) *The velocity of a car.*

   Solution. Vector. The velocity is defined to be both the speed of the car (how fast it's going) and the direction it's going.

   (e) *The speed of a car.*

   Solution. Scalar. The speed refers only to how fast the car is going; it is the magnitude of the velocity vector.

2. *Bert and Ernie are trying to drag a large box on the ground. Bert pulls the box toward the north with a force of 30 N, while Ernie pulls the box toward the east with a force of 40 N. What is the resultant force on the box?*

   Solution. The force Bert is applying can be described by the vector $\langle 0, 30\rangle$, while the force Ernie is applying is $\langle 40, 0\rangle$. We know that the resultant force can be obtained simply by summing the individual force vectors, so the resultant force is $\boxed{\langle 40, 30\rangle}$.

3. (a) *What is* $\langle 1, 2\rangle \cdot \langle 3, 4\rangle$*?*

   Solution. $1 \cdot 3 + 2 \cdot 4 = \boxed{11}$.

   (b) *What is* $\langle 1, 2, 3\rangle \cdot \langle 4, -5, 6\rangle$*?*

   Solution. $1 \cdot 4 + 2 \cdot -5 + 3 \cdot 6 = \boxed{12}$.

4. *True or false: if* $\vec{u}$, $\vec{v}$, *and* $\vec{w}$ *are vectors of the same dimension, then* $\vec{u} \cdot (\vec{v} \cdot \vec{w}) = (\vec{u} \cdot \vec{v}) \cdot \vec{w}$.

   Solution. Completely false. In fact, the statement doesn't even make sense! $\vec{v} \cdot \vec{w}$ is a scalar, and we can't dot a vector with a scalar, so $\vec{u} \cdot (\vec{v} \cdot \vec{w})$ is meaningless.

5. *What is the relationship between* $\vec{v} \cdot \vec{v}$ *and* $|\vec{v}|$*?*

   Solution. $\vec{v} \cdot \vec{v}$ is equal to $|\vec{v}|^2$. Again, this is easy to see from the component definition. For a two-dimensional vector $\vec{v} = \langle v_1, v_2\rangle$, $\vec{v} \cdot \vec{v} = v_1^2 + v_2^2 = |\vec{v}|^2$. For a three-dimensional vector $\vec{v} = \langle v_1, v_2, v_3\rangle$, $\vec{v} \cdot \vec{v} = v_1^2 + v_2^2 + v_3^2 = |\vec{v}|^2$.

6. *Find the angle between* $\langle 1, 2, 1\rangle$ *and* $\langle 1, -1, 1\rangle$.

Solution. Let $\vec{v} = \langle 1, 2, 1\rangle$ and $\vec{w} = \langle 1, -1, 1\rangle$, and let θ be the angle between $\vec{v}$ and $\vec{w}$. Then, we know that $\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos\theta$. We calculate that $\vec{v} \cdot \vec{w} = 1 \cdot 1 + 2 \cdot -1 + 1 \cdot 1 = 0$, so $0 = |\vec{v}||\vec{w}| \cos\theta$. Since the lengths $|\vec{v}|$ and $|\vec{w}|$ are both positive, $\cos\theta = 0$, so $\theta = \frac{\pi}{2}$.

7. *Find the vector projection of $\langle 0, 0, 1\rangle$ onto $\langle 1, 2, 3\rangle$.*

Solution. Let $\vec{v} = \langle 0, 0, 1\rangle$ and $\vec{w} = \langle 1, 2, 3\rangle$. We saw in class that the projection of $\vec{v}$ onto $\vec{w}$ is $\frac{\vec{v}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\vec{w}$. In this case, $\vec{v} \cdot \vec{w} = 3$ and $\vec{w} \cdot \vec{w} = 1^2 + 2^2 + 3^2 = 14$, so the projection is $\frac{3}{14}\langle 1, 2, 3\rangle = \left\langle \frac{3}{14}, \frac{6}{14}, \frac{9}{14}\right\rangle$.

8. *True or false: If $\vec{v}$ and $\vec{w}$ are parallel, then $|\vec{v} - \vec{w}| = |\vec{v}| - |\vec{w}|$.*

Solution. False. For example, let $\vec{v} = \langle 1, 0, 0\rangle$ and $\vec{w} = -\langle 1, 0, 0\rangle$. Then, $\vec{v} - \vec{w} = \langle 2, 0, 0\rangle$, which has length 2. On the other hand, $\vec{v}$ and $\vec{w}$ both have length 1, so $|\vec{v}| - |\vec{w}| = 0$.

9. *If $\vec{v}$ and $\vec{w}$ are vectors with the property that $|\vec{v}+\vec{w}|^2 = |\vec{v}|^2 + |\vec{w}|^2$, which of the following must be true?*

   (a) $\vec{v} = \vec{w}$.

   (b) $\vec{v} = \vec{0}$.

   (c) $\vec{v}$ *is orthogonal to* $\vec{w}$.

   (d) $\vec{v}$ *is parallel to* $\vec{w}$.

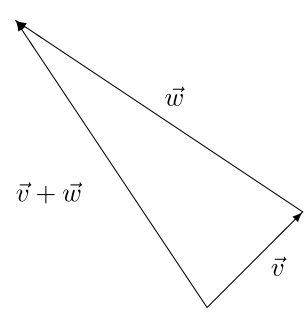
Solution. (c).

We can rewrite the equation $|\vec{v} + \vec{w}|^2 = |\vec{v}|^2 + |\vec{w}|^2$ using dot products:

$$\begin{aligned}
(\vec{v} + \vec{w}) \cdot (\vec{v} + \vec{w}) &= \vec{v} \cdot \vec{v} + \vec{w} \cdot \vec{w} \\
\cancel{\vec{v} \cdot \vec{v}} + 2\vec{v} \cdot \vec{w} + \cancel{\vec{w} \cdot \vec{w}} &= \cancel{\vec{v} \cdot \vec{v}} + \cancel{\vec{w} \cdot \vec{w}} \\
2\vec{v} \cdot \vec{w} &= 0 \\
\vec{v} \cdot \vec{w} &= 0
\end{aligned}$$

This is, of course, exactly what it means for $\vec{v}$ to be orthogonal to $\vec{w}$.

You could also think about this problem geometrically. If $\vec{v}$ and $\vec{w}$ are not parallel, then $\vec{v}$, $\vec{w}$, and $\vec{v} + \vec{w}$ form a triangle:



The equation $|\vec{v}+\vec{w}|^2 = |\vec{v}|^2 + |\vec{w}|^2$ says that the sides of the triangle satisfy the Pythagorean Theorem, so the triangle must be a right triangle with $\vec{v} + \vec{w}$ as the hypotenuse and $\vec{v}$ and $\vec{w}$ as the two legs. In other words, $\vec{v}$ and $\vec{w}$ must be orthogonal.

Cross Product and Triple Product

Algebraic definition of the cross product. If $\vec{v} = \langle v_1, v_2, v_3 \rangle$ and $\vec{w} = \langle w_1, w_2, w_3 \rangle$, then we define $\vec{v} \times \vec{w}$ to be $\langle v_2 w_3 - v_3 w_2, v_3 w_1 - v_1 w_3, v_1 w_2 - v_2 w_1 \rangle$.

There is a handy way of remembering this definition: the cross product $\vec{v} \times \vec{w}$ is equal to the determinant

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} \vec{i} - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} \vec{j} + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \vec{k}$$

Note: The cross product is only defined for three-dimensional vectors.

1. For this problem, let $\vec{v} = \langle 1, 2, 1 \rangle$ and $\vec{w} = \langle 0, -1, 3 \rangle$.

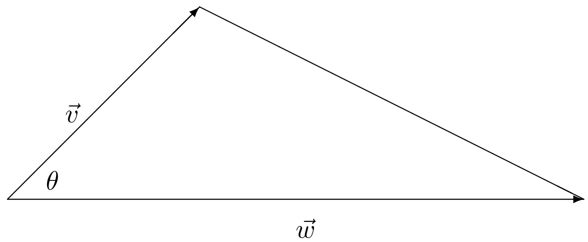
   (a) Compute $\vec{v} \times \vec{w}$.

   (b) Compute $\vec{w} \times \vec{v}$.

   (c) Let $\vec{u} = \vec{v} \times \vec{w}$, the vector you found in (a). What is the angle between $\vec{u}$ and $\vec{v}$? $\vec{u}$ and $\vec{w}$?

2. In general, what is the relationship between $\vec{v} \times \vec{w}$ and $\vec{w} \times \vec{v}$?

3. Any two vectors $\vec{v}$ and $\vec{w}$ which are not parallel determine a triangle, as shown. What is the relationship between the area of the triangle and $\vec{v} \times \vec{w}$?



4. If $\vec{v}$ and $\vec{w}$ are parallel, what is $\vec{v} \times \vec{w}$?

5. If the scalar triple product $\vec{u} \cdot (\vec{v} \times \vec{w})$ is equal to 0, what can you say about the vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$?

6. Find an equation for the plane which passes through the points $(1, 0, 1)$, $(0, 2, 0)$, and $(2, 1, 0)$.

7. True or false: If $\vec{u} \times \vec{v} = \vec{u} \times \vec{w}$, then $\vec{v} = \vec{w}$.

8. True or false: If $\vec{v} \times \vec{w} = \vec{0}$ and $\vec{v} \cdot \vec{w} = 0$, then at least one of $\vec{v}$ and $\vec{w}$ must be $\vec{0}$.

Cross Product and Triple Product

1. For this problem, let $\vec{v} = \langle 1, 2, 1 \rangle$ and $\vec{w} = \langle 0, -1, 3 \rangle$.

   (a) *Compute* $\vec{v} \times \vec{w}$.

   Solution. $\langle 7, -3, -1 \rangle$.

   (b) *Compute* $\vec{w} \times \vec{v}$.

   Solution. $\langle -7, 3, 1 \rangle$.

   (c) *Let* $\vec{u} = \vec{v} \times \vec{w}$*, the vector you found in (a). What is the angle between* $\vec{u}$ *and* $\vec{v}$*?* $\vec{u}$ *and* $\vec{w}$*?*

   Solution. To find the angle between two vectors, we use the dot product.

   $\vec{u} \cdot \vec{v} = \langle 7, -3, -1 \rangle \cdot \langle 1, 2, 1 \rangle = (7)(1) + (-3)(2) + (-1)(1) = 0$, so $\vec{u}$ is orthogonal to $\vec{v}$.

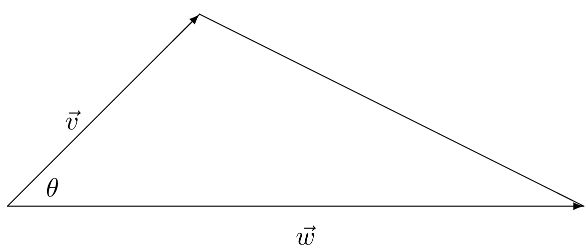
   $\vec{u} \cdot \vec{w} = \langle 7, -3, -1 \rangle \cdot \langle 0, -1, 3 \rangle = (7)(0) + (-3)(-1) + (-1)(3) = 0$, so $\vec{u}$ is also orthogonal to $\vec{w}$.

2. *In general, what is the relationship between* $\vec{v} \times \vec{w}$ *and* $\vec{w} \times \vec{v}$*?*

   Solution. From the definition of $\vec{v} \times \vec{w}$, you can compute directly that $\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$.

   Here's another way to get the same conclusion. Let θ be the angle between $\vec{v}$ and $\vec{w}$. Then, we know that the length of $\vec{v} \times \vec{w}$ is $|\vec{v}||\vec{w}| \sin\theta$, and the length of $\vec{w} \times \vec{v}$ is $|\vec{w}||\vec{v}| \sin\theta$. So, $\vec{v} \times \vec{w}$ and $\vec{w} \times \vec{v}$ have the same length. Also, $\vec{v} \times \vec{w}$ and $\vec{w} \times \vec{v}$ are both orthogonal to both $\vec{v}$ and $\vec{w}$, so they are either the same vector or negatives of each other. The right-hand rule tells us they must point in opposite directions, so $\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$.

3. *Any two vectors* $\vec{v}$ *and* $\vec{w}$ *which are not parallel determine a triangle, as shown. What is the relationship between the area of the triangle and* $\vec{v} \times \vec{w}$*?*



   Solution. The base of the triangle has length $|\vec{w}|$, and the height of the triangle is $|\vec{v}| \sin\theta$, so the area of the triangle is $\frac{1}{2}|\vec{v}||\vec{w}| \sin\theta$, which is equal to $\frac{1}{2}|\vec{v} \times \vec{w}|$.

   (Note that it is NOT correct to say that the area of the triangle is half of the **vector** $\vec{v} \times \vec{w}$; after all, area is a scalar, not a vector. Rather, the area of the triangle is half of the **length** of the vector $\vec{v} \times \vec{w}$).

4. *If* $\vec{v}$ *and* $\vec{w}$ *are parallel, what is* $\vec{v} \times \vec{w}$*?*

   Solution. If $\vec{v}$ and $\vec{w}$ are parallel, then the angle θ between them is either 0 or π. In either case, $\sin\theta = 0$, so $|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}| \sin\theta = 0$. This means that $\vec{v} \times \vec{w}$ must be the zero vector $\vec{0}$.

5. *If the scalar triple product* $\vec{u} \cdot (\vec{v} \times \vec{w})$ *is equal to* 0*, what can you say about the vectors* $\vec{u}$*,* $\vec{v}$*, and* $\vec{w}$*?*

Solution. The fact that $\vec{u} \cdot (\vec{v} \times \vec{w}) = 0$ means that $\vec{u}$ is orthogonal to $\vec{v} \times \vec{w}$. We also know that $\vec{v} \times \vec{w}$ is orthogonal to both $\vec{v}$ and $\vec{w}$, so this means that $\vec{u}, \vec{v}, \vec{w}$ are *all* orthogonal to the vector $\vec{v} \times \vec{w}$. In particular, if we stick the tails of $\vec{u}$, $\vec{v}$, and $\vec{w}$ at the same point, then $\vec{u}$, $\vec{v}$, and $\vec{w}$ all lie in the same plane. (We say that $\vec{u}$, $\vec{v}$, and $\vec{w}$ are coplanar.)

Note: Some students pointed out a special case: $\vec{u} \cdot (\vec{v} \times \vec{w})$ is equal to 0 if $\vec{v}$ and $\vec{w}$ are parallel (since then $\vec{v} \times \vec{w} = \vec{0}$, by #4). In this case, $\vec{u}$, $\vec{v}$, and $\vec{w}$ are still coplanar. To visualize this, imagine the plane that contains $\vec{u}$ and $\vec{v}$: $\vec{w}$ will automatically be in this plane because it's parallel to $\vec{v}$.

6. *Find an equation for the plane which passes through the points* $(1, 0, 1)$, $(0, 2, 0)$, *and* $(2, 1, 0)$.

   Solution. Let's give the three points names, say $P = (1, 0, 1)$, $Q = (0, 2, 0)$, and $R = (2, 1, 0)$. A point $S = (x, y, z)$ is in the plane if (and only if) the three vectors $\overrightarrow{PQ}$, $\overrightarrow{PR}$, and $\overrightarrow{PS}$ are coplanar. As we saw in #5, this is the same as saying that $\overrightarrow{PS} \cdot (\overrightarrow{PQ} \times \overrightarrow{PR}) = 0$.

   So now we compute some things: $\overrightarrow{PQ} = \langle -1, 2, -1 \rangle$ and $\overrightarrow{PR} = \langle 1, 1, -1 \rangle$, so $\overrightarrow{PQ} \times \overrightarrow{PR} = \langle -1, -2, -3 \rangle$. $\overrightarrow{PS} = \langle x-1, y, z-1 \rangle$, so $\overrightarrow{PS} \cdot (\overrightarrow{PQ} \times \overrightarrow{PR}) = 0$ can be rewritten as $-1(x-1) - 2y - 3(z-1) = 0$, or $\boxed{x + 2y + 3z = 4}$.

   Note that it is very easy to check that this answer is correct — the three points given in the problem all satisfy this equation.

7. *True or false: If* $\vec{u} \times \vec{v} = \vec{u} \times \vec{w}$, *then* $\vec{v} = \vec{w}$.

   Solution. False. There are lots of examples where this is not true. A simple one is to suppose that $\vec{u}$, $\vec{v}$, and $\vec{w}$ are all parallel but not equal to each other. Then, $\vec{u} \times \vec{v}$ and $\vec{u} \times \vec{w}$ are both $\vec{0}$, but $\vec{v} \neq \vec{w}$.

8. *True or false: If* $\vec{v} \times \vec{w} = \vec{0}$ *and* $\vec{v} \cdot \vec{w} = 0$, *then at least one of* $\vec{v}$ *and* $\vec{w}$ *must be* $\vec{0}$.

   Solution. True.

   Here's one way to think about it: $\vec{v} \times \vec{w} = \vec{0}$ means $\vec{v}$ and $\vec{w}$ are parallel. $\vec{v} \cdot \vec{w} = 0$ means $\vec{v}$ and $\vec{w}$ are perpendicular. The only way to have a pair of vectors that are both parallel to and perpendicular to each other is if at least one of them is the zero vector $\vec{0}$.

   If you prefer to write out equations, here's another way to think about the problem. Since $\vec{v} \times \vec{w} = \vec{0}$, $|\vec{v} \times \vec{w}| = 0$. If θ is the angle between $\vec{v}$ and $\vec{w}$, then this says that $|\vec{v}||\vec{w}| \sin\theta = 0$. On the other hand, $0 = \vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos\theta$. It's not possible for both $\sin\theta$ and $\cos\theta$ to be 0, so it must be the case that $|\vec{v}||\vec{w}| = 0$. That is, one of the vectors $\vec{v}$ and $\vec{w}$ must have length 0, and the only vector with 0 length is $\vec{0}$.

Lines and Planes

1. Find an equation describing the plane which passes through the points $(2, 2, 1)$, $(3, 1, 0)$, and $(0, -2, 1)$.

2. Find an equation describing the plane which goes through the point $(1, 3, 5)$ and is perpendicular to the vector $\langle 2, 1, -3 \rangle$.

3. Let L be the line which passes through the points $(1, -2, 3)$ and $(4, -5, 6)$.

   (a) Find a parametric vector equation for L.

   (b) Find parametric (scalar) equations for L.

   (c) Find symmetric equations for L.

4. Let L_1 be the line with parametric vector equation $\vec{r}_1(t) = \langle 7, 1, 3\rangle + t\langle 1, 0, -1\rangle$ and L_2 be the line described parametrically by $x = 5, y = 1 + 3t$, $z = t$. How many planes are there which contain L_2 and are parallel to L_1? Find an equation describing one such plane.

5. Find the distance from the point $(0, 1, 1)$ to the plane $2x + 3y + 4z = 15$.

6. Find the distance from the point $(1, 3, -2)$ to the line $\frac{x}{3} = y - 1 = z + 2$.

7. True or false: The line $x = 2t$, $y = 1 + 3t$, $z = 2 + 4t$ is parallel to the plane $x - 2y + z = 7$.

8. True or false: Let S be a plane normal to the vector $\vec{n}$, and let P and Q be points not on the plane S. If $\vec{n} \cdot \overrightarrow{PQ} = 0$, then P and Q lie on the same side of S.

Lines and Planes

1. *Find an equation describing the plane which passes through the points* $(2, 2, 1)$, $(3, 1, 0)$, *and* $(0, -2, 1)$.

   Solution. Let's give the three points names, say $P = (2, 2, 1)$, $Q = (3, 1, 0)$, and $R = (0, -2, 1)$. A point $S(x, y, z)$ is in the plane if (and only if) the three vectors $\overrightarrow{PQ}$, $\overrightarrow{PR}$, and $\overrightarrow{PS}$ are coplanar. As we saw before, this is the same as saying that the scalar triple product $\overrightarrow{PS} \cdot (\overrightarrow{PQ} \times \overrightarrow{PR})$ is equal to 0.

   Now, we can just write this all out: $\overrightarrow{PQ} = \langle 1, -1, -1\rangle$, $\overrightarrow{PR} = \langle -2, -4, 0\rangle$, so $\overrightarrow{PQ} \times \overrightarrow{PR} = \langle -4, 2, -6\rangle$. $\overrightarrow{PS} = \langle x - 2, y - 2, z - 1\rangle$, so the equation $\overrightarrow{PS} \cdot (\overrightarrow{PQ} \times \overrightarrow{PR}) = 0$ can be rewritten as $-4(x - 2) + 2(y - 2) - 6(z - 1) = 0$, or $4x - 2y + 6z = 10$. We can make this equation slightly simpler by dividing through by 2 to get $\boxed{2x - y + 3z = 5}$.

2. *Find an equation describing the plane which goes through the point* $(1, 3, 5)$ *and is perpendicular to the vector* $\langle 2, 1, -3\rangle$.

   Solution. Let's give the point $(1, 3, 5)$ a name; we'll call it P. A point $S(x, y, z)$ is on the plane if the vector $\overrightarrow{PS}$ is perpendicular to the vector $\langle 2, 1, -3\rangle$. That is, $\langle x - 1, y - 3, z - 5\rangle \cdot \langle 2, 1, -3\rangle = 0$. Simplifying, $2(x - 1) + 1(y - 3) - 3(z - 5) = 0$, which can also be written as $\boxed{2x + y - 3z = -10}$.

3. Let L be the line which passes through the points $(1, -2, 3)$ and $(4, -5, 6)$.

   (a) *Find a parametric vector equation for* L.

   Solution. Let's call the points $P(1, -2, 3)$ and $Q(4, -5, 6)$. The line we are looking for passes through P and is parallel to $\overrightarrow{PQ} = \langle 3, -3, 3\rangle$, so we can describe it using the parametric vector equation $\vec{r}(t) = \langle 1, -2, 3\rangle + t\langle 3, -3, 3\rangle$.

   (b) *Find parametric (scalar) equations for* L.

   Solution. From (a), $\vec{r}(t) = \langle 1 + 3t, -2 - 3t, 3 + 3t\rangle$, so we could also write the line as $x = 1 + 3t$, $y = -2 - 3t$, and $z = 3 + 3t$.

   (c) *Find symmetric equations for* L.

   Solution. To find symmetric equations, we solve the parametric scalar equations we found in (b) for t: $t = \frac{x-1}{3}$, $t = \frac{-2-y}{3}$, and $t = \frac{z-3}{3}$. Then, we set these equal to each other: $\frac{x-1}{3} = \frac{-2-y}{3} = \frac{z-3}{3}$. Of course, we could multiply these all by 3 to get the slightly simpler looking $\boxed{x - 1 = -2 - y = z - 3}$.

4. *Let* L_1 *be the line with parametric vector equation* $\vec{r}_1(t) = \langle 7, 1, 3\rangle + t\langle 1, 0, -1\rangle$ *and* L_2 *be the line described parametrically by* $x = 5, y = 1 + 3t$, $z = t$. *How many planes are there which contain* L_2 *and are parallel to* L_1*? Find an equation describing one such plane.*

   Solution. Let's first rewrite L_2 using a parametric vector equation $\vec{r}_2(t) = \langle 5, 1, 0\rangle + t\langle 0, 3, 1\rangle$.

   From the two parametric vector equations, we can see that:

   - L_1 goes through the point $P(7, 1, 3)$ and is parallel to the vector $\vec{u} = \langle 1, 0, -1\rangle$.
   - L_2 goes through $Q(5, 1, 0)$ and is parallel to the vector $\vec{v} = \langle 0, 3, 1\rangle$.

   Therefore, we are looking for a plane which:

- contains Q and is parallel to $\vec{v}$ (because it contains L_2)
- is parallel to $\vec{u}$ (because it is parallel to L_1)

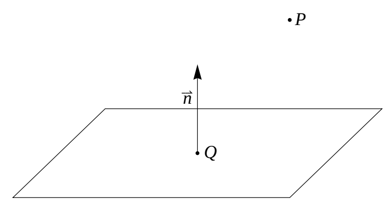
Since such a plane plane is parallel to both $\vec{u}$ and $\vec{v}$, it must be orthogonal to $\vec{u} \times \vec{v}$. We compute that $\vec{u} \times \vec{v} = \langle 3, -1, 3 \rangle$.

There is only one plane which is orthogonal to $\langle 3, -1, 3 \rangle$ and contains the point $(5, 1, 0)$. (We saw this idea already in #2.) Its equation is $\boxed{3x - y + 3z = 14}$.

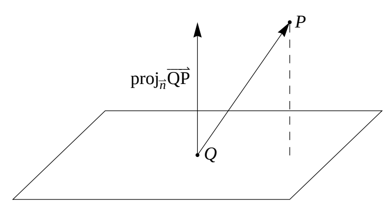
Note: If L_1 and L_2 had been parallel, there would have been infinitely many planes which contained L_2 and were parallel to L_1: any plane containing L_2 would automatically be parallel to L_1.

5. *Find the distance from the point* $(0, 1, 1)$ *to the plane* $2x + 3y + 4z = 15$.

Solution. We have a point $P(0, 1, 1)$ and a plane, and we want to find the distance between the two. Here is one method. Suppose Q is a point in the plane and $\vec{n}$ is a normal vector for the plane.



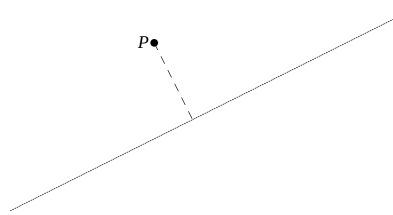
Then, the distance from P to the plane is simply the length of the projection vector $\mathrm{proj}_{\vec{n}} \overrightarrow{QP}$:



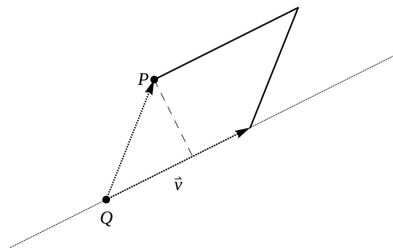
In this particular case, $Q = (0, 5, 0)$ is a point on the plane, and $\vec{n} = \langle 2, 3, 4 \rangle$ is a normal vector for the plane. Then, $\overrightarrow{QP} = \langle 0, -4, 1 \rangle$, and $\mathrm{proj}_{\vec{n}} \overrightarrow{QP} = \frac{\vec{n} \cdot \overrightarrow{QP}}{\vec{n} \cdot \vec{n}} \vec{n} = -\frac{8}{29} \langle 2, 3, 4 \rangle$. The length of this vector is $\boxed{\frac{8}{\sqrt{29}}}$.

6. *Find the distance from the point* $(1, 3, -2)$ *to the line* $\frac{x}{3} = y - 1 = z + 2$.

Solution. We have a point $P(1, 3, -2)$ and a line, and we want to find the distance between the two:



Here's one way to do that. Find a point Q on the line and a vector $\vec{v}$ parallel to the line; then, the distance is the height of the parallelogram determined by $\overrightarrow{QP}$ and $\vec{v}$:



The height of the parallelogram is equal to the area of the parallelogram divided by the length of the base, or $\frac{|\overrightarrow{QP}\times\vec{v}|}{|\vec{v}|}$.

For the given line, it will be easier to find a point on the line and a vector parallel to the line if we rewrite it using a parametric vector equation. To do this, let's set t equal to $\frac{x}{3} = y - 1 = z + 2$. Then, $x = 3t$, $y = 1 + t$, and $z = -2 + t$, so we can describe the line by the parametric vector equation $\langle 0, 1, -2\rangle + t\langle 3, 1, 1\rangle$. From this, we can see that $Q(0, 1, -2)$ is a point on the line and $\vec{v} = \langle 3, 1, 1\rangle$ is a vector parallel to the line.

Now, we just compute $\frac{|\overrightarrow{QP}\times\vec{v}|}{|\vec{v}|}$: $\overrightarrow{QP} = \langle 1, 2, 0\rangle$, so $\overrightarrow{QP} \times \vec{v} = \langle 2, -1, -5\rangle$ and $\dfrac{|\overrightarrow{QP} \times \vec{v}|}{|\vec{v}|} = \boxed{\sqrt{\dfrac{30}{11}}}$.

7. *True or false: The line $x = 2t$, $y = 1 + 3t$, $z = 2 + 4t$ is parallel to the plane $x - 2y + z = 7$.*

   Solution. True.

   A normal vector for the plane is $\vec{n} = \langle 1, -2, 1\rangle$. The line $x = 2t$, $y = 1 + 3t$, $z = 2 + 4t$ is parallel to the vector $\langle 2, 3, 4\rangle$, and this vector is orthogonal to $\vec{n}$, so this vector must be parallel to the plane.

   Another way to see that the line and plane are parallel is to try to compute the intersection. If (x, y, z) is in both the line and plane, then the four equations $x = 2t$, $y = 1+3t$, $z = 2+4t$, and $x-2y+z = 7$ must all be satisfied. Plugging the first three equations into the fourth, $2t - 2(1 + 3t) + (2 + 4t) = 7$, which simplifies to $0 = 7$, so there are no solutions to all 4 equations. This means that the line and plane do not intersect, so they must be parallel.

8. *True or false: Let S be a plane normal to the vector $\vec{n}$, and let P and Q be points not on the plane S. If $\vec{n} \cdot \overrightarrow{PQ} = 0$, then P and Q lie on the same side of S.*

   Solution. True. The fact that $\vec{n} \cdot \overrightarrow{PQ} = 0$ means that $\vec{n}$ is orthogonal to $\overrightarrow{PQ}$, so $\overrightarrow{PQ}$ is parallel to the plane.

Using Projections

1 In the diagram are the vectors $\vec{u}$ and $\vec{v}$. Draw and label $\text{proj}_{\vec{u}}\vec{v}$ and $\text{proj}_{\vec{v}}\vec{u}$

2 (a) What is the distance from $(2, -1, 2)$ to the plane $x + 3y - z = 11$? Can you express it as the length of some vector?

What is the distance from the plane $x + 3y - z = 11$ to the plane $x + 3y - z = 15$?

(b) What is the distance from the plane $x + 3y - z = 11$ to the plane $2x - y - 7z = 10$?

3. What point on the sphere $(x-3)^2 + y^2 + z^2 = 4$ is closest to the plane $5x + 2y + z = -17$?

4. Let L_1 be the line $\frac{x-1}{2} = \frac{y+1}{4} = \frac{z-2}{3}$ and L_2 be the line $\frac{x+2}{2} = \frac{y-1}{-5} = -z$.

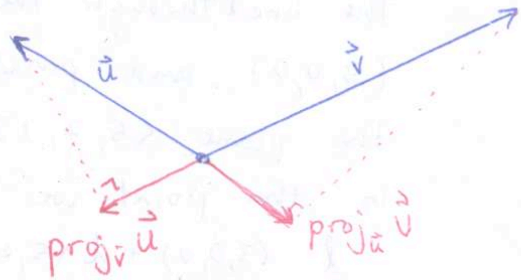
(a) What is the relationship of these two lines L_1 and L_2? Parallel? Skew? Intersecting? How can you be sure?

Check point: What are the possible relationship between any two lines in space?

(b) What is the distance between L_1 and L_2?

Using Projections

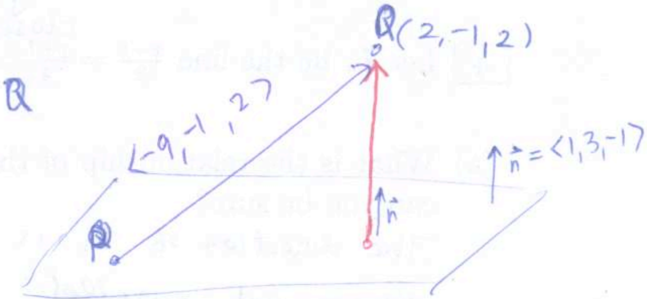
1. In the diagram are the vectors $\vec{u}$ and $\vec{v}$. Draw and label $\text{proj}_{\vec{u}}\vec{v}$ and $\text{proj}_{\vec{v}}\vec{u}$



2. (a) What is the distance from $(2, -1, 2)$ to the plane $x + 3y - z = 11$? Can you express it as the length of some vector?

Another pt. on the plane is P(11, 0, 0)

The vector $\vec{PQ}$ goes from the plane to Q but not perpendicular to the plane



$\text{proj}_{\vec{n}} \vec{PQ}$ is the component of $\vec{PQ}$ that's orthogonal to the plane so the distance is $\left|\text{proj}_{\vec{n}} \vec{PQ}\right| = \left|\text{comp}_{\vec{n}} \vec{PQ}\right| = \left|\frac{\vec{PQ}\cdot\vec{n}}{|\vec{n}|}\right| = \left|\frac{\langle -9,-1,2\rangle\cdot\langle 1,3,-1\rangle}{\sqrt{1+9+1}}\right| = \left|\frac{-14}{\sqrt{11}}\right| = \frac{14}{\sqrt{11}}$

What is the distance from the plane $x + 3y - z = 11$ to the plane $x + 3y - z = 15$?

To find the distance between the two parallel planes, it's enough to find the distance from the first plane to any point on the second. The point (0, 5, 0) is on the second plane. We can compute the distance from (0, 5, 0) to the first plane the same way as in part (a). The answer is $\frac{4}{\sqrt{11}}$

(b) What is the distance from the plane $x + 3y - z = 11$ to the plane $2x - y - 7z = 10$?

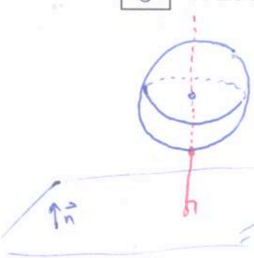
The normal vectors of these planes are

$\langle 1, 3, -1 \rangle$ and $\langle 2, -1, -7 \rangle$

respectively.

Since the planes aren't parallel, they must intersect. The distance between them is 0.

3 What point on the sphere $(x-3)^2+y^2+z^2=4$ is closest to the plane $5x+2y+z=-17$?



The line $\ell: (3,0,0)+t\langle 5,2,1\rangle$ through the center of the sphere and parallel to the plane's normal intersects the sphere in the point we want (and one other!)

Solve $((3+5t)-3)^2+(2t)^2+(t)^2=4$ for this intersection and get $t=\pm\frac{2}{\sqrt{30}}$. You can check for yourself that the point $\left(3-\frac{10}{\sqrt{30}}, \frac{-4}{\sqrt{30}}, -\frac{2}{\sqrt{30}}\right)$ is closer to the plane than $\left(3+\frac{10}{\sqrt{30}}, \frac{4}{\sqrt{30}}, \frac{2}{\sqrt{30}}\right)$.

4 Let L_1 be the line $\frac{x-1}{2}=\frac{y+1}{4}=\frac{z-2}{3}$ and L_2 be the line $\frac{x+2}{2}=\frac{y-1}{-5}=-z$.

direction: $\langle 2,4,3\rangle$ $\qquad$ $\langle 2,-5,-1\rangle$

(a) What is the relationship of these two lines L_1 and L_2? Parallel? Skew? Intersecting? How can you be sure?

Their direction vectors are'nt parallel, so L_1 & L_2 are either skew (if they're not coplanar) or intersect in one point (if they are coplanar)

One vector from a point $(1,-1,2)$ on L_1 to a point on L_2 $(-2,1,0)$ is $\langle -3,2,-2\rangle$. The lines are coplanar if their directions and this third vector are coplanar.

But $\langle -3,2,2\rangle\cdot(\langle 2,4,3\rangle\times\langle 2,-5,-1\rangle)= \qquad \neq 0$ so the vectors aren't coplanar. The lines are skew.

Check point: What are the possible relationship between any two lines in space?

they can be coplanar and either parallel or intersecting

they can be non-coplanar and thus skew

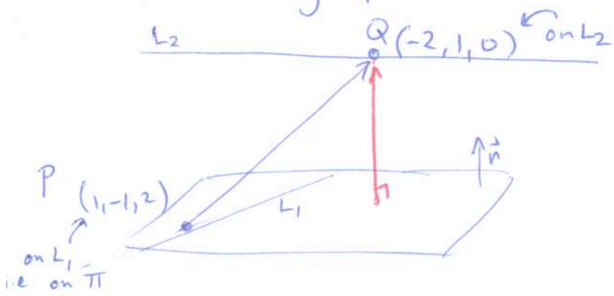
they can be the same line

(b) What is the distance between L_1 and L_2?

The plane that contains L_1 and is parallel to L_2 has parametric eq'n

$$\pi: (1,-1,2)+t\langle 2,4,3\rangle+s\langle 2,-5,-1\rangle$$

The distance from L_2 to L_1 is the same as the distance from L_2 to this plane, π, which is the same as the distance from any point on L_2 to π. The normal to π is $\langle 2,4,3\rangle\times\langle 2,-5,-1\rangle = \langle 11,8,-18\rangle$

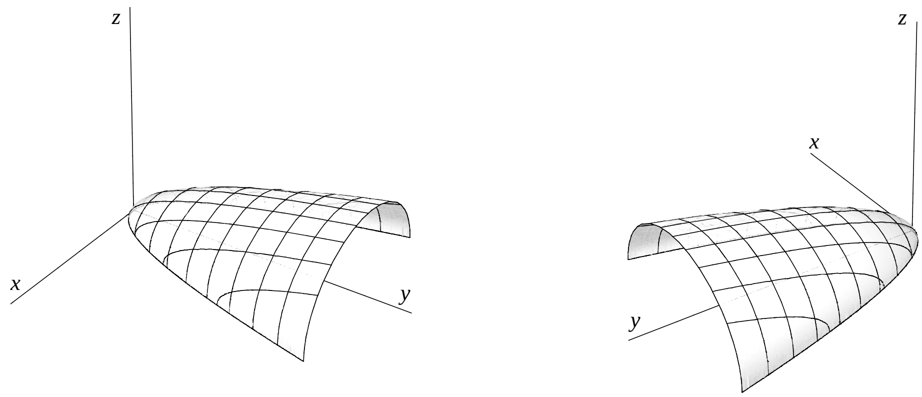


The distance we want is

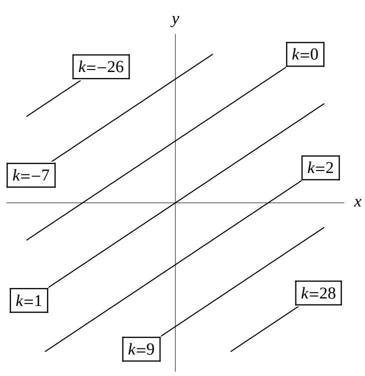
$$\left|\operatorname{proj}_{\vec n}\vec{PQ}\right| = \frac{\langle -3,2,-2\rangle\cdot\langle 11,8,-18\rangle}{\sqrt{11^2+8^2+18^2}} = \frac{19}{\sqrt{509}}.$$

Functions and Graphs

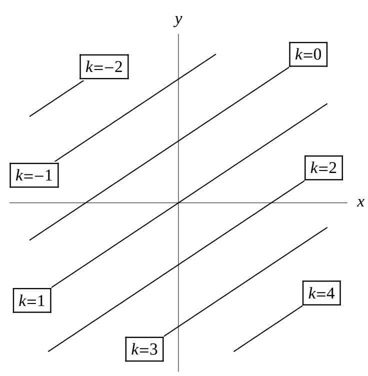
Here is the graph $z = \sqrt{y - x^2}$ of the function $f(x, y) = \sqrt{y - x^2}$, shown from two different angles.



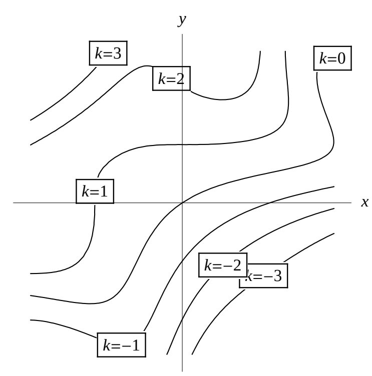
1. The first row shows traces of three graphs $z = f(x, y)$ in the planes $z = k$. (Traces of the graph $z = f(x, y)$ in $z = k$ are also known as level sets of $f(x, y)$.) Match each diagram with the graph of the function.



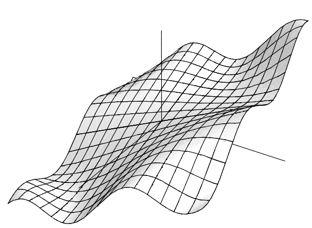
(I)



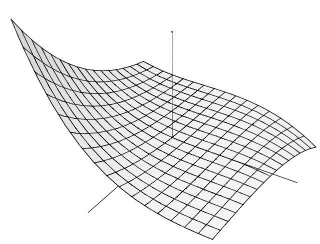
(II)



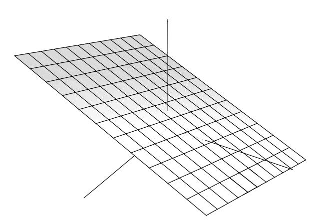
(III)



(a)

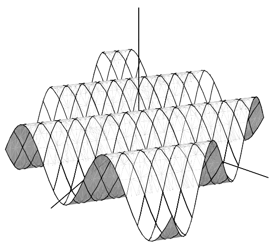


(b)

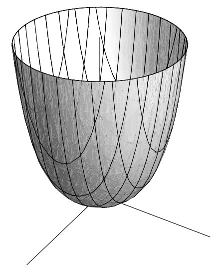


(c)

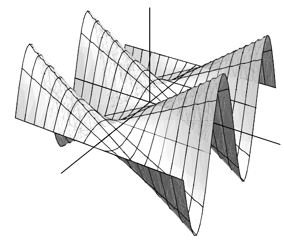
2. Here are several surfaces.



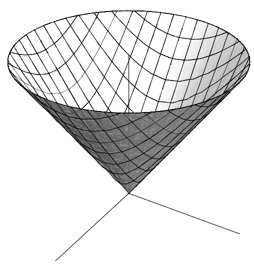
(I)



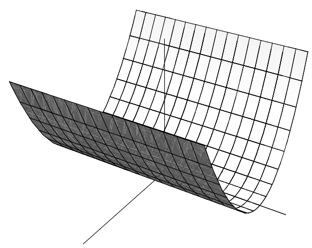
(II)



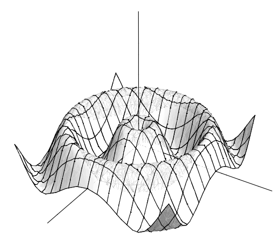
(III)



(IV)



(V)



(VI)

Match each function with its graph.

(a) $f(x,y) = x^2$.

(b) $f(x,y) = \sqrt{x^2+y^2}$.

(c) $f(x,y) = e^{x^2+y^2} - 1$.

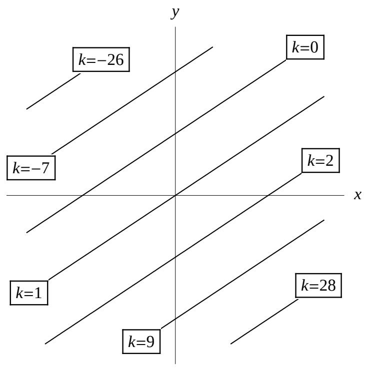
(d) $f(x,y) = y \sin x$.

(e) $f(x,y) = \sin(x+y)$.

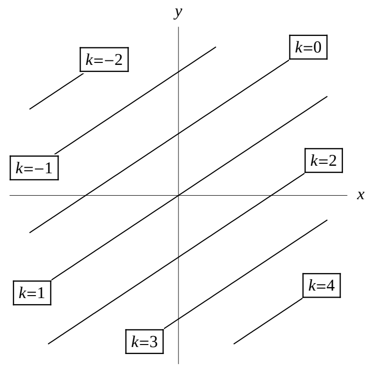
(f) $f(x,y) = \sin\left(\sqrt{x^2+y^2}\right)$.

Functions and Graphs

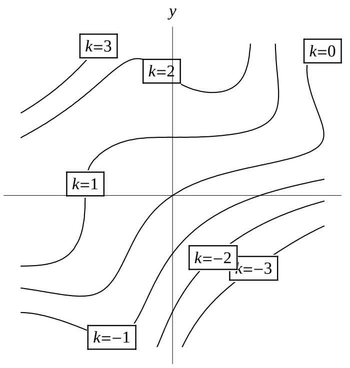
1. The first row shows traces of three graphs $z = f(x, y)$ in the planes $z = k$. (Traces of the graph $z = f(x, y)$ in $z = k$ are also known as level sets of $f(x, y)$.) Match each diagram with the graph of the function.



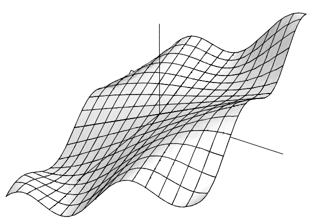
(I)



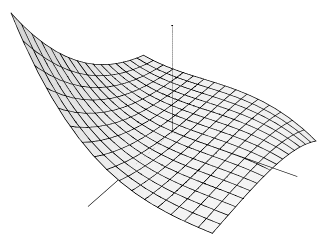
(II)



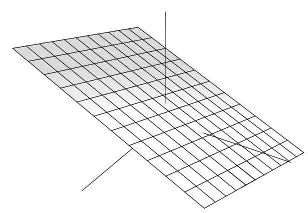
(III)



(a)



(b)



(c)

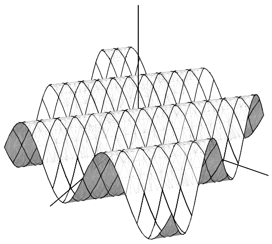
Solution.

(a) The traces of this surface in $z = k$ are not straight lines, which only matches (III). Another way to think about it as that the surface is highest where y is positive and x is negative, which only matches (III).

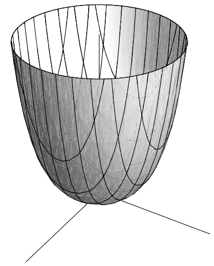
(b) We need to decide whether this graph matches (I) or (II). Both have traces in $z = k$ which are straight lines. The difference is in the z-values. The values in (II) are evenly spaced, while the values in (I) change more quickly the further out we go. In graph (b), the surface changes height more quickly the further out we go, so this matches (I).

(c) Using the same reasoning as in the previous part (or process of elimination), this must match (II).

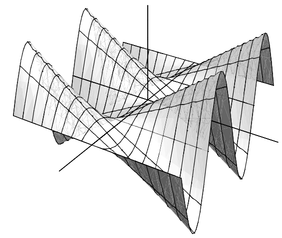
2. Here are several surfaces.



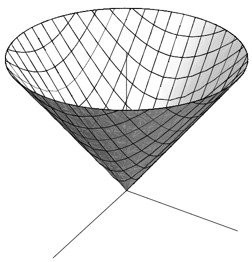
(I)



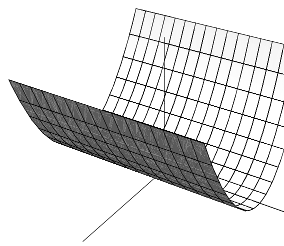
(II)



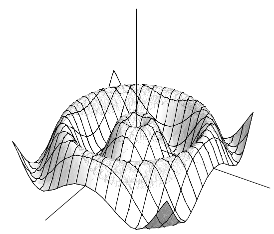
(III)



(IV)



(V)



(VI)

Match each function with its graph.

(a) $f(x, y) = x^2$.

Solution. Notice that $f(x, y)$ does not depend on y. Therefore, each trace of the surface $z = x^2$ in $y = k$ is the same: the parabola $z = x^2$. This matches only $\boxed{\text{(V)}}$. (Also notice that each trace of the surface $z = x^2$ in $x = k$ is just a line $z = k^2$.)

(b) $f(x, y) = \sqrt{x^2 + y^2}$.

Solution. The level set $\sqrt{x^2 + y^2} = k$ is a circle of radius k centered at the origin (as long as $k \geq 0$). Both (II) and (IV) have level sets which are just circles, so let's look at some other traces to help us tell the two apart.

The traces in $x = k$ of the graph $z = \sqrt{x^2 + y^2}$ are $z = \sqrt{y^2 + k^2}$. In particular, the trace in $x = 0$ is $z = \sqrt{y^2} = |y|$, which matches $\boxed{\text{(IV)}}$ but does not match (II).

(c) $f(x, y) = e^{x^2+y^2} - 1$.

Solution. The level set $e^{x^2+y^2} - 1 = k$ can be rewritten as $x^2 + y^2 = \ln(k + 1)$, so it is a circle of radius $\sqrt{\ln(k + 1)}$ centered at the origin (as long as $k \geq 0$). There are two graphs whose level sets are circles, (II) and (IV), but we have already eliminated (IV), so this must match $\boxed{\text{(II)}}$.

(d) $f(x, y) = y \sin x$.

Solution. In this case, it looks like the traces in $x = k$ and $y = k$ may be easier to understand than the level sets, so let's start there. Traces of $z = y \sin x$ in $x = k$ are lines $z = (\sin k)y$. Traces of $z = y \sin x$ in $y = k$ are sinusoids $z = k \sin x$ (graphs of $\sin x$ scaled vertically by a constant k). This matches $\boxed{\text{(III)}}$.

(e) $f(x, y) = \sin(x + y)$.

Solution. The level set $\sin(x + y) = k$ consists of infinitely many parallel lines, which matches $\boxed{\text{(I)}}$.

To see why each level set is a bunch of parallel lines, look at the example $\sin(x + y) = \frac{1}{2}$. This is satisfied when $x + y = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \ldots$. (A more precise way of writing this is to say that $x + y = \frac{\pi}{6} + 2\pi n$ or $\frac{5\pi}{6} + 2\pi n$ for any integer n. That is, $x + y$ differs from either $\frac{\pi}{6}$ or $\frac{5\pi}{6}$ by a multiple of 2π.) This describes a bunch of parallel lines.

(f) $f(x, y) = \sin\left(\sqrt{x^2 + y^2}\right)$.

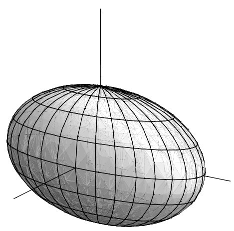
Solution. Let's look at the level set $\sin\left(\sqrt{x^2 + y^2}\right) = k$. This is a set of concentric circles centered at the origin, which matches graph $\boxed{\text{(VI)}}$.

To see why each level set is a bunch of concentric circles, look at the example $\sin\left(\sqrt{x^2 + y^2}\right) = \frac{1}{2}$. This equation is satisfied when $\sqrt{x^2 + y^2} = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \ldots$, which describes circles centered at the origin with radii $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \ldots$. So, the level set $\sin\left(\sqrt{x^2 + y^2}\right) = \frac{1}{2}$ consists of infinitely many circles centered at the origin.

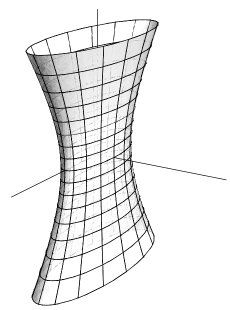
Quadric Surfaces

Six basic types of quadric surfaces:

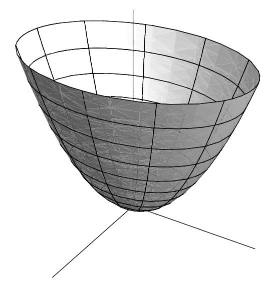
- ellipsoid
- cone
- elliptic paraboloid
- hyperboloid of one sheet
- hyperboloid of two sheets
- hyperbolic paraboloid



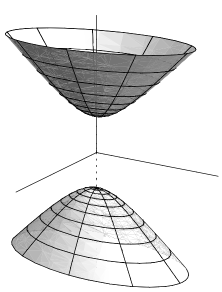
(A)



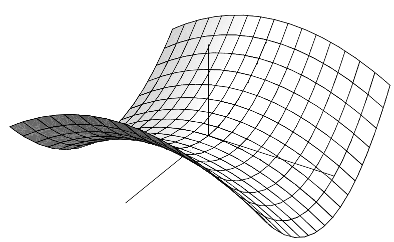
(B)



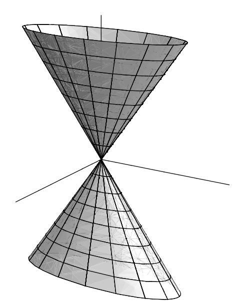
(C)



(D)



(E)



(F)

Quick reminder:

- $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = r$ describes ...
  - an ellipse if $r > 0$.
  - a point if $r = 0$ (we consider this a "degenerate" ellipse).
  - nothing if $r < 0$.
- $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = r$ describes ...
  - a hyperbola if $r \neq 0$.
  - a pair of lines if $r = 0$ (we consider this a "degenerate" hyperbola).
- $y = ax^2 + b$ describes a parabola.

1. For each surface, describe the traces of the surface in $x = k$, $y = k$, and $z = k$. Then pick the term from the list above which seems to most accurately describe the surface (we haven't learned any of these terms yet, but you should be able to make a good educated guess), and pick the correct picture of the surface.

   (a) $\dfrac{x^2}{9} - \dfrac{y^2}{16} = z.$

   - Traces in $x = k$:
   - Traces in $y = k$:
   - Traces in $z = k$:

   (b) $\dfrac{x^2}{4} + \dfrac{y^2}{25} + \dfrac{z^2}{9} = 1.$

   - Traces in $x = k$:
   - Traces in $y = k$:
   - Traces in $z = k$:

   (c) $\dfrac{x^2}{4} + \dfrac{y^2}{9} = \dfrac{z}{2}.$

   - Traces in $x = k$:
   - Traces in $y = k$:
   - Traces in $z = k$:

   (d) $\dfrac{z^2}{4} - x^2 - \dfrac{y^2}{4} = 1.$

   - Traces in $x = k$:
   - Traces in $y = k$:
   - Traces in $z = k$:

   (e) $x^2 + \dfrac{y^2}{9} = \dfrac{z^2}{16}.$

   - Traces in $x = k$:
   - Traces in $y = k$:
   - Traces in $z = k$:

(f) $\dfrac{x^2}{9} + y^2 - \dfrac{z^2}{16} = 1.$

- Traces in $x = k$:
- Traces in $y = k$:
- Traces in $z = k$:

2. Sketch the surface $9y^2 + 4z^2 = 36$. What type of quadric surface is it?

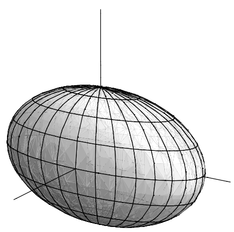
3. Sketch the surface $y^2 + 2y + z^2 = x^2$. What type of quadric surface is it?

4. What type of quadric surface is $4x^2 - y^2 + z^2 + 9 = 0$?

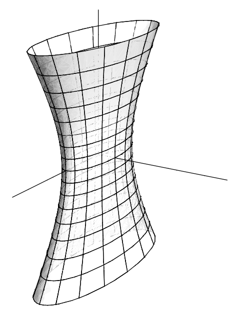
Quadric Surfaces

Six basic types of quadric surfaces:

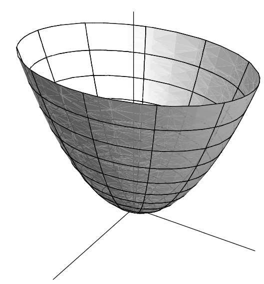
- ellipsoid
- cone
- elliptic paraboloid
- hyperboloid of one sheet
- hyperboloid of two sheets
- hyperbolic paraboloid



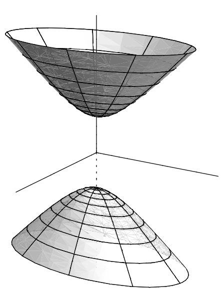
(A)



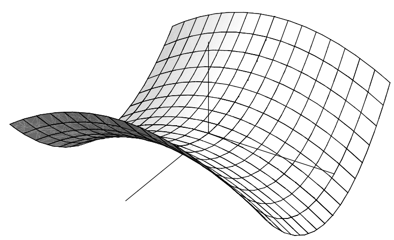
(B)



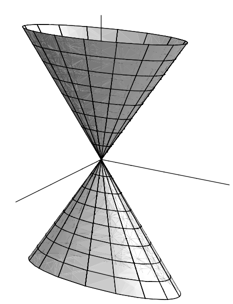
(C)



(D)



(E)



(F)

1. For each surface, describe the traces of the surface in $x = k$, $y = k$, and $z = k$. Then pick the term from the list above which seems to most accurately describe the surface (we haven't learned any of these terms yet, but you should be able to make a good educated guess), and pick the correct picture of the surface.

   (a) $\dfrac{x^2}{9} - \dfrac{y^2}{16} = z.$

- *Traces in $x = k$:* **parabolas**
- *Traces in $y = k$:* **parabolas**
- *Traces in $z = k$:* **hyperbolas (possibly a pair of lines)**

Solution. The trace in $x = k$ of the surface is $z = -\frac{y^2}{16} + \frac{k^2}{9}$, which is a downward-opening parabola.

The trace in $y = k$ of the surface is $z = \frac{x^2}{9} - \frac{k^2}{16}$, which is an upward-opening parabola.

The trace in $z = k$ of the surface is $\frac{x^2}{9} - \frac{y^2}{16} = k$, which is a hyperbola if $k \neq 0$ and a pair of lines (a degenerate hyperbola) if $k = 0$.

This surface is called a $\boxed{\text{hyperbolic paraboloid}}$, and it looks like picture $\boxed{\text{(E)}}$. It is also sometimes called a saddle.

(b) $\dfrac{x^2}{4} + \dfrac{y^2}{25} + \dfrac{z^2}{9} = 1.$

- *Traces in $x = k$:* **ellipses (possibly a point) or nothing**
- *Traces in $y = k$:* **ellipses (possibly a point) or nothing**
- *Traces in $z = k$:* **ellipses (possibly a point) or nothing**

Solution. The trace in $x = k$ of the surface is $\frac{y^2}{25} + \frac{z^2}{9} = 1 - \frac{k^2}{4}$. Since $1 - \frac{k^2}{4}$ can be positive, zero, or negative (depending on what k is), this intersection can be an ellipse, a point (a degenerate ellipse), or nothing.

The trace in $y = k$ of the surface is $\frac{x^2}{4} + \frac{z^2}{9} = 1 - \frac{k^2}{25}$, which is an ellipse, a point, or nothing.

The trace in $z = k$ of the surface is $\frac{x^2}{4} + \frac{y^2}{25} = 1 - \frac{k^2}{9}$, which is an ellipse, a point, or nothing.

The picture which matches this description is $\boxed{\text{(A)}}$, and this surface is an $\boxed{\text{ellipsoid.}}$

(c) $\dfrac{x^2}{4} + \dfrac{y^2}{9} = \dfrac{z}{2}.$

- *Traces in $x = k$:* **parabolas**
- *Traces in $y = k$:* **parabolas**
- *Traces in $z = k$:* **ellipses (possibly a point) or nothing**

Solution. The trace in $x = k$ of this surface is $z = \frac{2y^2}{9} - \frac{k^2}{2}$, which is a parabola.

The trace in $y = k$ of this surface is $z = \frac{x^2}{2} - \frac{2k^2}{9}$, which is a parabola.

The trace in $z = k$ of this surface is $\frac{x^2}{4} + \frac{y^2}{9} = \frac{k}{2}$, which is an ellipse when $k > 0$. If $k = 0$, it is a point (a degenerate ellipse), and if $k < 0$, it is nothing.

This matches picture $\boxed{\text{(C)}}$, and this surface is called an $\boxed{\text{elliptic paraboloid}}$.

(d) $\dfrac{z^2}{4} - x^2 - \dfrac{y^2}{4} = 1.$

- *Traces in $x = k$:* **hyperbolas (never a pair of lines)**
- *Traces in $y = k$:* **hyperbolas (never a pair of lines)**
- *Traces in $z = k$:* **ellipses (possibly a point) or nothing**

Solution. The trace in $x = k$ of this surface is $\frac{z^2}{4} - \frac{y^2}{4} = 1 + k^2$, which is a hyperbola. Since $1 + k^2$ is always positive, this is never a pair of lines.

The trace in $y = k$ of this surface is $\frac{z^2}{4} - x^2 = 1 + \frac{k^2}{4}$, which is a hyperbola (and never a pair of lines).

The trace in $z = k$ of this surface is $x^2 + \frac{y^2}{4} = \frac{k^2}{4} - 1$. Since $\frac{k^2}{4} - 1$ can be positive, zero, or negative, this intersection can be an ellipse, a point, or nothing.

The pictures (B), (D), and (F) all show surfaces whose traces in $x = k$ and $y = k$ are hyperbolas and whose traces in $z = k$ are ellipses. Of these three graphs, (D) is the only one in which some traces in $z = k$ are nothing, so the correct picture is $\boxed{\text{(D)}}$. This surface is an example of a $\boxed{\text{hyperboloid of two sheets}}$.

(e) $x^2 + \dfrac{y^2}{9} = \dfrac{z^2}{16}$.

- *Traces in $x = k$:* **hyperbolas (possibly a pair of lines)**
- *Traces in $y = k$:* **hyperbolas (possibly a pair of lines)**
- *Traces in $z = k$:* **ellipses (possibly a point)**

Solution. The trace in $x = k$ of this surface is $\frac{z^2}{16} - \frac{y^2}{9} = k^2$. Since k^2 can be either non-zero or zero, this is either a hyperbola or a pair of lines. (It is only a pair of lines when $k = 0$; otherwise, it is a hyperbola.)

The trace in $y = k$ of this surface is $\frac{z^2}{16} - x^2 = \frac{k^2}{9}$, which is again either a hyperbola or a pair of lines (a pair of lines only when $k = 0$).

The trace in $z = k$ of this surface is $x^2 + \frac{y^2}{9} = \frac{k^2}{16}$. Since $\frac{k^2}{16}$ is either positive or 0, this intersection is either an ellipse or a point (but never nothing, and it is only a point when $k = 0$).

This matches picture $\boxed{\text{(F)}}$ and is an example of an $\boxed{\text{cone}}$.

(f) $\dfrac{x^2}{9} + y^2 - \dfrac{z^2}{16} = 1$.

- *Traces in $x = k$:* **hyperbolas (possibly a pair of lines)**
- *Traces in $y = k$:* **hyperbolas (possibly a pair of lines)**
- *Traces in $z = k$:* **ellipses (never a point)**

Solution. The trace in $x = k$ of this surface is $y^2 - \frac{z^2}{16} = 1 - \frac{k^2}{9}$. Since $1 - \frac{k^2}{9}$ can be non-zero or zero, this is either a hyperbola or a pair of lines.

The trace in $y = k$ of this surface is $\frac{x^2}{9} - \frac{z^2}{16} = 1 - k^2$, which is either a hyperbola or a pair of lines.

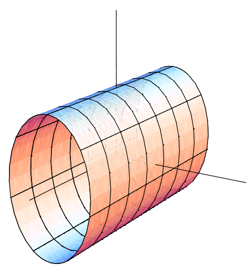
The trace in $z = k$ of this surface is $\frac{x^2}{9} + y^2 = 1 + \frac{k^2}{16}$. Since $1 + \frac{k^2}{16}$ is always positive, this is always an ellipse (and never a point or nothing).

This description matches picture $\boxed{\text{(B)}}$, and this surface is a $\boxed{\text{hyperboloid of one sheet}}$.

2. *Sketch the surface $9y^2 + 4z^2 = 36$. What type of quadric surface is it?*

Solution. Notice that the equation describing this surface does not involve x at all. Each trace of the surface in $x = k$ is the ellipse $9y^2 + 4z^2 = 36$, which can also be written as $\frac{y^2}{4} + \frac{z^2}{9} = 1$.

Therefore, the surface looks like this:



This is not any of the quadrics we have seen so far; it is a $\boxed{\text{(quadric) cylinder}}$.

3. *Sketch the surface $y^2 + 2y + z^2 = x^2$. What type of quadric surface is it?*

Solution. We have a y^2 term as well as a y term, and we can make the equation simpler by completing the square. In this case, we accomplish that by adding 1 to both sides of the equation:

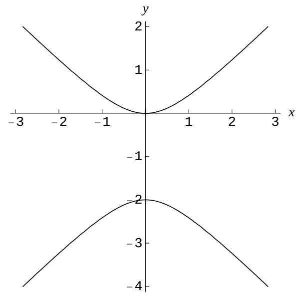
$$\begin{aligned} y^2 + 2y + 1 + z^2 &= x^2 + 1 \\ (y+1)^2 + z^2 &= x^2 + 1 \\ -x^2 + (y+1)^2 + z^2 &= 1 \end{aligned}$$

To sketch, let's look at the traces:

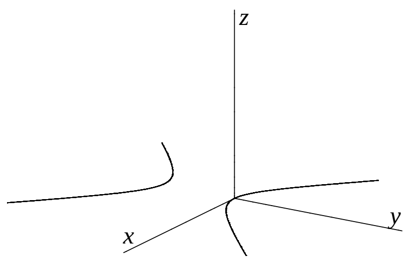
- The trace in $x = k$ of the surface is $(y+1)^2 + z^2 = 1 + k^2$, which describes a circle (centered at $y = -1$, $z = 0$ and having radius $\sqrt{1+k^2}$).
- The trace in $y = k$ of the surface is $-x^2 + z^2 = 1 - (k+1)^2$, which is a hyperbola (or a pair of lines).
- The trace in $z = k$ of the surface $-x^2 + (y+1)^2 = 1 - k^2$, which is a hyperbola (or a pair of lines).

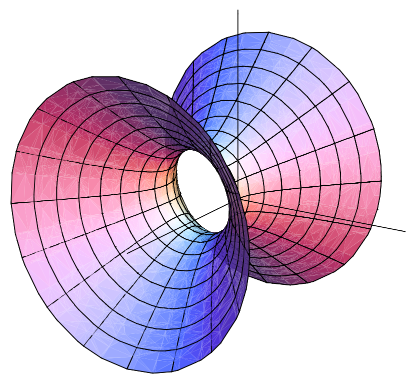
Since we know the traces in $x = k$ are circles centered at $y = -1$, $z = 0$, we can think of the surface as being obtained by rotating some sort of curve about the line $y = -1$, $z = 0$ (this line is parallel to the x-axis). To figure out what that curve is, we can look at the trace of the surface in either $y = -1$ or $z = 0$.

Let's do the trace in $z = 0$: this is $(y+1)^2 - x^2 = 1$, which is the following hyperbola:



If we draw this in space (in the plane $z = 0$, since we're talking about the trace in $z = 0$), it looks like the left figure below. Therefore, the surface looks like the right figure.





This is a hyperboloid of one sheet.

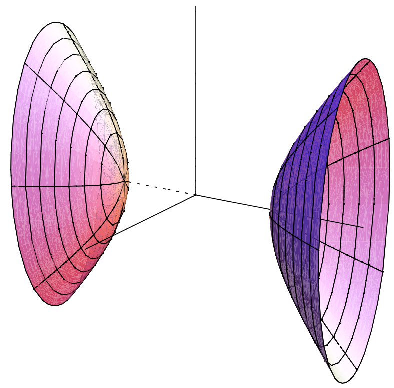
4. *What type of quadric surface is* $4x^2 - y^2 + z^2 + 9 = 0$*?*

Solution. Let's look at the traces:

- The trace in $x = k$ of this surface is $y^2 - z^2 = 4k^2 + 9$, which is always a hyperbola.
- The trace in $y = k$ of this surface is $4x^2 + z^2 = k^2 - 9$, which is an ellipse, a point, or nothing, depending on k.
- The trace in $z = k$ of this surface is $y^2 - 4x^2 = k^2 + 9$, which is always a hyperbola.

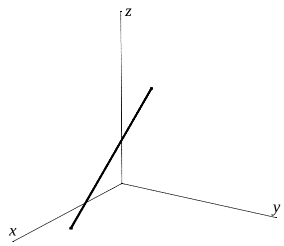
Since the traces in $x = k$ and $z = k$ are hyperbolas, we know the surface is either a hyperboloid or a cone. The fact that the traces in $y = k$ can be nothing tells us that the surface must be a hyperboloid of two sheets.

The surface looks like this:

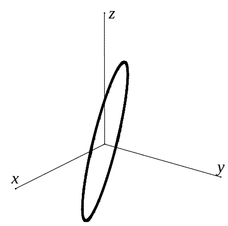


Vector-Valued Functions

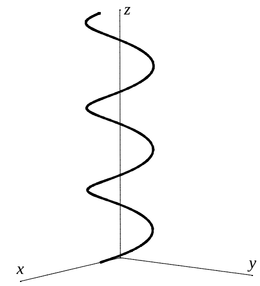
1. Here are several curves.



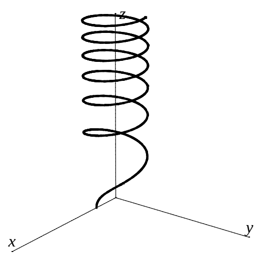
(I)



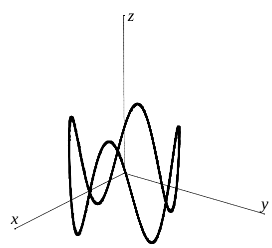
(II)



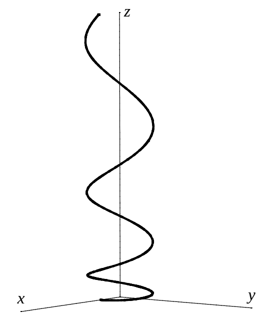
(III)



(IV)



(V)



(VI)

Find the curve parameterized by each vector-valued function.

(a) $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$.

(b) $\vec{r}(s) = \langle \cos s, \sin s, \sin 4s \rangle$.

(c) $\vec{r}(s) = \langle \cos s, \sin s, 4 \sin s \rangle$.

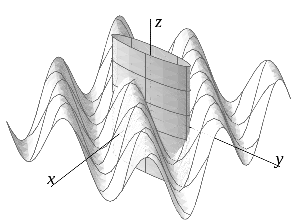
(d) $\vec{r}(u) = \langle \cos u^3, \sin u^3, u^3 \rangle$.

(e) $\vec{r}(u) = \langle 3 + 2\cos u, 1 + 4\cos u, 2 + 5\cos u \rangle$.

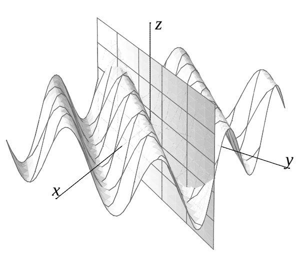
2. Let L be the line tangent to curve (III) at the point $(1, 0, 2\pi)$. Find parametric equations for L.

3. A fly is sitting on the wall at the point $(0, 1, 3)$. At time $t = 0$, he starts flying; his velocity at time t is given by $\vec{v}(t) = \langle \cos 2t, e^t, \sin t \rangle$. Find the fly's location at time t.

4. (a) The surfaces $9x^2 + \frac{y^2}{4} = 1$ and $z = \sin(x - y)$ intersect in a curve. Find a parameterization of the curve.

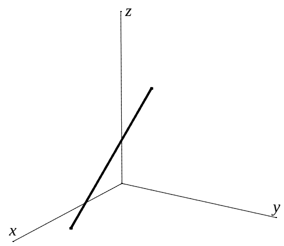


(b) The surfaces $z = \sin(x - y)$ and $y = 2x$ intersect in a curve. Find a parameterization of the curve.

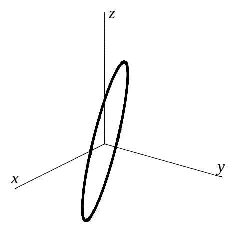


Vector-Valued Functions

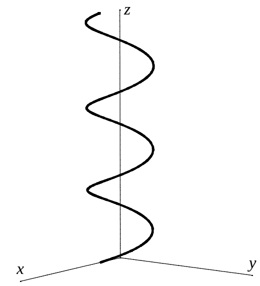
1. Here are several curves.



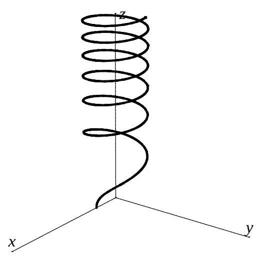
(I)



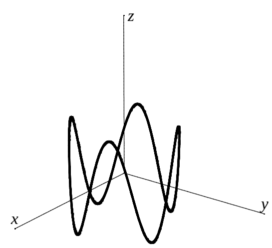
(II)



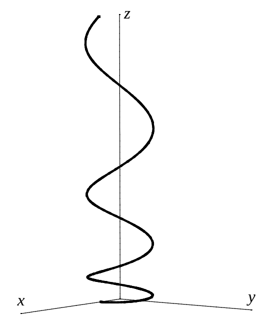
(III)



(IV)



(V)



(VI)

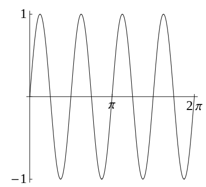
Find the curve parameterized by each vector-valued function.

(a) $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$.

Solution. Remember that we visualize this by imagining a particle in space whose position at time t is $(\cos t, \sin t, t)$. Let's first just think about what the particle's x- and y-components are doing. (One way to visualize this is to imagine looking down on the particle from above; then you can't really see what its height is, so you're looking only at what its x- and y-components are doing.) We know that $x = \cos t$, $y = \sin t$ traces out a circle (the unit circle $x^2 + y^2 = 1$), repeating its path every 2π. Since the z component is just t, each time the particle's x- and y-components trace out a circle, the particle rises by 2π. This matches picture $\boxed{\text{(III)}}$.

(b) $\vec{r}(s) = \langle \cos s, \sin s, \sin 4s \rangle$.

Solution. The x- and y-components here are the same as in (a). However, from time $s = 0$ to $s = 2\pi$, the particle's height is given by $\sin 4s$, whose graph looks like this:



That is, each time the x- and y-components of the particle make a loop, the height should rise and fall four times. This matches (V).

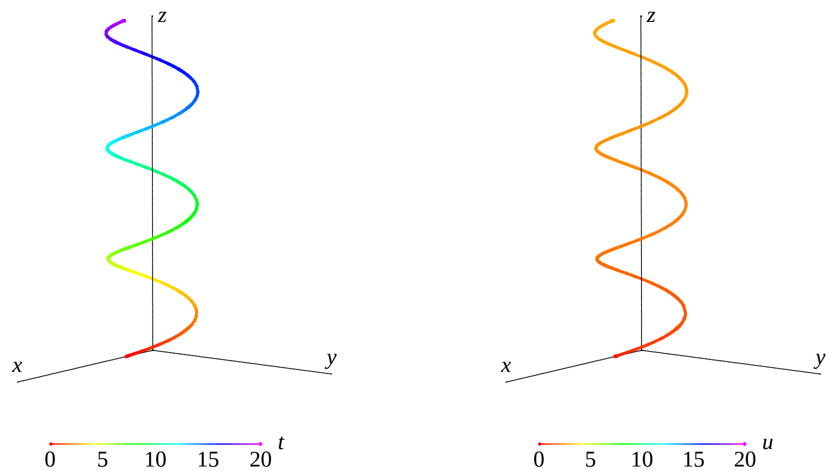
(c) $\vec{r}(s) = \langle \cos s, \sin s, 4\sin s\rangle$.

Solution. Using the same line of reasoning as in (a) and (b), we see that this is (II). Notice that the entire curve appears to lie in a single plane. This is indeed the case, which is easy to check from the equations: if $x = \cos s$, $y = \sin s$, and $z = 4\sin s$, then z is always equal to $4y$, so the curve sits in the plane $z = 4y$.

(d) $\vec{r}(u) = \langle \cos u^3, \sin u^3, u^3\rangle$.

Solution. Observe that the particle's position at time $t = u^3$ is just $(\cos t, \sin t, t)$, which is exactly the same as in (a). Thus, the curve traced out by this function is again (III). The difference between this function and the one in part (a) is the time it takes the particle to reach a given point on the curve. (Another way of saying the same thing is that a particle traveling according to the function in (a) and a particle traveling according to the function here travel the same path, but they go at different speeds.)

Here is a visual illustration:



The left picture shows $\vec{r}(t) = \langle \cos t, \sin t, t\rangle$, with the value of t at a particular point on the curve indicated by color. A particle traveling according to this parameterization reaches the top point at time $t = 6\pi$ ($6\pi \approx 18.8$, so this point is colored purple).

On the other hand, a particle traveling according to the parameterization $\vec{r}(u) = \langle \cos u^3, \sin u^3, u^3\rangle$ reaches the top point much more quickly (in fact, at time $u = \sqrt[3]{6\pi} \approx 2.7$), as shown in the right picture.

(e) $\vec{r}(u) = \langle 3 + 2\cos u, 1 + 4\cos u, 2 + 5\cos u \rangle$.

Solution. This is simpler than it looks. At time $t = \cos u$, the particle is at the point $(3 + 2t, 1 + 4t, 2 + 5t)$. You should recognize this as parameterizing a line, so the correct picture is $\boxed{\text{(I)}}$.

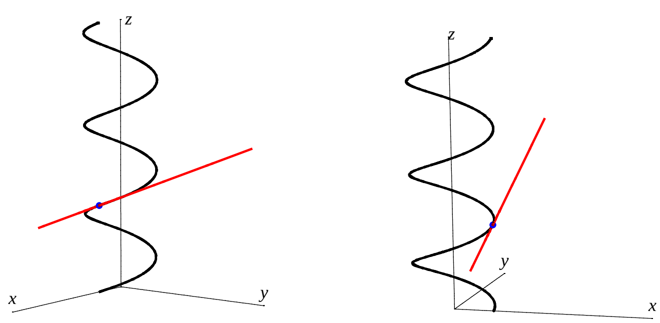
Notice that, when $u = 0$, the particle is at $(5, 5, 7)$; when $u = \pi$, the particle is at $(1, -3, -3)$. When $u = 2\pi$, the particle is back at $(5, 5, 7)$. Since $t = \cos u$ oscillates between 1 and -1 forever, the particle just travels back and forth along the line segment between $(5, 5, 7)$ and $(1, -3, -3)$.

2. *Let L be the line tangent to curve (III) at the point $(1, 0, 2\pi)$. Find parametric equations for L.*

Solution. In #1, we saw that curve (III) could be parameterized by the vector-valued function $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$. If we use this parameterization, then the point $(1, 0, 2\pi)$ corresponds to $t = 2\pi$. The derivative $\vec{r}'(t)$ is equal to $\langle -\sin t, \cos t, 1 \rangle$, so $\vec{r}'(2\pi) = \langle 0, 1, 1 \rangle$. This gives a vector which is parallel to the tangent line, and $(1, 0, 2\pi)$ is a point on the tangent line.

To summarize, the tangent line contains the point $(1, 0, 2\pi)$ and is parallel to $\langle 0, 1, 1 \rangle$. We know that such a line is given parametrically by the vector equation $\vec{f}(t) = \langle 1, 0, 2\pi \rangle + t\langle 0, 1, 1 \rangle$, or by the scalar equations $\boxed{x = 1,\ y = t,\ z = 2\pi + t}$.

Here are the line and the curve (the blue point is $(1, 0, 2\pi)$), shown from two different angles.



3. *A fly is sitting on the wall at the point $(0, 1, 3)$. At time $t = 0$, he starts flying; his velocity at time t is given by $\vec{v}(t) = \langle \cos 2t, e^t, \sin t \rangle$. Find the fly's location at time t.*

Solution. Let $\vec{r}(t)$ be the fly's location at time t. We know that $\vec{r}'(t) = \vec{v}(t)$, so $\vec{r}(t)$ is an antiderivative of $\vec{v}(t)$. Since we differentiate component by component, we can also antidifferentiate component by component, so $\vec{r}(t) = \left\langle \frac{1}{2}\sin 2t + C_1, e^t + C_2, -\cos t + C_3 \right\rangle$ where C_1, C_2, and C_3 are constants which are still to be determined.

To find the constants, we need an initial condition. In this case, we know that the fly's position at time 0 is $(0, 1, 3)$, so $\vec{r}(0) = \langle 0, 1, 3 \rangle$. Plugging $t = 0$ into our expression for $\vec{r}(t)$ gives $\langle 0, 1, 3 \rangle = \vec{r}(0) = \langle C_1, 1 + C_2, -1 + C_3 \rangle$, so $C_1 = 0$, $C_2 = 0$, and $C_3 = 4$. Therefore, the fly's location at time t is $\vec{r}(t) = \left\langle \frac{1}{2}\sin 2t, e^t, -\cos t + 4 \right\rangle$.

4. (a) *The surfaces $9x^2 + \frac{y^2}{4} = 1$ and $z = \sin(x - y)$ intersect in a curve. Find a parameterization of the curve.*

Solution. Notice that it is easy to express z in terms of x and y: $z = \sin(x - y)$. Therefore, if we can express both x and y in terms of a parameter t, we will automatically be able to express z in terms of t as well.

So, let's focus on the relationship between x and y, which is given by the equation $9x^2 + \frac{y^2}{4} = 1$. If we rewrite this as $(3x)^2 + (y/2)^2 = 1$, then we see that we can write $3x = \cos t$, $y/2 = \sin t$, or $x = \frac{1}{3}\cos t$ and $y = 2\sin t$.

Since $z = \sin(x - y)$, we now have $z = \sin\left(\frac{1}{3}\cos t - 2\sin t\right)$. We can also write this as the vector-valued function $\boxed{\vec{r}(t) = \left\langle \frac{1}{3}\cos t, 2\sin t, \sin\left(\frac{1}{3}\cos t - 2\sin t\right)\right\rangle}$.

(This is certainly not the only correct answer; there are infinitely many ways to parameterize a given curve.)

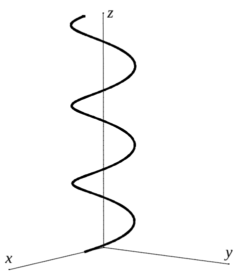
(b) *The surfaces* $z = \sin(x - y)$ *and* $y = 2x$ *intersect in a curve. Find a parameterization of the curve.*

Solution. As in the previous part, it's easy to express z in terms of x and y, so we should focus on writing x and y in terms of a parameter t. Notice, however, that this time it's also easy to write y in terms of x: $y = 2x$.

Therefore, we can simply let x be the parameter, $x = t$. Then, $y = 2x = 2t$, and $z = \sin(x - y) = \sin(t - 2t) = \sin(-t)$. Written as a vector-valued function, $\boxed{\vec{r}(t) = \langle t, 2t, \sin(-t)\rangle}$.

Arc Length and Curvature

1. Last time, we saw that $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$ parameterized the pictured curve.



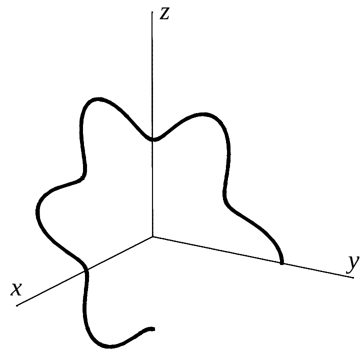
(a) Find the arc length of the curve between $(1, 0, 0)$ and $(1, 0, 2\pi)$.

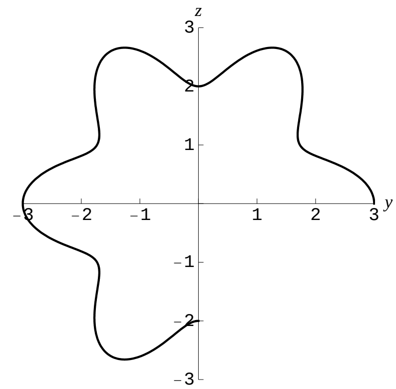
(b) Find the unit tangent vector at the point $(1, 0, 2\pi)$.

(c) Find the curvature at the point $(1, 0, 2\pi)$.

(d) Find the unit normal vector at the point $(1, 0, 2\pi)$.

2. Suppose that $\vec{r}(t)$, $0 \le t \le 3$, parameterizes the following curve in space, with $\vec{r}(0) = \langle 0, 3, 0\rangle$ and $\vec{r}(0) = \langle 0, 0, -2\rangle$. The curve lies entirely in the plane $x = 0$, and the right picture shows just that plane. We are told that the arc length of the curve is approximately 15.3.





Find each of the following, or explain why there is not enough information to do so.

(a) A sketch of the arc length function $s(t)$.

(b) The unit tangent vector $\vec{T}$ at the point $(0, 0, 2)$.

(c) The unit tangent vector $\vec{T}(2)$.

(d) The osculating plane at $(0, 0, 2)$.

(e) The unit normal vector $\vec{N}$ at the point $(0, 0, 2)$.

(f) The unit normal vector $\vec{N}(2)$.

(g) The binormal vector $\vec{B}$ at the point $(0, 0, 2)$.

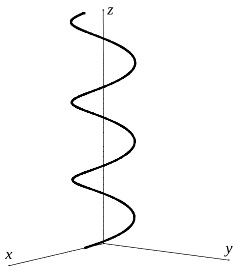
(h) The normal plane at $(0, 0, 2)$.

(i) Which of the following is the best estimate for the curvature of the curve at $(0, -3, 0)$?

$\frac{1}{10}$ $\qquad$ $\frac{1}{2}$ $\qquad$ 2 $\qquad$ 10

Arc Length and Curvature

1. Last time, we saw that $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$ parameterized the pictured curve.



   (a) *Find the arc length of the curve between* $(1,0,0)$ *and* $(1,0,2\pi)$.

   Solution. We'd like to find the arc length of the curve parameterized by $\vec{r}(t)$ between $t=0$ (the point $(1,0,0)$) and $t=2\pi$ (the point $(1,0,2\pi)$). If we imagine $\vec{r}(t)$ as describing the position of a particle in space, the particle's velocity at time t is $\vec{r'}(t) = \langle -\sin t, \cos t, 1\rangle$, so the particle's speed at time t is $|\vec{r'}(t)| = \sqrt{2}$. To find the distance traveled by the particle between times 0 and 2π, we simply integrate the speed between $t=0$ and $t=2\pi$: the arc length is $\int_0^{2\pi} \left|\vec{r'}(t)\right| \, dt = \boxed{2\pi\sqrt{2}}$.

   (b) *Find the unit tangent vector at the point* $(1,0,2\pi)$.

   Solution. As we computed already, $\vec{r'}(t) = \langle -\sin t, \cos t, 1\rangle$ and $|\vec{r'}(t)| = \sqrt{2}$, so $\vec{T}(t) = \frac{\vec{r'}(t)}{|\vec{r'}(t)|} = \left\langle -\frac{\sin t}{\sqrt{2}}, \frac{\cos t}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$. The point $(1,0,2\pi)$ corresponds to $t=2\pi$, and $\vec{T}(2\pi) = \boxed{\left\langle 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle}$.

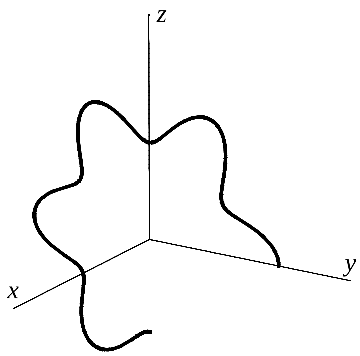
   (c) *Find the curvature at the point* $(1,0,2\pi)$.

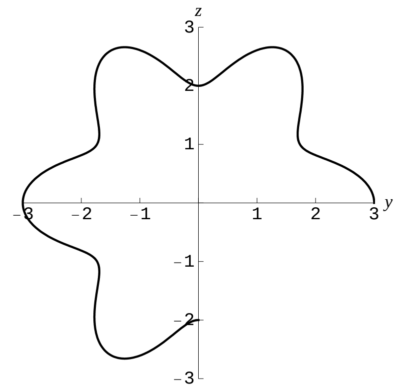
   Solution. We've already found that $\vec{T}(t) = \left\langle -\frac{\sin t}{\sqrt{2}}, \frac{\cos t}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$, so, $\vec{T}'(t) = \left\langle -\frac{\cos t}{\sqrt{2}}, -\frac{\sin t}{\sqrt{2}}, 0 \right\rangle$ and $|\vec{T}'(t)| = \frac{1}{\sqrt{2}}$. We found in (a) that $|\vec{r'}(t)| = \sqrt{2}$, so $\kappa(t) = \frac{1}{2}$. This means that the curvature is $\frac{1}{2}$ everywhere along the curve, so the curvature at the point $(1,0,2\pi)$ is $\boxed{\frac{1}{2}}$.

   (d) *Find the unit normal vector at the point* $(1,0,2\pi)$.

   Solution. We know that the unit normal vector $\vec{N}(t)$ is equal to $\frac{\vec{T}'(t)}{|\vec{T}'(t)|}$. Using the calculations from the previous part, this is equal to $\langle -\cos t, -\sin t, 0\rangle$. In particular, $\vec{N}(2\pi) = \boxed{\langle -1, 0, 0\rangle}$.

2. Suppose that $\vec{r}(t)$, $0 \le t \le 3$, parameterizes the following curve in space, with $\vec{r}(0) = \langle 0,3,0\rangle$ and $\vec{r}(0) = \langle 0,0,-2\rangle$. The curve lies entirely in the plane $x=0$, and the right picture shows just that plane. We are told that the arc length of the curve is approximately 15.3.





Find each of the following, or explain why there is not enough information to do so.

(a) *A sketch of the arc length function* $s(t)$.

Solution. There is $\boxed{\text{not enough information}}$. If we are imagining $\vec{r}(t)$ as describing the position of a particle in space, $s(t)$ describes the distance the particle has traveled by time t. However, we don't know how fast the particle is moving along its path, so it is impossible for us to know what this function looks like.

We do, however, know that the particle has traveled nowhere at time $t = 0$, so $s(0) = 0$, and that the particle has traveled approximately 15.3 units at time $t = 3$, so $s(3) \approx 15.3$. In addition, the distance the particle has traveled certainly increases with time, so $s(t)$ is an increasing function.

(b) *The unit tangent vector* $\vec{T}$ *at the point* $(0, 0, 2)$.

Solution. Since the curve lies entirely in the plane $x = 0$, we know the unit tangent vector must also lie in the plane $x = 0$. From looking at the picture, we can see that the tangent line at $(0, 0, 2)$ is flat in the yz-plane. Since the particle is moving counter-clockwise around the curve, the unit tangent vector is therefore $\boxed{\langle 0, -1, 0 \rangle}$.

(c) *The unit tangent vector* $\vec{T}(2)$.

Solution. This is asking us to find the unit tangent vector at time $t = 2$, but since we don't know the exact parameterization, we have no idea where the particle is at time $t = 2$. Therefore, there is $\boxed{\text{not enough information}}$ to say what the unit tangent vector is.

(d) *The osculating plane at* $(0, 0, 2)$.

Solution. The osculating plane at $(0, 0, 2)$ is the plane that comes closest to containing the curve near $(0, 0, 2)$. Of course, our curve sits entirely in the plane $\boxed{x = 0}$, so that must be the osculating plane.

(e) *The unit normal vector* $\vec{N}$ *at the point* $(0, 0, 2)$.

Solution. Remember that the unit normal vector sits in the osculating plane, is perpendicular to the unit tangent vector, has length 1, and points in the direction that the curve is turning. Therefore, the unit normal vector in this case must be $\boxed{\langle 0, 0, 1 \rangle}$.

(f) *The unit normal vector* $\vec{N}(2)$.

Solution. For the same reason as in (c), there is $\boxed{\text{not enough information}}$ to determine this.

(g) *The binormal vector $\vec{B}$ at the point* $(0,0,2)$.

Solution. We know $\vec{B}$ is $\vec{T} \times \vec{N}$; from parts (b) and (e), we know that $\vec{T} = \langle 0,-1,0\rangle$ and $\vec{N} = \langle 0,0,1\rangle$ at the point $(0,0,2)$, so $\vec{B} = \boxed{\langle -1,0,0\rangle}$.

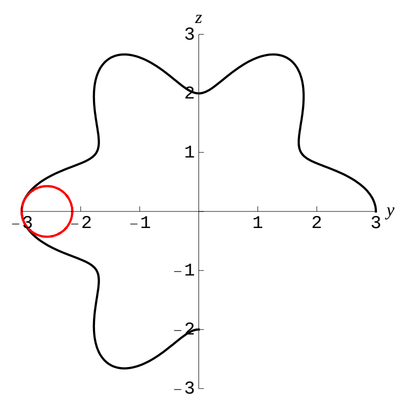
(h) *The normal plane at* $(0,0,2)$.

Solution. This is the plane which is normal to the unit tangent vector at $(0,0,2)$ and which contains the point $(0,0,2)$. We found in (b) that the unit tangent vector at $(0,0,2)$ is $\langle 0,-1,0\rangle$. The plane normal to $\langle 0,-1,0\rangle$ which contains the point $(0,0,2)$ is $\boxed{y=0}$.

(i) *Which of the following is the best estimate for the curvature of the curve at* $(0,-3,0)$*?*

$$\frac{1}{10} \qquad \frac{1}{2} \qquad 2 \qquad 10$$

Solution. Remember that the curvature of a circle of radius a is $\frac{1}{a}$. So, if we can find the circle which best matches the curve near $(-3,0,0)$, the curvature should be 1 over the radius of that circle. If we draw in such a circle, its diameter looks a little less than 1, so its radius should be slightly less than $\frac{1}{2}$. Therefore, its curvature should be slightly more than $\boxed{2}$.

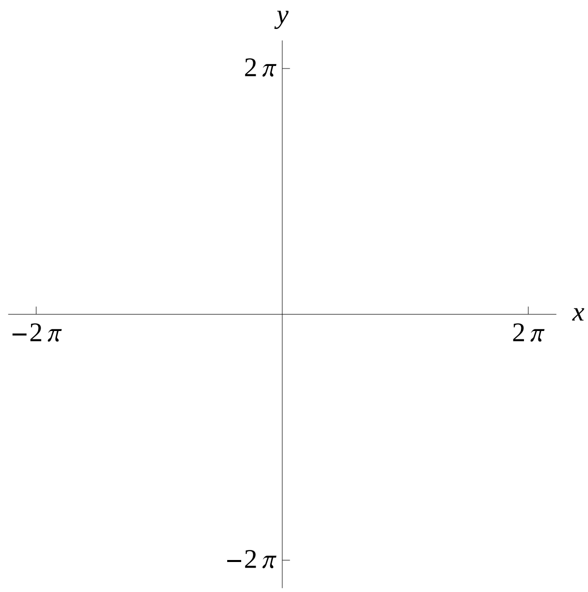


(In fact, the actual curvature is $\frac{7}{3}$.)

Polar, Cylindrical, and Spherical Coordinates

1. (a) In polar coordinates, what shapes are described by $r = k$ and $\theta = k$, where k is a constant?

   (b) Draw $r = 0$, $r = \frac{2\pi}{3}$, $r = \frac{4\pi}{3}$, $r = 2\pi$, $\theta = 0$, $\theta = \frac{2\pi}{3}$, and $\theta = \frac{4\pi}{3}$ on the following axes. (Why can't we draw $\theta = 2\pi$?)



   (c) On the axes in (b), sketch the curve with polar equation $r = \theta$.

2. In cylindrical coordinates, what shapes are described by $r = k$, $\theta = k$, and $z = k$, where k is a constant?

3. In spherical coordinates, what shapes are described by $\rho = k$, $\theta = k$, and $\phi = k$, where k is a constant?

4. (a) In cylindrical coordinates, let's look at the surface $r = 5$. What does $z = k$ look like on this surface? How about $\theta = k$? (k is a constant.)

   (b) In spherical coordinates, let's look at the surface $\rho = 5$. What does $\theta = k$ look like on this surface? How about $\phi = k$?

5. Write the point $(x, y, z) = (\sqrt{6}, -\sqrt{6}, -2)$ in cylindrical and spherical coordinates.

6. Consider the surface whose equation in cylindrical coordinates is $z = r$. How could you describe this surface in Cartesian coordinates? Spherical? Can you sketch the surface?

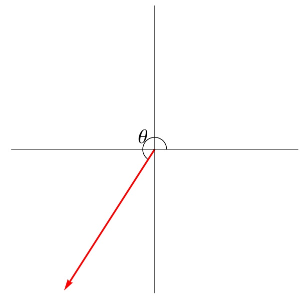
7. Most of the time, a single equation like $2x + 3y + 4z = 5$ in Cartesian coordinates or $\rho = 1$ in spherical coordinates defines a surface. Can you find examples in Cartesian, cylindrical, and spherical coordinates where this is not the case?

Polar, Cylindrical, and Spherical Coordinates

1. (a) *In polar coordinates, what shapes are described by $r = k$ and $\theta = k$, where k is a constant?*

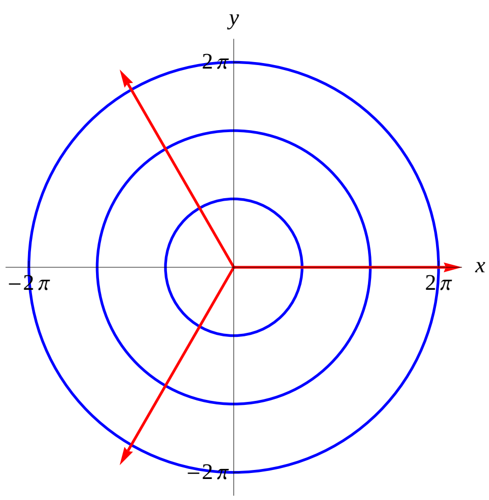
   Solution. $r = k$ describes a circle of radius k centered at the origin.

   $\theta = k$ describes a ray from the origin which makes an angle of θ when measured counter-clockwise from the x-axis.



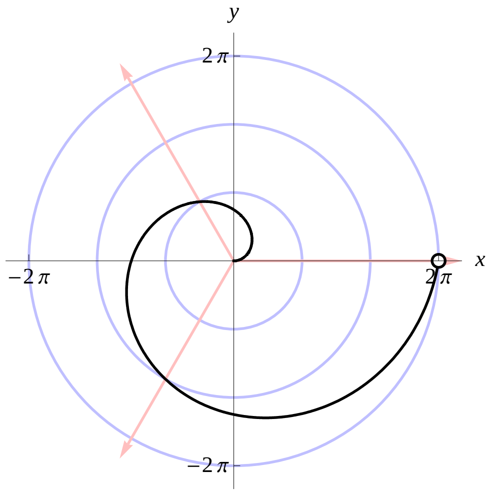
   (b) *Draw $r = 0$, $r = \frac{2\pi}{3}$, $r = \frac{4\pi}{3}$, $r = 2\pi$, $\theta = 0$, $\theta = \frac{2\pi}{3}$, and $\theta = \frac{4\pi}{3}$ on the following axes. (Why can't we draw $\theta = 2\pi$?)*

   Solution. Here is the picture, with $r =$ constant curves drawn in blue and $\theta =$ constant drawn in red. We only allow θ to be in $[0, 2\pi)$, so we can't actually have $\theta = 2\pi$.



   (c) *On the axes in (b), sketch the curve with polar equation $r = \theta$.*

   Solution. The curve is a spiral starting from $(x, y) = (0, 0)$. Notice that it does not actually contain the point $(x, y) = (2\pi, 0)$, since θ cannot actually equal 2π.



2. *In cylindrical coordinates, what shapes are described by $r = k$, $\theta = k$, and $z = k$, where k is a constant?*

   Solution. $r = k$ describes a cylinder of radius k centered around the z-axis.

   $\theta = k$ describes a vertical half-plane whose intersection with the plane $z = 0$ is just the ray $\theta = k$ in polar coordinates.

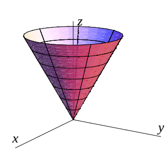
   $z = k$ describes a plane parallel to the xy-plane.

3. *In spherical coordinates, what shapes are described by $\rho = k$, $\theta = k$, and $\phi = k$, where k is a constant?*

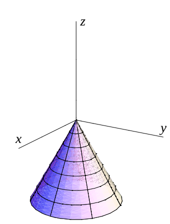
   Solution. $\rho = k$ describes a sphere of radius k centered at the origin.

   $\theta = k$ describes exactly the same thing as it does in cylindrical coordinates. (After all, θ means the same thing in both coordinate systems.)

   $\phi = k$ describes a half-cone. (Remember that the quadric surface that we call a cone actually opens in two directions. A half-cone is what we typically think of as a cone; it opens only in one direction.) For instance, here are pictures of $\phi = \frac{\pi}{6}$ and $\phi = \frac{5\pi}{6}$.



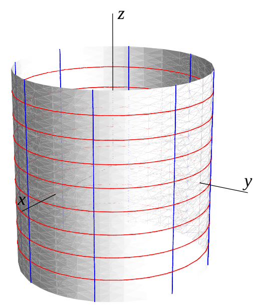
$\phi = \frac{\pi}{6}$



$\phi = \frac{5\pi}{6}$

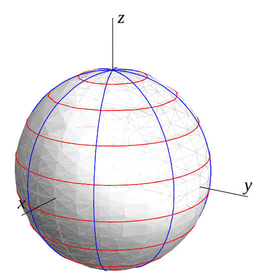
4. (a) *In cylindrical coordinates, let's look at the surface $r = 5$. What does $z = k$ look like on this surface? How about $\theta = k$? (k is a constant.)*

   Solution. The surface $r = 5$ is a cylinder of radius 5 centered about the z-axis. On this surface, each curve $z = k$ is a circle (shown in red), and each curve $\theta = k$ is a vertical line (shown in blue):



(b) *In spherical coordinates, let's look at the surface $\rho = 5$. What does $\theta = k$ look like on this surface? How about $\phi = k$?*

Solution. The surface $\rho = 5$ is a sphere of radius 5 centered about the origin. On this surface, each curve $\theta = k$ is a half-circle running from the top point of the sphere to the bottom point. (If you imagine the sphere as a globe, these are like lines of longitude.) These are shown in blue on the picture. Each curve $\phi = k$ is a circle (these are like lines of latitude), and these are shown in red.



5. *Write the point $(x, y, z) = (\sqrt{6}, -\sqrt{6}, -2)$ in cylindrical and spherical coordinates.*

Solution. To write the point in cylindrical coordinates, we essentially convert $x = \sqrt{6}, y = -\sqrt{6}$ from Cartesian coordinates to polar coordinates. Since $x = r\cos\theta$ and $y = r\sin\theta$, we have $r = \sqrt{x^2+y^2} = 2\sqrt{3}$ and $\tan\theta = \frac{y}{x} = -1$. Now, just knowing $\tan\theta = -1$ tells us that θ is either $\frac{3\pi}{4}$ or $\frac{7\pi}{4}$. But we also know that $\cos\theta = \frac{x}{r} = \frac{\sqrt{2}}{2}$, so θ must be $\frac{7\pi}{4}$. Thus, in cylindrical coordinates, the point is $\boxed{(r, \theta, z) = \left(2\sqrt{3}, \frac{7\pi}{4}, -2\right)}$.

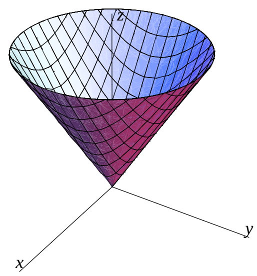
Next, we need to find the point in spherical coordinates. We know that $\rho = \sqrt{x^2+y^2+z^2} = 4$ and that $\tan\phi = \frac{r}{z} = -\sqrt{3}$, so $\phi = \frac{2\pi}{3}$. So, in spherical coordinates, the point is $\boxed{(\rho, \theta, \phi) = \left(4, \frac{7\pi}{4}, \frac{2\pi}{3}\right)}$.

6. *Consider the surface whose equation in cylindrical coordinates is $z = r$. How could you describe this surface in Cartesian coordinates? Spherical? Can you sketch the surface?*

Solution. To write in Cartesian coordinates, we can use the fact that $r = \sqrt{x^2+y^2}$, so $z = r$ is the same as $\boxed{z = \sqrt{x^2 + y^2}}$. This is the portion of $z^2 = x^2 + y^2$ with $z \geq 0$, so it is the top half of an elliptic cone.

To write in spherical coordinates, remember that $z = \rho \cos \phi$ and $r = \rho \sin \phi$, so this is the surface with $\rho \cos \phi = \rho \sin \phi$. In order for this to be true, $\cos \phi = \sin \phi$, so $\tan \phi = 1$. Since $0 \leq \phi \leq 1$, $\boxed{\phi = \frac{\pi}{4}}$.

Here is the surface:



7. *Most of the time, a single equation like* $2x + 3y + 4z = 5$ *in Cartesian coordinates or* $\rho = 1$ *in spherical coordinates defines a surface. Can you find examples in Cartesian, cylindrical, and spherical coordinates where this is not the case?*

Solution. In Cartesian coordinates, $xyz = 0$ is just the three coordinate axes. $x^2 + y^2 + z^2 = 0$ is a point.

In cylindrical coordinates, $r = 0$ is just the z-axis.

In spherical coordinates, $\rho = 0$ is a single point, the origin. Also, $\phi = 0$ is the non-negative portion of the z-axis. If you have trouble visualizing this, you might want to change to Cartesian coordinates:

$$\begin{aligned} x &= \rho \sin \phi \cos \theta \\ y &= \rho \sin \phi \sin \theta \\ z &= \rho \cos \phi \end{aligned}$$

When $\phi = 0$, $\sin \phi = 0$ and $\cos \phi = 1$, so we have $x = 0$, $y = 0$, and $z = \rho$.

Of course, these are not the only examples.

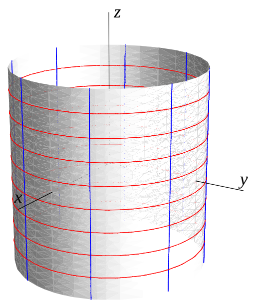
Parametric Surfaces

Last time, we learned that we could go from cylindrical coordinates (r, θ, z) or spherical coordinates (ρ, θ, ϕ) to Cartesian coordinates (x, y, z) using

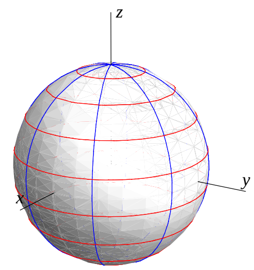
cylindrical		spherical
$x = r\cos\theta$		$x = \rho\sin\phi\cos\theta$
$y = r\sin\theta$		$y = \rho\sin\phi\sin\theta$
$z = z$		$z = \rho\cos\phi$

In the last problem we did in class, we looked at the cylinder $r = 5$ in cylindrical coordinates and saw that $\theta = k$ and $z = k$ (k a constant) formed a grid on the cylinder. Similarly, in spherical coordinates, we looked at the sphere $\rho = 5$ and saw that $\theta = k$ and $\phi = k$ formed a grid on the sphere.

cylindrical $r = 5$



spherical $\rho = 5$

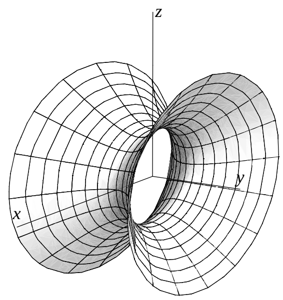


1. (a) Parameterize the elliptic paraboloid $z = x^2 + y^2 + 1$. Sketch the grid curves defined by your parameterization.

   (b) If we only want to parameterize the part of the elliptic paraboloid under the plane $z = 10$, what restrictions would you place on the parameters you used in (a)?

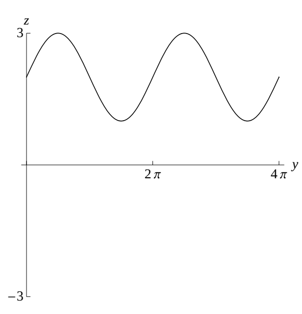
2. Parameterize the plane that contains the 3 points $P(1, 0, 1)$, $Q(2, -2, 2)$, and $R(3, 2, 4)$.

3. Parameterize the hyperboloid $x^2 - 4y^2 + z^2 = 1$.

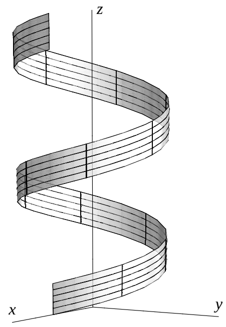


4. Parameterize the ellipsoid $9x^2 + 4y^2 + z^2 = 36$.

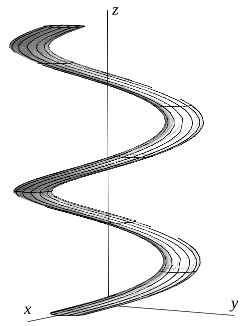
5. Consider the curve $z = 2 + \sin y$, $0 \le y \le 4\pi$ in the yz-plane. Let S be the surface obtained by rotating this curve about the y-axis. Find a parameterization of S.



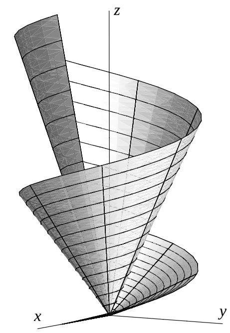
6. Here are three surfaces.



(I)



(II)



(III)

Match each function with the surface it parameterizes. Which curves are where u is constant and which curves are where v is constant?

(a) $\vec{r}(u,v) = \left\langle \dfrac{\cos u}{4} + \cos v, \dfrac{\sin u}{4} + \sin v, v \right\rangle$, $0 \le u \le 2\pi$, $0 \le v \le 4\pi$.

(b) $\vec{r}(u,v) = \left\langle \cos u, \sin u, u + \dfrac{v}{4} \right\rangle$, $0 \le u \le 4\pi$, $0 \le v \le 2\pi$.

(c) $\vec{r}(u,v) = \langle u \cos v, u \sin v, uv \rangle$, $0 \le u \le 2\pi$, $0 \le v \le 4\pi$.

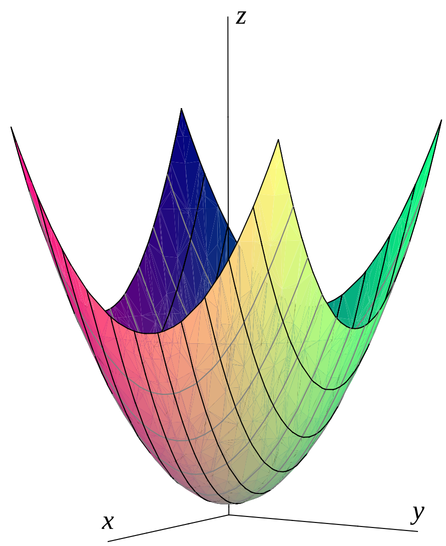
Parametric Surfaces

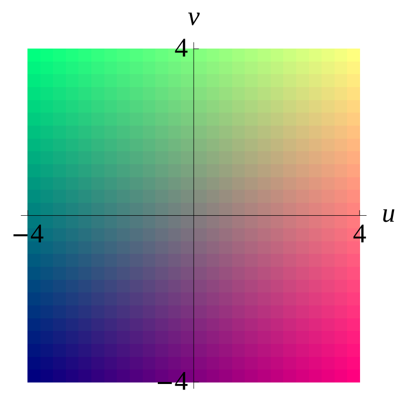
1. (a) *Parameterize the elliptic paraboloid $z = x^2 + y^2 + 1$. Sketch the grid curves defined by your parameterization.*

Solution. There are several ways to parameterize this. Here are a few.

i. One way to think of parameterizing is simply that we want to describe the surface using 2 variables. This amounts to describing x, y, and z using just 2 variables. In this case, z is already written in terms of x, y, and z, so we can describe the surface just using x and y. That is, we can use the parameterization $\vec{r}(x, y) = \langle x, y, x^2 + y^2 + 1\rangle$. We often use the variables u and v as the parameters (just as we usually used t for the parameter when parameterizing curves), so we could also write this as $\boxed{\vec{r}(u, v) = \langle u, v, u^2 + v^2 + 1\rangle}$. (It is certainly not necessary to use u and v though.)

Here is a picture of this parameterization:

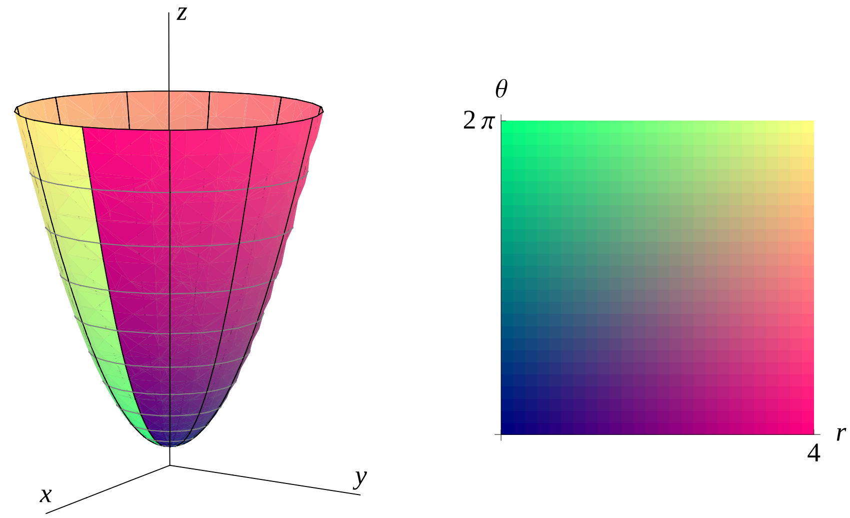




The gray curves are where u is constant, and the black curves are where v is constant. The right picture shows a way of coloring the uv-plane; the paraboloid is colored according to the corresponding u and v value at each point. For instance, from the right picture, we see that $u = 4$, $v = -4$ is colored pink. Therefore, the point $\vec{r}(4, -4) = \langle 4, -4, 17\rangle$ on the paraboloid is colored pink.

ii. Another possibility is to use cylindrical coordinates to rewrite the surface as $z = r^2 + 1$. Then, every point can be described in terms of r and θ since $z = r^2 + 1$. Converting back to Cartesian coordinates, $x = r\cos\theta$, $y = r\sin\theta$, and $z = r^2 + 1$, so we have the parameterization $\boxed{\vec{r}(r, \theta) = \langle r\cos\theta, r\sin\theta, r^2 + 1\rangle}$.

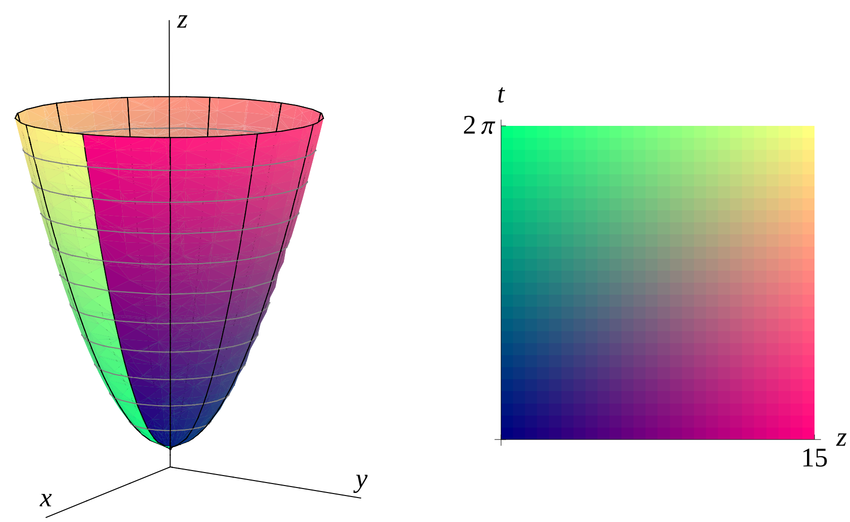
Here is a picture of this parameterization:



The gray curves are where r is constant, and the black curves are where θ is constant.

iii. Yet another possibility is to think of slicing (or taking cross-sections or traces). This approach is slightly more difficult than the previous one, but it's also more flexible. Looking at the surface, we know that taking traces in $z = k$ gives us circles, and circles are curves that we know how to parameterize. If we imagine slicing at a particular z-value, then the slice is the circle $x^2 + y^2 = z - 1$, which is a circle centered at $(x, y) = (0, 0)$ with radius $\sqrt{z-1}$. Therefore, we know that x and y can be described by $x = \sqrt{z-1}\cos t, y = \sqrt{z-1}\sin t$. This gives the parameterization $\boxed{\vec{r}(z,t) = \langle \sqrt{z-1}\cos t, \sqrt{z-1}\sin t, z\rangle}$.

Here is a picture of this parameterization:



The gray curves are where z is constant, and the black curves are where t is constant.

(b) *If we only want to parameterize the part of the elliptic paraboloid under the plane $z = 10$, what restrictions would you place on the parameters you used in (a)?*

Solution.

i. For the parameterization $\vec{r}(u, v) = \langle u, v, u^2 + v^2 + 1\rangle$, we need to restrict u and v. Since we want z (the last component) to be less than 10, we need $\boxed{u^2 + v^2 < 9}$.

ii. For the parameterization $\vec{r}(r, \theta) = \langle r\cos\theta, r\sin\theta, r^2 + 1\rangle$, we need to restrict r and θ. Since the paraboloid was written as $z = r^2 + 1$ in cylindrical coordinates and we want $z < 10$, we need $r < 3$. We know that θ can be anything, so our restrictions are $\boxed{0 \le r < 3,\ 0 \le \theta < 2\pi}$.

iii. For the parameterization $\vec{r}(z, t) = \langle\sqrt{z-1}\cos t, \sqrt{z-1}\sin t, z\rangle$, we need to restrict z and t. We already know that we want $z < 10$. Looking at the paraboloid, we also want $z \ge 1$.(1) Looking back, we used t to parameterize a circle, and the parameterization we chose means $0 \le t < 2\pi$ is a good restriction. So, for this parameterization, we have $\boxed{1 \le z < 10,\ 0 \le t < 2\pi}$.

2. *Parameterize the plane that contains the 3 points $P(1, 0, 1)$, $Q(2, -2, 2)$, and $R(3, 2, 4)$.*

Solution. One way to parameterize the plane is to let the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ define our grid. We can think of P as an "origin" for the plane and the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ as a set of "axes" for the plane. That is, we can reach any point in the plane by starting at P, going in the direction of $\overrightarrow{PQ}$ for a while, and then going in the direction of $\overrightarrow{PR}$ for a while.

In this case, $\overrightarrow{PQ} = \langle 1, -2, 1\rangle$ and $\overrightarrow{PR} = \langle 2, 2, 3\rangle$, so our parameterization is

$$\boxed{\vec{r}(u, v) = \langle 1, 0, 1\rangle + u\langle 1, -2, 1\rangle + v\langle 2, 2, 3\rangle = \langle 1 + u + 2v, -2u + 2v, 1 + u + 3v\rangle}$$

(You can think of this as saying: start at $P(1, 0, 1)$, go off in the direction of $\overrightarrow{PQ} = \langle 1, -2, 1\rangle$ for a bit — how long is determined by u — and then go off in the direction of $\overrightarrow{PR} = \langle 2, 2, 3\rangle$ for a bit.)

Alternatively, you could find the equation of the plane (see the worksheet "Lines and Planes") — it is $8x + y - 6z = 2$. Then, we can write any one of the variables in terms of the other two and use those other two as parameters. For instance, $y = 2 - 8x + 6z$ expresses y in terms of x and z. If we want to use u and v as our parameters, then we can just have $x = u$, $z = v$, and $y = 2 - 8u + 6v$, which gives the parameterization $\boxed{\vec{r}(u, v) = \langle u, 2 - 8u + 6v, v\rangle.}$

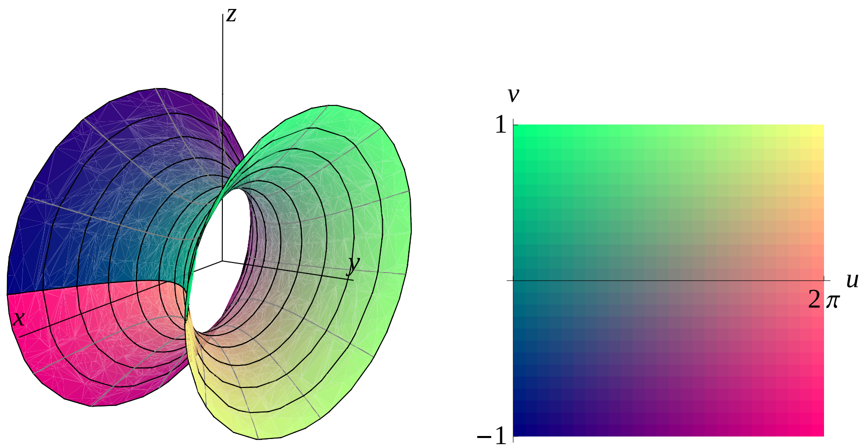
Of course, there are many other parameterizations. One way to check whether your parameterization is reasonable is to remember that you are supposed to be parameterizing $8x+y-6z = 2$. So, however you parameterize, this relationship should be satisfied. For instance, in our first parameterization $\vec{r}(u, v) = \langle 1+u+2v, -2u+2v, 1+u+3v\rangle$, you can easily check that $8(1+u+2v)+(-2u+2v)-6(1+u+3v) = 2$.

3. *Parameterize the hyperboloid $x^2 - 4y^2 + z^2 = 1$.*

Solution. The traces in $y = k$ of this surface will be circles. In particular, since $x^2 + z^2 = 1 + 4y^2$, the trace in $y = k$ will be a circle centered at $(x, z) = (0, 0)$ with radius $\sqrt{1 + 4y^2}$. We can parameterize this by taking $x = \sqrt{1 + 4y^2}\cos u$, $z = \sqrt{1 + 4y^2}\cos u$ with $0 \le u < 2\pi$. Our other parameter is just y; if we rename it v, then we have the parameterization $\boxed{\vec{r}(u, v) = \langle\sqrt{1 + 4v^2}\cos u, v, \sqrt{1 + 4v^2}\sin u\rangle}$ with $\boxed{0 \le u < 2\pi}$ (v can be anything).

(1)Notice that this is implied in our parameterization since $\sqrt{z-1}$ is not defined if $z < 1$.

Here is a picture of the parameterization.

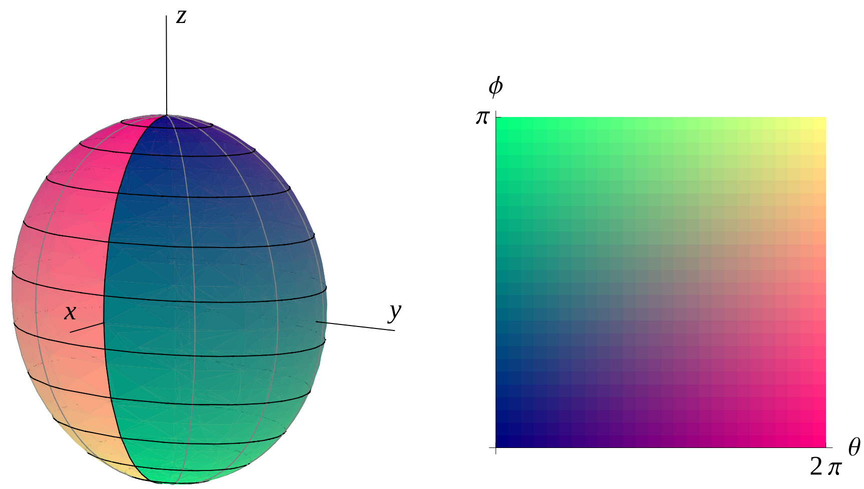


The gray curves are where u is constant, and the black curves are where v is constant.

4. *Parameterize the ellipsoid* $9x^2 + 4y^2 + z^2 = 36$.

Solution. There is not much work to do here if we take a clever approach. Let's rewrite the given equation as $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 + \left(\frac{z}{6}\right)^2 = 1$. We know that $X^2 + Y^2 + Z^2 = 1$ can be parameterized as $X = \sin\phi\cos\theta$, $Y = \sin\phi\sin\theta$, and $Z = \cos\phi$, and if we think of X as being $\frac{x}{2}$, Y as being $\frac{y}{3}$, and Z as being $\frac{z}{6}$, then we get $x = 2\sin\phi\cos\theta$, $y = 3\sin\phi\sin\theta$, $z = 6\cos\phi$. That is, our parameterization is $\boxed{\vec{r}(\theta,\phi) = \langle 2\sin\phi\cos\theta, 3\sin\phi\sin\theta, 6\cos\phi\rangle}$ with $\boxed{0 \le \theta < 2\pi,\ 0 \le \phi \le \pi}$.

Here is a picture of the parameterization.

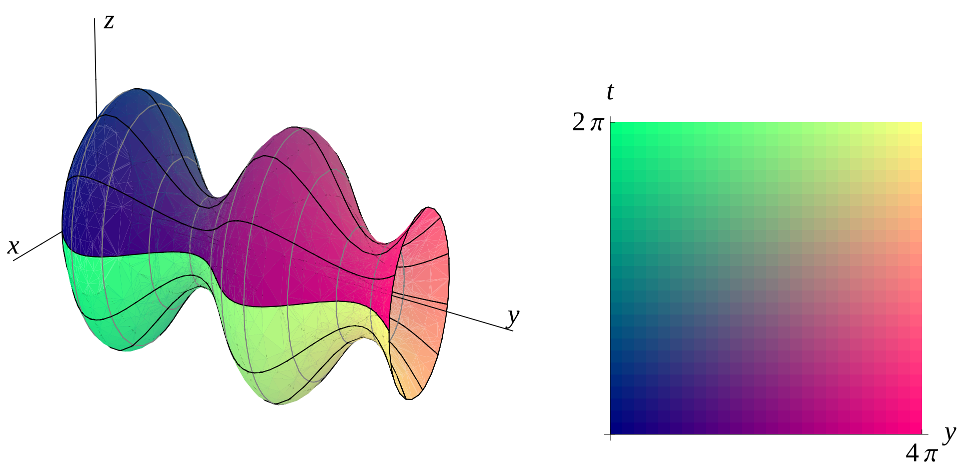


The gray curves are where θ is constant, and the black curves are where ϕ is constant.

5. *Consider the curve $z = 2 + \sin y$, $0 \le y \le 4\pi$ in the yz-plane. Let S be the surface obtained by rotating this curve about the y-axis. Find a parameterization of S.*

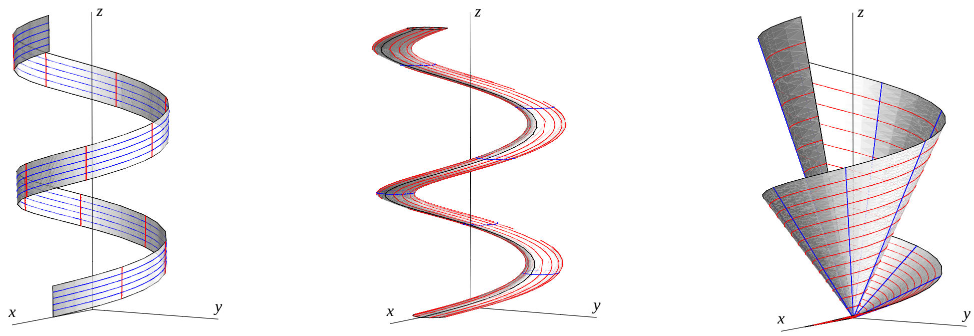
   Solution. The traces in $y = k$ of this surface will be circles. In particular, if we look at the trace in $y = k$, we see a circle centered at $(x, z) = (0, 0)$ with radius $2 + \sin y$. We can parameterize this by taking $x = (2 + \sin y)\cos t$, $z = (2 + \sin y)\sin t$ with $0 \le t < 2\pi$. Thus, a parameterization of the surface is $\boxed{\vec{r}(y,t) = \langle (2+\sin y)\cos t, y, (2+\sin y)\sin t\rangle}$ with $\boxed{0 \le y \le 4\pi,\ 0 \le t < 2\pi}$.

   Here is a picture of the parameterization.



The gray curves are where y is constant, and the black curves are where t is constant.

6. Here are three surfaces.



(I) (II) (III)

Match each function with the surface it parameterizes. Which curves are where u is constant and which curves are where v is constant?

(a) $\vec{r}(u,v) = \left\langle \dfrac{\cos u}{4} + \cos v, \dfrac{\sin u}{4} + \sin v, v \right\rangle$, $0 \le u \le 2\pi$, $0 \le v \le 4\pi$.

Solution. If u is a constant, then $\vec{r}(u,v)$ has the form $\langle C_1 + \cos v, C_2 + \sin v, v\rangle$ where C_1 and C_2 are constants. You should recognize this as a helix (remember $\langle \cos t, \sin t, t\rangle$), shifted.

On the other hand, if v is a constant, then $\vec{r}(u,v)$ has the form $\left\langle \frac{1}{4}\cos u + C_1, \frac{1}{4}\sin u + C_2, C_3 \right\rangle$ where C_1, C_2, and C_3 are constants. You should recognize this as parameterizing a circle (which is parallel to the xy-plane since the z-component does not vary with u.

The surface which has helices as one set of grid curves and circles as the other is (II). The grid curves with u constant are shown in red; the grid curves with v constant are shown in blue.

(b) $\vec{r}(u,v) = \left\langle \cos u, \sin u, u + \dfrac{v}{4} \right\rangle$, $0 \le u \le 4\pi$, $0 \le v \le 2\pi$.

Solution. If u is a constant, then $\vec{r}(u,v)$ has the form $\left\langle C_1, C_2, C_3 + \frac{v}{4} \right\rangle$ where C_1, C_2, and C_3 are constants. This simply parameterizes a vertical line segment (of length $\frac{\pi}{2}$ since v varies between 0 and 2π).

On the other hand, if v is a constant, then $\vec{r}(u,v)$ has the form $\langle \cos u, \sin u, u + C\rangle$ where C is a constant. This parameterizes a helix.

The surface which has vertical line segments as one set of grid curves and helices as the other is (I). The grid curves with u constant are shown in red; the grid curves with v constant are shown in blue.

(c) $\vec{r}(u,v) = \langle u\cos v, u \sin v, uv\rangle$, $0 \le u \le 2\pi$, $0 \le v \le 4\pi$.

Solution. If u is a constant, then $\vec{r}(u,v)$ has the form $\langle C\cos v, C\sin v, Cv\rangle = C\langle \cos v, \sin v, v\rangle$ where C is a constant. This parameterizes a helix. Notice that, unlike in the two previous parts, the helices here can be different sizes.

On the other hand, if v is a constant, then $\vec{r}(u,v)$ has the form $\langle C_1 u, C_2 u, C_3 u\rangle$ where C_1, C_2, and C_3 are constants. You should recognize this as parameterizing a line segment; since u starts at 0, this line segment always starts at the origin.

The matching surface is (III). The grid curves with u constant are shown in red; the grid curves with v constant are shown in blue.

Functions, Limits, and Continuity

1. Describe the level sets of the following functions. What shape are they?

   (a) $f(x, y) = x^2 + 4y^2$.

   (b) $f(x, y, z) = x^2 + 4y^2 + 9z^2$.

   (c) $f(x, y) = y - x$.

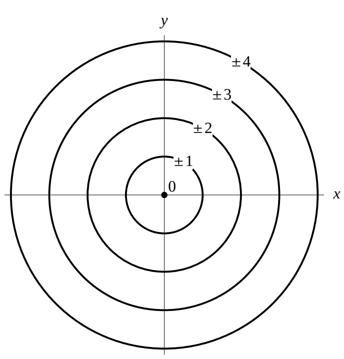
   (d) $f(x, y, z) = 2x + 3y + 4z$.

   (e) $f(x, y, z) = 4x^2 + 9y^2$.

2. Let S be the unit sphere centered at $(0, 0, 0)$. Is S the graph of a function? If so, what function?

   Is S a level set of a function? If so, what function?

3. Is the following picture the <u>level set diagram</u> (also known as <u>contour map</u>) of a function? If so, sketch the graph of the function.

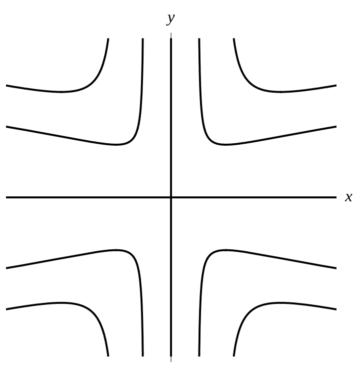


4. Match each function with its level set diagram and its graph. (Note that each function is undefined at $(0,0)$.)

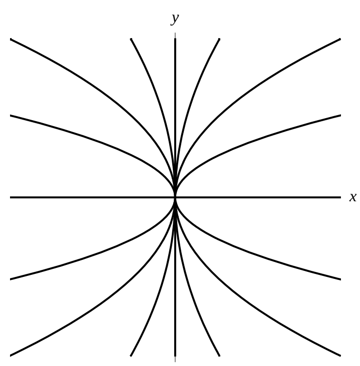
(a) $f(x,y) = \dfrac{y^2}{x^2+y^2}$. (Hint: What are the level sets $f(x,y)=0$, $f(x,y)=\frac{1}{2}$, and $f(x,y)=1$?)

(b) $f(x,y) = -\dfrac{xy^2}{x^2+y^4}$. (Hint: What are the level sets $f(x,y)=\frac{1}{2}$ and $f(x,y)=-\frac{1}{2}$?)

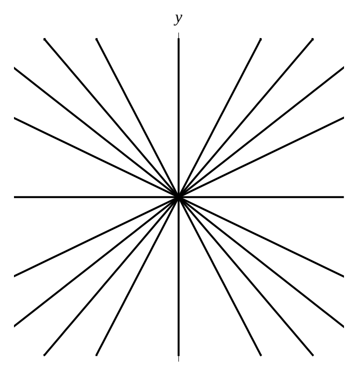
(c) $f(x,y) = -\dfrac{xy^2}{x^2+y^2}$. (Hint: Process of elimination!)



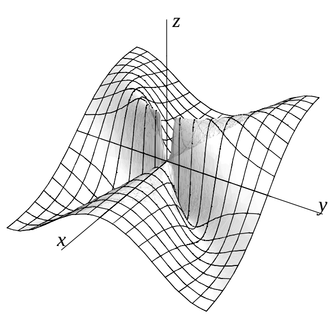
(I)



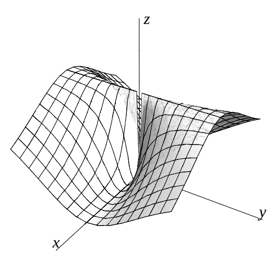
(II)



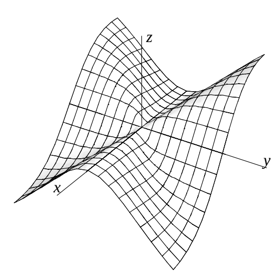
(III)



(A)



(B)



(C)

Definition. The limit of $f(x, y)$ as (x, y) approaches (a, b) is L if we can make the values of $f(x, y)$ as close to L as we like by taking the point (x, y) sufficiently close to the point (a, b), but not equal to (a, b). We write this as $\lim_{(x,y)\to(a,b)} f(x, y) = L$.

Strategy.

- To show that a limit $\lim_{(x,y)\to(a,b)} f(x, y)$ does *not* exist, we usually try to find two different paths approaching (a, b) on which $f(x, y)$ has different limits.
- Showing that a limit $\lim_{(x,y)\to(a,b)} f(x, y)$ *does* exist is generally harder. If the point (a, b) is $(0, 0)$, one strategy is to rewrite the limit in polar coordinates. Then, no matter how (x, y) approaches $(0, 0)$, r tends to 0, so if the limit $\lim_{r\to 0^+} f(r\cos\theta, r\sin\theta)$ exists, then the original limit $\lim_{(x,y)\to(0,0)} f(x, y)$ also exists.

5. Using the contour maps from #4, first guess whether $\lim_{(x,y)\to(0,0)} f(x, y)$ exists for each of the following functions. Then show that your guess is correct using the strategy described above.

   (a) $f(x, y) = \dfrac{y^2}{x^2 + y^2}$.

   (b) $f(x, y) = -\dfrac{xy^2}{x^2 + y^4}$.

   (c) $f(x, y) = -\dfrac{xy^2}{x^2 + y^2}$.

Functions, Limits, and Continuity

1. Describe the level sets of the following functions. What shape are they?

   (a) $f(x,y) = x^2 + 4y^2$.

   Solution. The level set $f(x,y) = k$ consists of points (x,y) satisfying $x^2 + 4y^2 = k$. Each level set is an $\boxed{\text{ellipse}}$.

   (b) $f(x,y,z) = x^2 + 4y^2 + 9z^2$.

   Solution. The level set $f(x,y,z) = k$ consists of points (x,y,z) satisfying $x^2 + 4y^2 + 9z^2 = k$. Therefore, each level set is an $\boxed{\text{ellipsoid}}$.

   (c) $f(x,y) = y - x$.

   Solution. The level set $f(x,y) = k$ consists of points (x,y) satisfying $y - x = k$. Each level set is a $\boxed{\text{line}}$. (And all of the level sets are parallel to each other.)

   (d) $f(x,y,z) = 2x + 3y + 4z$.

   Solution. The level set $f(x,y,z) = k$ consists of points (x,y,z) satisfying $2x + 3y + 4z = k$. Therefore, each level set is a $\boxed{\text{plane}}$. (And all of the level sets are parallel to each other.)

   (e) $f(x,y,z) = 4x^2 + 9y^2$.

   Solution. The level set $f(x,y,z) = k$ consists of points (x,y,z) satisfying $4x^2 + 9y^2 = k$. Therefore, each level set is a $\boxed{\text{cylinder}}$. (Each cylinder is centered around the z-axis.)

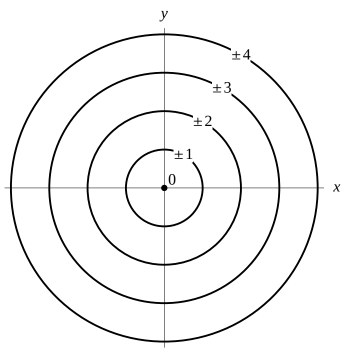
2. *Let S be the unit sphere centered at $(0,0,0)$. Is S the graph of a function? If so, what function?*

   Is S a level set of a function? If so, what function?

   Solution. Asking whether S is the graph of a function is the same as asking whether S can be described by a single equation of the form $z = f(x,y)$. (If so, S is the graph of f.) If a surface is described as $z = f(x,y)$, then there can only be one z value for each (x,y) value. The sphere does not satisfy this: for instance $(0,0,1)$ and $(0,0,-1)$ are both points on S, and they have the same x and y values but different z values. So, S is $\boxed{\text{not the graph of a function}}$.

   On the other hand, S $\boxed{\text{is a level set}}$. To express S as a level set means to write S as $g(x,y,z) = k$ for some function g and some constant k. We can write S as $x^2+y^2+z^2 = 1$, so S is a level set of $g(x,y,z) = x^2 + y^2 + z^2$. This answer is not unique either; S is also a level set of $\boxed{g(x,y,z) = x^2 + y^2 + z^2 + c}$ for any constant c. (For instance, we can write S as $x^2 + y^2 + z^2 - 7 = -6$, which shows that it is a level set of $x^2 + y^2 + z^2 - 7$.)

3. *Is the following picture the <u>level set diagram</u> (also known as <u>contour map</u>) of a function? If so, sketch the graph of the function.*



Solution. The picture is $\boxed{\text{not the level set diagram}}$ of a function.

Why? Well, pretend that it is in fact the level set diagram of a function $f(x, y)$. Let's pick a point (x, y) on the outer-most circle of the diagram. What is the value of f at that point? According to the diagram, it has to be both 4 and -4, which doesn't make sense — a function can only spit out 1 output for each input.

So, we can conclude that the picture is not the level set diagram of any function.

4. Match each function with its level set diagram and its graph. (Note that each function is undefined at $(0, 0)$.)

   (a) $f(x, y) = \dfrac{y^2}{x^2 + y^2}$. *(Hint: What are the level sets* $f(x, y) = 0$, $f(x, y) = \frac{1}{2}$, *and* $f(x, y) = 1$*?)*

   Solution. Following the hint, we'll look at the level sets $f(x, y) = 0$, $f(x, y) = \frac{1}{2}$, and $f(x, y) = 1$.

   $f(x, y) = 0$ when $y = 0$. This is just the x-axis.

   $f(x, y) = 1$ when $y^2 = x^2 + y^2$, or $x = 0$. This is just the y-axis.

   $f(x, y) = \frac{1}{2}$ when $\frac{y^2}{x^2+y^2} = \frac{1}{2}$, or $2y^2 = x^2 + y^2$. This is the same as $y^2 = x^2$, or $y = \pm x$. So, this level set is a pair of lines.

   The only level set diagram that matches this is $\boxed{\text{(III)}}$.

   To pick the right graph, notice that we said $f(x, y) = 1$ along the y-axis. The only picture that matches that is $\boxed{\text{(B)}}$.

   (b) $f(x, y) = -\dfrac{xy^2}{x^2 + y^4}$. *(Hint: What are the level sets* $f(x, y) = \frac{1}{2}$ *and* $f(x, y) = -\frac{1}{2}$*?)*

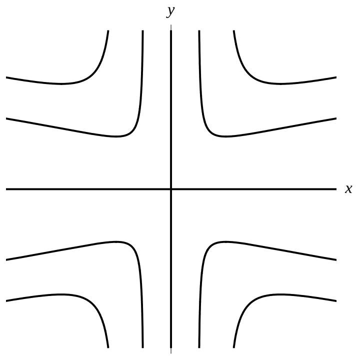
   Solution. $f(x, y) = \frac{1}{2}$ when $-\frac{xy^2}{x^2+y^4} = \frac{1}{2}$, or $x^2 + y^4 = -2xy^2$. We can rewrite this as $x^2 + 2xy^2 + y^4 = 0$ and factor to get $(x + y^2)^2 = 0$, or simply $x = -y^2$. Similarly, $f(x, y) = -\frac{1}{2}$ is just $x = y^2$.

   The only level set diagram that has parabolas as level sets is $\boxed{\text{(II)}}$.

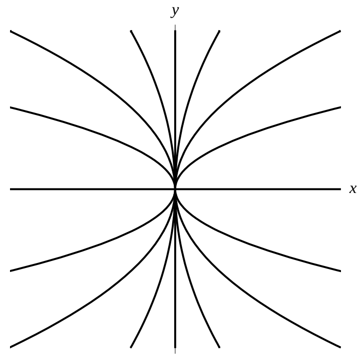
   The correct graph is $\boxed{\text{(A)}}$.

(c) $f(x,y) = -\dfrac{xy^2}{x^2+y^2}$. *(Hint: Process of elimination!)*

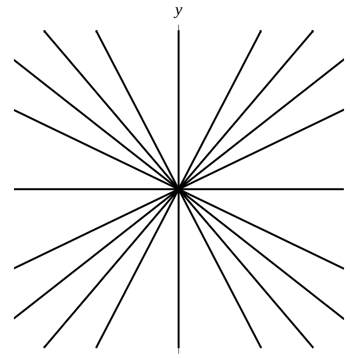
Solution. The level sets of this function are hard to describe, but process of elimination tells us that this matches $\boxed{\text{(I)}}$ and $\boxed{\text{(C)}}$.



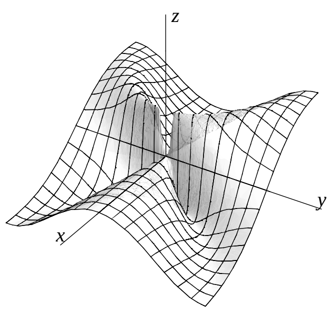
(I)



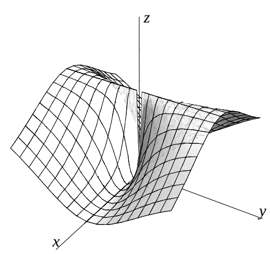
(II)



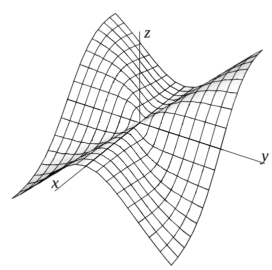
(III)



(A)



(B)



(C)

5. Using the contour maps from #4, first guess whether $\lim_{(x,y)\to(0,0)} f(x,y)$ exists for each of the following functions. Then show that your guess is correct using the strategy described above.

(a) $f(x,y) = \dfrac{y^2}{x^2+y^2}$.

Solution. This was contour map (III), and we can see in that contour map that several contours of different values tend toward the origin. This means that the limit $\lim_{(x,y)\to(0,0)} f(x,y)$ should not exist because, as (x,y) approaches $(0,0)$ along each contour, $f(x,y)$ has a different value.

To show this, let's have (x,y) approach $(0,0)$ along the positive x-axis and along the positive y-axis. If (x,y) approaches $(0,0)$ along the positive x-axis, then y is always 0 on this path and $x \to 0^+$. So, we are looking at the limit $\lim_{x\to 0^+} \dfrac{0^2}{x^2+0^2} = 0$.

On the other hand, if (x,y) approaches $(0,0)$ along the positive y-axis, then x is always 0 on this path and $y \to 0^+$. So, we are looking at the limit $\lim_{y\to 0^+} \dfrac{y^2}{0^2+y^2} = 1$.

Since these two limits are different, we can conclude that $\lim_{(x,y)\to(0,0)} f(x,y)$ $\boxed{\text{does not exist}}$.

(b) $f(x,y) = -\dfrac{xy^2}{x^2+y^4}$.

Solution. This was contour map (II), and we can see in that contour map that several contours of different values tend toward the origin. This means that the limit $\lim_{(x,y)\to(0,0)} f(x,y)$ should not exist because, as (x,y) approaches $(0,0)$ along each contour, $f(x,y)$ has a different value.

To show this, let's have (x,y) approach $(0,0)$ along the paths $x=-y^2$ and $x=y^2$. As we showed in #4, along the path $x=-y^2$, $f(x,y)=\frac{1}{2}$; along the path $x=y^2$, $f(x,y)=-\frac{1}{2}$. Therefore, the limit along the path $x=-y^2$ will be $\frac{1}{2}$, while the limit along the path $x=y^2$ will be $-\frac{1}{2}$.

Since these two limits are different, we can conclude that $\lim_{(x,y)\to(0,0)} f(x,y)$ $\boxed{\text{does not exist}}$.

(c) $f(x,y) = -\dfrac{xy^2}{x^2+y^2}$.

Solution. Looking at contour map (I), it seems entirely plausible that the limit exists because we don't see different contours heading to $(0,0)$.

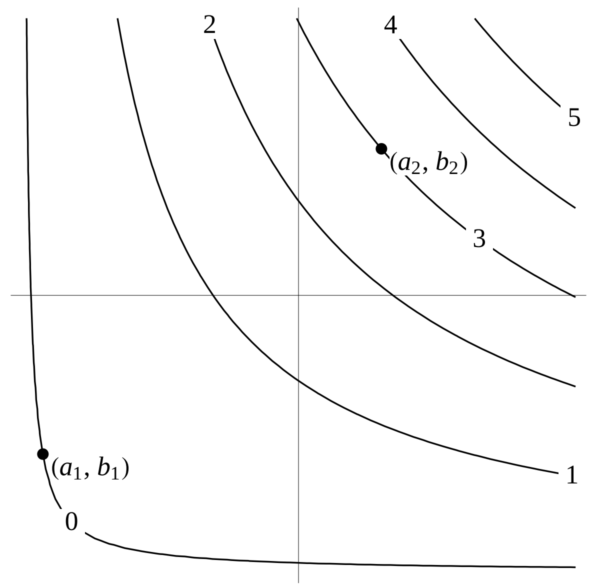
To check this, let's rewrite the limit in polar coordinates. If we write $(x,y)=(r\cos\theta, r\sin\theta)$, then (x,y) approaches $(0,0)$ as $r\to 0^+$. So, we can rewrite this limit as $\lim_{r\to 0^+} -\dfrac{(r\cos\theta)(r\sin\theta)^2}{r^2} = \lim_{r\to 0^+} -\dfrac{r^3\cos\theta\sin^2\theta}{r^2} = \lim_{r\to 0^+} -r\cos\theta\sin\theta = 0$, no matter what θ is.

So, the limit $\boxed{\text{exists and is equal to 0}}$.

Note: What would have happened in (a) if we had mistakenly guessed that the limit existed and tried to use polar coordinates to show that? Well, we would have gotten $\lim_{r\to 0^+} \dfrac{r^2\sin^2\theta}{r^2} = \lim_{r\to 0^+} \sin^2\theta$. However, this limit clearly depends on θ, which says exactly that the limit changes depending on the path of approach.

Partial Derivatives

Here is the level set diagram of a function $f(x, y)$. The value of f on each level set is labeled.



Based on the level set diagram, decide whether each of the statements should be true or false. (For which can you be totally sure, and for which would you need more information to be totally sure?)

1. $f_x(a_1, b_1) \geq 0$.

2. $f_y(a_2, b_2) \geq 0$.

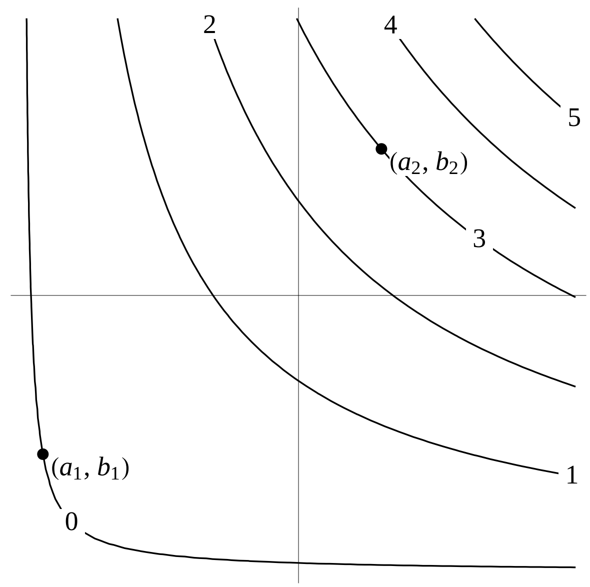
3. $f_x(a_1, b_1) \geq f_x(a_2, b_2)$.

4. $f_{xx}(a_2, b_2) \geq 0$.

5. $f_{xy}(a_2, b_2) \geq 0$.

Partial Derivatives

Here is the level set diagram of a function $f(x, y)$. The value of f on each level set is labeled.



Based on the level set diagram, decide whether each of the statements should be true or false. (For which can you be totally sure, and for which would you need more information to be totally sure?)

1. $f_x(a_1, b_1) \geq 0$.

   Solution. True; if we go right from (a_1, b_1), $f(x, y)$ is increasing.

2. $f_y(a_2, b_2) \geq 0$.

   Solution. True; if we go up from (a_2, b_2), $f(x, y)$ is increasing.

3. $f_x(a_1, b_1) \geq f_x(a_2, b_2)$.

   Solution. False.

   If we go right from (a_1, b_1), it takes more distance for f to increase by 1 (reach the next contour line) than if we go right from (a_2, b_2). Another way of saying this is that the contour lines near (a_2, b_2) are spaced more closely together than near (a_1, b_1). This suggests that f is increasing more quickly at (a_2, b_2) than at (a_1, b_1) (if we travel to the right).

   However, we cannot tell for sure since the level set diagram gives us limited information: from the level set diagram, we can only estimate the *average* rate of change between one contour and the next; we cannot be absolutely sure what the *instantaneous* rate of change (which is what the derivative is) is doing.

4. $f_{xx}(a_2, b_2) \geq 0$.

   Solution. True.

We can think of f_{xx} as $\frac{\partial}{\partial x} f_x$, which measures the rate of change of f_x as we move to the right. $\frac{\partial}{\partial x} f_x$ being positive would indicate that f_x is increasing as we move to the right. To see if this appears to be true, we do pretty much the same thing as in #3 except that we compare f_x at a point slightly to the right of (a_2, b_2) with f_x at a point slightly to the left of (a_2, b_2). Here, it appears that f_x at a point slightly to the right is larger, so f_{xx} appears to be positive.

5. $f_{xy}(a_2, b_2) \geq 0$.

   Solution. True.

   Let's think about this the same way that we thought about #4. That is, think of f_{xy} as $\frac{\partial}{\partial y} f_x$, so f_{xy} is positive if f_x is increasing as we move up. To decide whether this is true, we compare f_x at a point slightly below (a_2, b_2) to f_x at a point slightly above (a_2, b_2). Here, it appears f_x is higher at the point slightly above, which suggests that f_{xy} is indeed positive.

Tangent Planes and Linear Approximation

1. Let S be the cylinder $x^2 + y^2 = 4$. Find the plane tangent to S at the point $(1, \sqrt{3}, 5)$.

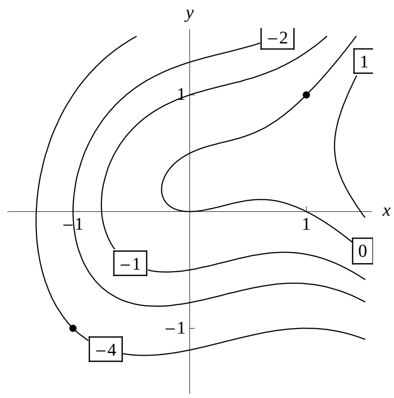
2. Let S be the surface $z = y \sin x$. Find the plane tangent to S at the point $\left(\frac{\pi}{6}, 2, 1\right)$.

3. Let S be the graph of $f(x, y)$; that is, S is the surface $z = f(x, y)$. Find the plane tangent to S at the point $(a, b, f(a, b))$.

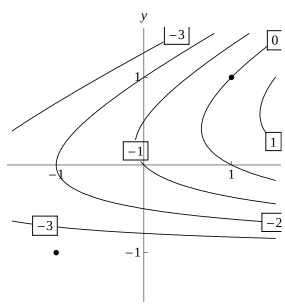
   (Notice that #2 was a special case of this, with $f(x, y) = y \sin x$, $a = \frac{\pi}{6}$, and $b = 2$.)

4. Let $f(x, y, z) = \sqrt{x} + xyz$. Use linear approximation to approximate the value of $f(1.1, 1.9, 3.1)$.

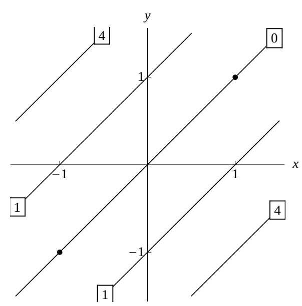
5. Suppose the mysterious function $f(x, y)$ has the following level set diagram (contour map).



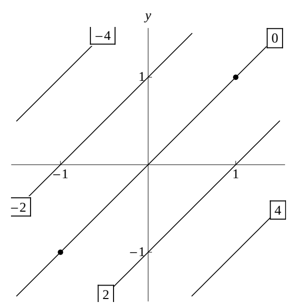
The points $(1, 1)$ and $(-1, -1)$ are marked with dots. Let $L_1(x, y)$ be the linearization of f at $(1, 1)$ and $L_2(x, y)$ be the linearization of f at $(-1, -1)$. Which of the following is the level set diagram of $L_1(x, y)$? Which of the following is the level set diagram of $L_2(x, y)$?



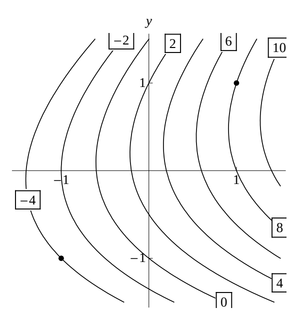
(A)



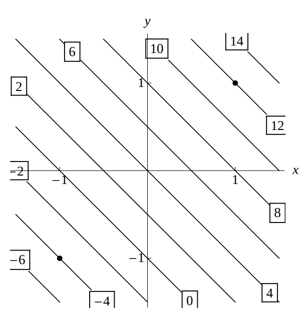
(B)



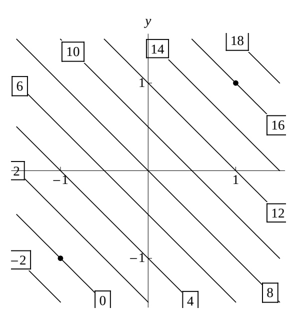
(C)



(D)



(E)



(F)

Tangent Planes and Linear Approximation

1. *Let S be the cylinder $x^2 + y^2 = 4$. Find the plane tangent to S at the point $(1, \sqrt{3}, 5)$.*

   Solution. The first thing we need to do is to parameterize the surface S. Since S can be expressed in cylindrical coordinates as $r = 2$, we know that we can parameterize S using θ and z from cylindrical coordinates. This gives us the parameterization $\vec{r}(\theta, z) = \langle 2\cos\theta, 2\sin\theta, z\rangle$. The point $(1, \sqrt{3}, 5)$ is then $\vec{r}\left(\frac{\pi}{3}, 5\right)$.

   $$\begin{aligned} \vec{r}_\theta &= \langle -2\sin\theta, 2\cos\theta, 0\rangle &\Rightarrow \quad \vec{r}_\theta\left(\tfrac{\pi}{3}, 5\right) &= \langle -\sqrt{3}, 1, 0\rangle \\ \vec{r}_z &= \langle 0, 0, 1\rangle &\Rightarrow \quad \vec{r}_z\left(\tfrac{\pi}{3}, 5\right) &= \langle 0, 0, 1\rangle \end{aligned}$$

   Since both $\vec{r}_\theta\left(\frac{\pi}{3}, 5\right)$ and $\vec{r}_z\left(\frac{\pi}{3}, 5\right)$ will lie in the tangent plane, their cross product $\langle 1, \sqrt{3}, 0\rangle$ is a normal vector for the plane. In addition, we know that the point $(1, \sqrt{3}, 5)$ lies on the tangent plane, so the plane can be described as the set of points (x, y, z) such that $\langle 1, \sqrt{3}, 0\rangle \cdot \langle x-1, y-\sqrt{3}, z-5\rangle = 0$, or $\boxed{1(x-1) + \sqrt{3}(y-\sqrt{3}) = 0}$.

2. *Let S be the surface $z = y\sin x$. Find the plane tangent to S at the point $\left(\frac{\pi}{6}, 2, 1\right)$.*

   Solution. The first thing we need to do is to parameterize the surface. Since we can easily write z in terms of x and y, it makes sense to use x and y as our parameters. This gives us the parameterization $\vec{r}(x, y) = \langle x, y, y\sin x\rangle$. The point $\left(\frac{\pi}{6}, 2, 1\right)$ is then $\vec{r}\left(\frac{\pi}{6}, 2\right)$.

   Now, we have:

   $$\begin{aligned} \vec{r}_x &= \langle 1, 0, y\cos x\rangle &\Rightarrow \quad \vec{r}_x\left(\tfrac{\pi}{6}, 2\right) &= \left\langle 1, 0, \sqrt{3}\right\rangle \\ \vec{r}_y &= \langle 0, 1, \sin x\rangle &\Rightarrow \quad \vec{r}_y\left(\tfrac{\pi}{6}, 2\right) &= \left\langle 0, 1, \tfrac{1}{2}\right\rangle \end{aligned}$$

   Therefore, $\langle 1, 0, \sqrt{3}\rangle$ and $\left\langle 0, 1, \frac{1}{2}\right\rangle$ are two vectors parallel to the plane, and we can use their cross product $\langle 1, 0, \sqrt{3}\rangle \times \left\langle 0, 1, \frac{1}{2}\right\rangle = \left\langle -\sqrt{3}, -\frac{1}{2}, 1\right\rangle$ as a normal vector for the plane.

   We know that the plane should contain the point $\left(\frac{\pi}{6}, 2, 1\right)$, so the plane consists of all points (x, y, z) such that $\left\langle -\sqrt{3}, -\frac{1}{2}, 1\right\rangle \cdot \left\langle x - \frac{\pi}{6}, y-2, z-1\right\rangle = 0$, or $\boxed{-\sqrt{3}\left(x - \frac{\pi}{6}\right) - \frac{1}{2}(y-2) + 1(z-1) = 0}$.

3. *Let S be the graph of $f(x, y)$; that is, S is the surface $z = f(x, y)$. Find the plane tangent to S at the point $(a, b, f(a, b))$.*

   (Notice that #2 was a special case of this, with $f(x, y) = y\sin x$, $a = \frac{\pi}{6}$, and $b = 2$.)

   Solution. The first thing we need to do is to parameterize the surface. Since z is already expressed in terms of x and y, it makes sense to use x and y as our parameters. This gives us the parameterization $\vec{r}(x, y) = \langle x, y, f(x, y)\rangle$. Then, the point $(a, b, f(a, b))$ is $\vec{r}(a, b)$. We have

   $$\begin{aligned} \vec{r}_x &= \langle 1, 0, f_x\rangle &\Rightarrow \quad \vec{r}_x(a, b) &= \langle 1, 0, f_x(a, b)\rangle \\ \vec{r}_y &= \langle 0, 1, f_y\rangle &\Rightarrow \quad \vec{r}_y(a, b) &= \langle 0, 1, f_y(a, b)\rangle \end{aligned}$$

   Therefore, a normal vector for the tangent plane is the cross product $\langle 1, 0, f_x(a, b)\rangle \times \langle 0, 1, f_y(a, b)\rangle = \langle -f_x(a, b), -f_y(a, b), 1\rangle$. A point on the tangent plane is $(a, b, f(a, b))$, so the tangent plane consists of all points (x, y, z) such that

   $$\begin{aligned} \langle -f_x(a, b), -f_y(a, b), 1\rangle \cdot \langle x-a, y-b, z-f(a, b)\rangle &= 0 \\ -f_x(a, b)(x-a) - f_y(a, b)(y-b) + (z - f(a, b)) &= 0 \end{aligned}$$

   We often rearrange this as $\boxed{z = f(a, b) + f_x(a, b)(x-a) + f_y(a, b)(y-b)}$.

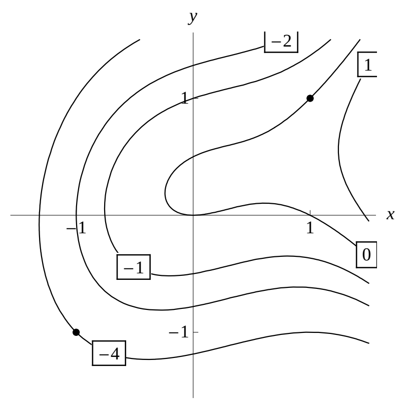
4. *Let $f(x,y,z) = \sqrt{x} + xyz$. Use linear approximation to approximate the value of $f(1.1, 1.9, 3.1)$.*

Solution. First, we should pick a point near $(1.1, 1.9, 3.1)$ at which we can easily compute the linear approximation. Let's use $(1, 2, 3)$. Then, we know that $f(x,y,z) \approx f(1,2,3) + f_x(1,2,3)(x-1) + f_y(1,2,3)(y-2) + f_z(1,2,3)(z-3)$ for (x,y,z) near $(1,2,3)$. Let's find the necessary partial derivatives:

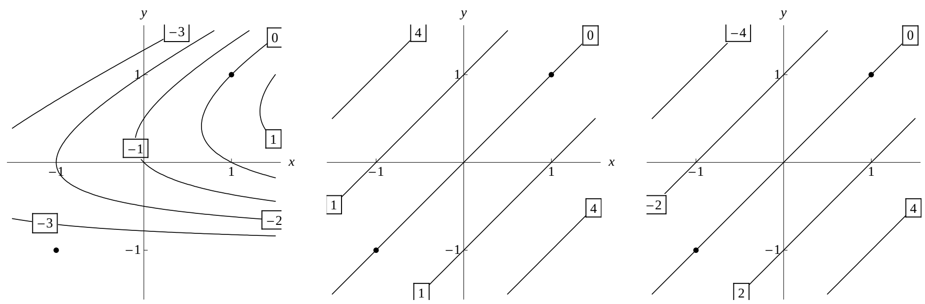
$$\begin{aligned} f_x &= \tfrac{1}{2}x^{-1/2} + yz &\Rightarrow\quad f_x(1,2,3) &= 6.5 \\ f_y &= xz &\Rightarrow\quad f_y(1,2,3) &= 3 \\ f_z &= xy &\Rightarrow\quad f_z(1,2,3) &= 2 \end{aligned}$$

Also, $f(1,2,3) = 7$. So, for (x,y,z) near $(1,2,3)$, $f(x,y,z) \approx 7 + 6.5(x-1) + 3(y-2) + 2(z-3)$. Plugging in $(x,y,z) = (1.1, 1.9, 3.1)$ gives $f(1.1, 1.9, 3.1) \approx 7 + 6.5(.1) + 3(-.1) + 2(.1) = \boxed{7.55}$. (The actual value of $f(1.1, 1.9, 3.1)$ is about 7.5278.)

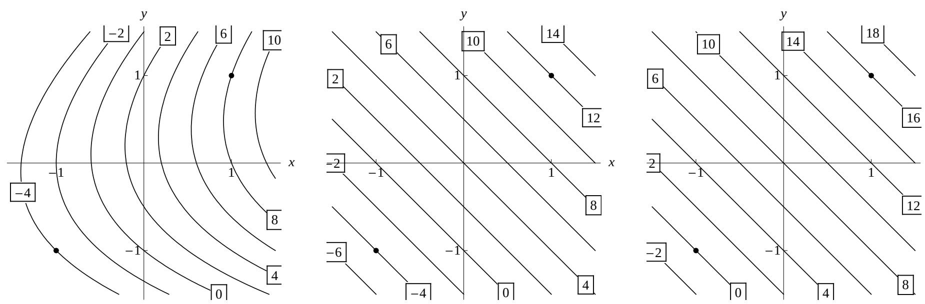
5. *Suppose the mysterious function $f(x,y)$ has the following level set diagram (contour map).*



The points $(1,1)$ and $(-1,-1)$ are marked with dots. Let $L_1(x,y)$ be the linearization of f at $(1,1)$ and $L_2(x,y)$ be the linearization of f at $(-1,-1)$. Which of the following is the level set diagram of $L_1(x,y)$? Which of the following is the level set diagram of $L_2(x,y)$?



(A) (B) (C)



(D) (E) (F)

Solution. The level set diagram of $L_1(x,y)$ is $\boxed{\text{(C)}}$, and the level set diagram of $L_2(x,y)$ is $\boxed{\text{(E)}}$.

Why? Let's think about what the linearization $L(x,y)$ of a function $f(x,y)$ at a point (a,b) represents. One way to think about it is that $L(x,y)$ is the linear function which best approximates f around (a,b). Another way to think about it is that the graph $z = L(x,y)$ is simply the plane tangent to the graph $z = f(x,y)$ of f at $(a,b,f(a,b))$. Either way you think about it, the level set diagram of $L(x,y)$ should consist of parallel lines, with the values of $L(x,y)$ being evenly spaced.(1) This eliminates diagrams (A), (B), and (D) from consideration.

Since the tangent plane and the graph of f touch at $(a,b,f(a,b))$, we know that the function f and its linearization L should have the same value at (a,b).(2) From this, we can conclude that (C) must be the level set diagram of $L_1(x,y)$ and (E) must be the level set diagram of $L_2(x,y)$.

(1) We thought about this before in #1 on the worksheet "Functions and Graphs", although we were in a slightly different context there.

(2) You can also see this from the formula we derived for the linearization: $L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$, so if we plug in $(x,y) = (a,b)$, we just get $L(a,b) = f(a,b)$.

The Chain Rule

1. Warm-up problem: A clown is inflating a spherical balloon so that its radius at time t is $\ln(1+t)$. Find the rate at which the volume of the balloon is changing at time t. (Remember that the volume of a sphere of radius x is $\frac{4}{3}\pi x^3$.)

2. An ant is walking around on the blackboard. The temperature on the blackboard at the point (x, y) is x^4y^2. The ant's position at time t is given by the vector-valued function $\vec{r}(t) = \langle \cos t, e^t \rangle$. What is the rate of change of temperature experienced by the ant (with respect to time) at any time t?

3. Quick gradient practice: Find the gradient ∇f of the following functions f.

   (a) $f(x, y) = x^2 + y^2$.

   (b) $f(x, y, z) = x^2 + y^2 + z^2$.

   (c) $f(x, y) = xy$.

   (d) $f(x, y, z) = xyz$.

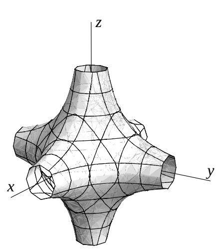
4. A fly is flying around a room; his position at time t is $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$. The temperature in the room is given by the function $f(x, y, z) = xyz$. What is the rate of change of the temperature experienced by the fly at time t?

5. Suppose $z = x^3 + xy + \cos y$, $x = t^2$, and $y = e^t$. Find $\dfrac{dz}{dt}$.

6. Suppose $u = x^2 + y^2 + z^2$, $x = s^2$, $y = \sin s$, and $z = e^s$. Find $\dfrac{du}{ds}$.

7. Suppose $z = x^2 - y^2$, $x = \sin st$, and $y = te^s$. Find $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$.

8. (Implicit differentiation.) The equation $x^2y^2 + y^2z^2 + x^2z^2 = 9$ describes the surface shown. Find $\frac{\partial z}{\partial x}$ at the point $(-1, 1, -2)$.



The Chain Rule

1. *Warm-up problem: A clown is inflating a spherical balloon so that its radius at time t is $\ln(1+t)$. Find the rate at which the volume of the balloon is changing at time t. (Remember that the volume of a sphere of radius x is $\frac{4}{3}\pi x^3$.)*

   Solution. Let's write this using function notation. (Normally, there isn't much reason to, but I'm doing it here so that you can clearly see the analogy with the multi-variable chain rule.)

   Let $f(x) = \frac{4}{3}\pi x^3$; this function describes the volume in terms of the radius x. Let $r(t) = \ln(1+t)$, the radius of the balloon at time t.

   The volume of the balloon at time t is $f(r(t))$, so we are looking for the derivative $\frac{d}{dt}f(r(t))$, which the Chain Rule says is the same as $f'(r(t))r'(t)$. Now, we just compute:

$$\begin{aligned} f'(x) &= 4\pi x^2 \\ f'(r(t)) &= 4\pi[\ln(1+t)]^2 \\ r'(t) &= \frac{1}{1+t} \\ f'(r(t))r'(t) &= \boxed{4\pi[\ln(1+t)]^2\frac{1}{1+t}} \end{aligned}$$

2. *An ant is walking around on the blackboard. The temperature on the blackboard at the point (x, y) is x^4y^2. The ant's position at time t is given by the vector-valued function $\vec{r}(t) = \langle \cos t, e^t \rangle$. What is the rate of change of temperature experienced by the ant (with respect to time) at any time t?*

   Solution. If we write the temperature as a function $f(x, y) = x^4y^2$, then we are looking for $\frac{d}{dt}f(\vec{r}(t))$. The Chain Rule says that $\frac{d}{dt}f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r'}(t)$. Let's calculate:

$$\begin{aligned} \nabla f &= \langle 4x^3y^2, 2x^4y \rangle \\ \nabla f(\vec{r}(t)) &= \langle 4(\cos t)^3e^{2t}, 2(\cos t)^4e^t \rangle \\ \vec{r'}(t) &= \langle -\sin t, e^t \rangle \\ \nabla f(\vec{r}(t)) \cdot \vec{r'}(t) &= \boxed{4(\cos t)^3e^{2t}(-\sin t) + 2(\cos t)^4e^te^t} \end{aligned}$$

3. Quick gradient practice: Find the gradient ∇f of the following functions f.

   (a) $f(x, y) = x^2 + y^2$.

   Solution. $\nabla f = \langle 2x, 2y \rangle$.

   (b) $f(x, y, z) = x^2 + y^2 + z^2$.

   Solution. $\nabla f = \langle 2x, 2y, 2z \rangle$.

   (c) $f(x, y) = xy$.

   Solution. $\nabla f = \langle y, x \rangle$.

   (d) $f(x, y, z) = xyz$.

   Solution. $\nabla f = \langle yz, xz, xy \rangle$.

4. *A fly is flying around a room; his position at time t is $\vec{r}(t) = \langle \cos t, \sin t, t\rangle$. The temperature in the room is given by the function $f(x,y,z) = xyz$. What is the rate of change of the temperature experienced by the fly at time t?*

   Solution. The temperature experienced by the fly at time t is $f(\vec{r}(t))$, so we are looking for $\frac{d}{dt}f(\vec{r}(t))$. By the Chain Rule, this is equal to $\nabla f(\vec{r}(t)) \cdot \vec{r'}(t)$. Let's calculate:

$$\begin{aligned}
\nabla f &= \langle yz, xz, xy\rangle \\
\nabla f(\vec{r}(t)) &= \langle t\sin t, t\cos t, \cos t \sin t\rangle \\
\vec{r'}(t) &= \langle -\sin t, \cos t, 1\rangle \\
\nabla f(\vec{r}(t)) \cdot \vec{r'}(t) &= \boxed{-t\sin^2 t + t\cos^2 t + \cos t \sin t}
\end{aligned}$$

5. *Suppose $z = x^3 + xy + \cos y$, $x = t^2$, and $y = e^t$. Find $\dfrac{dz}{dt}$.*

   Solution. Let's write this in function notation. Let $f(x,y) = x^3 + xy + \cos y$ and $\vec{r}(t) = \langle t^2, e^t\rangle$. Then, we are looking for $\frac{d}{dt}f(\vec{r}(t))$. By the Chain Rule, this is equal to $\nabla f(\vec{r}(t)) \cdot \vec{r'}(t)$. Let's calculate:

$$\begin{aligned}
\nabla f &= \langle 3x^2 + y, x - \sin y\rangle \\
\nabla f(\vec{r}(t)) &= \langle 3t^4 + e^t, t^2 - \sin e^t\rangle \\
\vec{r'}(t) &= \langle 2t, e^t\rangle \\
\nabla f(\vec{r}(t)) \cdot \vec{r'}(t) &= \boxed{(3t^4 + e^t)2t + (t^2 - \sin e^t)e^t}
\end{aligned}$$

6. *Suppose $u = x^2 + y^2 + z^2$, $x = s^2$, $y = \sin s$, and $z = e^s$. Find $\dfrac{du}{ds}$.*

   Solution. Let's write this in function notation. Let $f(x,y,z) = x^2 + y^2 + z^2$ and $\vec{r}(s) = \langle s^2, \sin s, e^s\rangle$. We are looking for $\frac{d}{ds}f(\vec{r}(s))$. By the Chain Rule, this is equal to $\nabla f(\vec{r}(s)) \cdot \vec{r'}(s)$. Let's calculate:

$$\begin{aligned}
\nabla f &= \langle 2x, 2y, 2z\rangle \\
\nabla f(\vec{r}(s)) &= \langle 2s^2, 2\sin s, 2e^s\rangle \\
\vec{r'}(s) &= \langle 2s, \cos s, e^s\rangle \\
\nabla f(\vec{r}(s)) \cdot \vec{r'}(s) &= \boxed{4s^3 + 2\sin s \cos s + 2e^{2s}}
\end{aligned}$$

7. *Suppose $z = x^2 - y^2$, $x = \sin st$, and $y = te^s$. Find $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$.*

   Solution. This may look more complicated than the previous problems because there are two variables s and t to worry about, but it really isn't. After all, when finding $\frac{\partial z}{\partial s}$, we just treat t as a constant, so we're really thinking only about one variable s. Similarly, when finding $\frac{\partial z}{\partial t}$, we can pretty much ignore s and just focus on t.

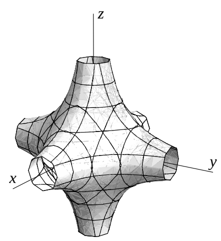
   Let's write $f(x,y) = x^2 - y^2$, so $\nabla f = \langle 2x, -2y\rangle$.

   To find $\frac{\partial z}{\partial s}$, we'll treat t as a constant and use $\vec{r}(s) = \langle \sin st, te^s\rangle$. Then, we're looking for $\frac{d}{ds}f(\vec{r}(s)) =$

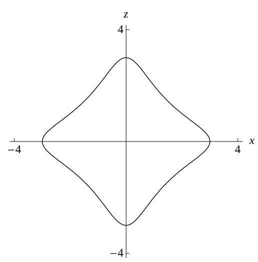
$\nabla f(\vec{r}(s)) \cdot \vec{r'}(s) = \langle 2\sin st, -2te^s\rangle \cdot \langle t\cos st, te^s\rangle$, which simplifies to $\boxed{2t\sin st\cos st - 2t^2e^{2s}}$.[(1)]

Similarly, to find $\frac{\partial z}{\partial t}$, we'll treat s as a constant and use $\vec{r}(t) = \langle \sin st, te^s\rangle$. Then, we're looking for $\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r'}(t) = \langle 2\sin st, -2te^s\rangle \cdot \langle s\cos st, e^s\rangle$, or $\boxed{(2\sin st)(s\cos st) - (2te^s)(e^s)}$.

8. *(Implicit differentiation.) The equation* $x^2y^2 + y^2z^2 + x^2z^2 = 9$ *describes the surface shown. Find* $\frac{\partial z}{\partial x}$ *at the point* $(-1, 1, -2)$.



Solution. One way to think about what this problem is asking is to remember what partial derivatives mean. When we're looking for $\frac{\partial z}{\partial x}$, we're really thinking of keeping y constant (in this case, $y = 1$). Here is a picture of the trace in $y = 1$ of the surface:



We're looking for $\frac{\partial z}{\partial x}$, which is the slope of this curve at $(x, z) = (-1, -2)$. So, we should think of y as being constant but z as depending on x.

There are two ways we could do this:

- **Just using the single-variable chain rule:**

  Let's take the partial derivative of both sides of the equation $x^2y^2 + y^2z^2 + x^2z^2 = 9$ with respect to x. Remember that we are thinking of y as being constant but z as depending on x. Therefore,

  $$2xy^2 + y^2(2z)\frac{\partial z}{\partial x} + x^2(2z)\frac{\partial z}{\partial x} + 2xz^2 = 0.$$

  (This is exactly how you do implicit differentiation in single-variable calculus.) Rearranging,

  $$(2y^2z + 2x^2z)\frac{\partial z}{\partial x} = -2xy^2 - 2xz^2,$$

  so

  $$\frac{\partial z}{\partial x} = -\frac{2xy^2 + 2xz^2}{2y^2z + 2x^2z}.$$

  Plugging in $x = -1$, $y = 1$, $z = -2$ gives $\boxed{\frac{\partial z}{\partial x} = -\frac{5}{4}}$.

[(1)] If you prefer, you can certainly write $\vec{r}(s, t) = \langle \sin st, te^s\rangle$. Then, use partial derivatives instead of ordinary derivatives and write $\frac{\partial}{\partial s} f(\vec{r}(s, t)) = \nabla f(\vec{r}(s, t)) \cdot \vec{r}_s$. This amounts to exactly the same thing.

- **Using the multi-variable chain rule:**

  Alternatively, we can use the multi-variable chain rule. Let's write $F(x, y, z) = x^2y^2 + y^2z^2 + x^2z^2$ and differentiate both sides of the equation $F(x, y, z) = 9$ with respect to x. Using the multi-variable chain rule, we get

  $$\frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0.$$

  Of course, $\frac{\partial x}{\partial x}$ is just 1. Since we are thinking of y as a constant, $\frac{\partial F}{\partial y} = 0$. Finally, $\frac{\partial z}{\partial x}$ is just $\frac{\partial z}{\partial x}$, the thing we are looking for. So, the above equation simplifies to

  $$F_x + F_z\frac{\partial z}{\partial x} = 0,$$

  and we can solve for $\frac{\partial z}{\partial x}$ to get

  $$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}.$$

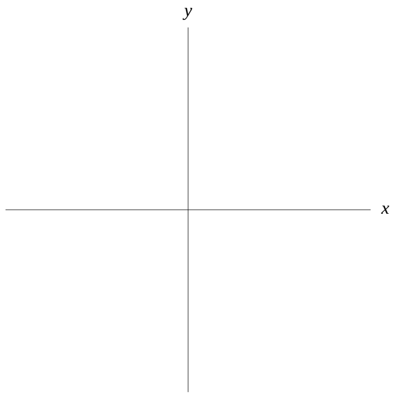
  If we differentiate and plug in $x = -1$, $y = 1$, $z = -2$, we again get that $\frac{\partial z}{\partial x} = -\frac{5}{4}$.

The Gradient and Level Sets

1. Let $f(x, y) = x^2 + y^2$.

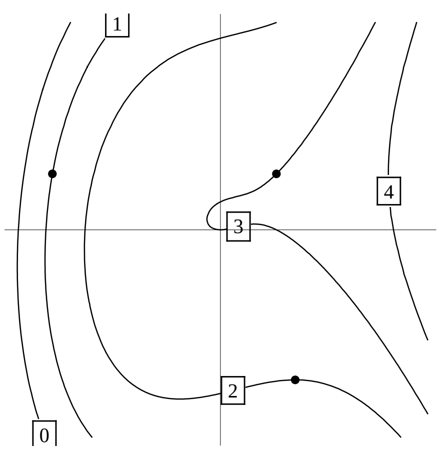
   (a) Find the gradient ∇f.

   (b) Pick your favorite positive number k, and let $\mathcal{C}$ be the curve $f(x, y) = k$. Draw the curve on the axes below. Now pick a point (a, b) on the curve $\mathcal{C}$. What is the vector $\nabla f(a, b)$? Draw the vector $\nabla f(a, b)$ with its tail at the point (a, b). What relationship does the vector have to the curve?



   (c) Let $\vec{r}(t)$ be any parameterization of your curve $\mathcal{C}$. What is $f(\vec{r}(t))$? What happens if you use the Chain Rule to find $\frac{d}{dt} f(\vec{r}(t))$? Use this to explain your observation from (b).

2. Here is the level set diagram (contour map) of a function $f(x, y)$. The value of $f(x, y)$ on each level set is labeled. For each of the three points (a, b) marked in the picture, draw a vector showing the direction of $\nabla f(a, b)$. (Don't worry about the magnitude of $\nabla f(a, b)$.)



3. Let S be the cylinder $x^2 + y^2 = 4$. Find the plane tangent to S at the point $(1, \sqrt{3}, 5)$.

4. Let S be the surface $z = y \sin x$. Find the plane tangent to S at the point $\left(\frac{\pi}{6}, 2, 1\right)$.

5. Suppose that $3x + 4y - 5z = -4$ is the plane tangent to the graph of $f(x, y)$ at the point $(1, 2, 3)$.

   (a) Find $\nabla f(1, 2)$.

   (b) Use linear approximation to approximate $f(1.1, 1.9)$.

The Gradient and Level Sets

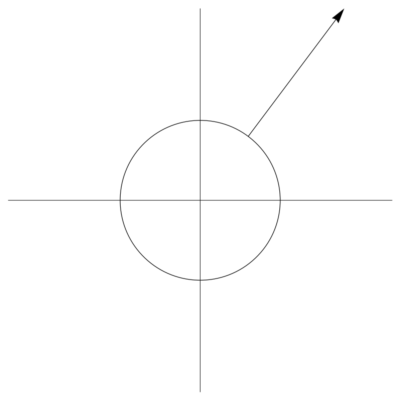
1. Let $f(x, y) = x^2 + y^2$.

   (a) *Find the gradient* ∇f.

   Solution. $\nabla f(x, y) = \langle 2x, 2y \rangle$.

   (b) *Pick your favorite positive number* k*, and let* $\mathcal{C}$ *be the curve* $f(x, y) = k$*. Draw the curve on the axes below. Now pick a point* (a, b) *on the curve* $\mathcal{C}$*. What is the vector* $\nabla f(a, b)$*? Draw the vector* $\nabla f(a, b)$ *with its tail at the point* (a, b)*. What relationship does the vector have to the curve?*

   Solution. The curve $f(x, y) = k$ is a circle in the xy-plane centered at the origin. The vector $\nabla f(a, b)$ is equal to $\langle 2a, 2b \rangle$, and this vector is perpendicular to the circle (no matter what value of k you picked and what point (a, b) you picked).

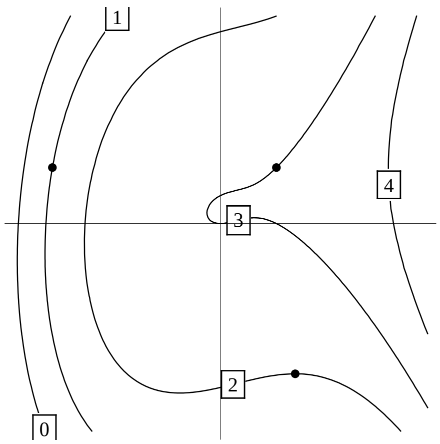


   (c) *Let* $\vec{r}(t)$ *be any parameterization of your curve* $\mathcal{C}$*. What is* $f(\vec{r}(t))$*? What happens if you use the Chain Rule to find* $\frac{d}{dt} f(\vec{r}(t))$*? Use this to explain your observation from (b).*

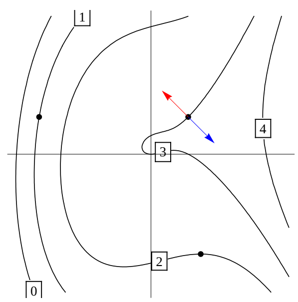
   Solution. $f(\vec{r}(t))$ is always equal to k, since $\vec{r}(t)$ parameterizes the curve $\mathcal{C}$, and the curve $\mathcal{C}$ is exactly the set of points where $f(x, y) = k$.

   Since $f(\vec{r}(t)) = k$ for all t, $\frac{d}{dt} f(\vec{r}(t)) = \frac{d}{dt} k = 0$. By the Chain Rule, we know that $\frac{d}{dt} f(\vec{r}(t))$ is also equal to $\nabla f(\vec{r}(t)) \cdot \vec{r}'(t)$, so the gradient $\nabla f(\vec{r}(t))$ is always perpendicular to $\vec{r'}(t)$. Recall that $\vec{r'}(t)$ gives the direction of the tangent vector, so ∇f is always perpendicular to the tangent vector.

2. *Here is the level set diagram (contour map) of a function* $f(x, y)$*. The value of* $f(x, y)$ *on each level set is labeled. For each of the three points* (a, b) *marked in the picture, draw a vector showing the direction of* $\nabla f(a, b)$*. (Don't worry about the magnitude of* $\nabla f(a, b)$*.)*

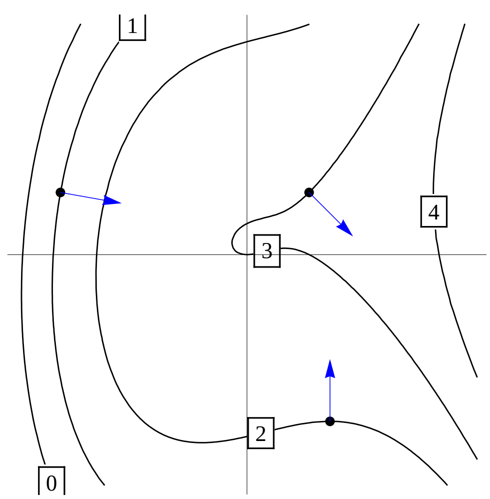


Solution. Let's look at the upper right point. We know that $\nabla f(a,b)$ is perpendicular to the curve at (a,b), but there are still two possible directions:



To determine which is correct, we use the definition of $\nabla f(a,b)$: it is defined to be $\langle f_x(a,b), f_y(a,b)\rangle$. We can tell from the original picture that $f_x(a,b) > 0$ (see the worksheet "Partial Derivatives" for more explanation), so the vector $\nabla f(a,b)$ must point to the right.

Using the same reasoning for the other two points, we get the following vectors:



3. *Let S be the cylinder $x^2 + y^2 = 4$. Find the plane tangent to S at the point $(1, \sqrt{3}, 5)$.*

Solution. If we let $f(x, y, z) = x^2 + y^2$, then S is the level set $f(x, y, z) = 4$. Therefore, $\nabla f(1, \sqrt{3}, 5)$ will be a normal vector for the tangent plane we want.

$\nabla f = \langle 2x, 2y, 0\rangle$, so $\nabla f(1, \sqrt{3}, 5) = \langle 2, 2\sqrt{3}, 0\rangle$. Since $(1, \sqrt{3}, 5)$ is a point on the tangent plane, the tangent plane has equation $\langle 2, 2\sqrt{3}, 0\rangle \cdot \langle x - 1, y - \sqrt{3}, z - 5\rangle = 0$, or $\boxed{2(x-1) + 2\sqrt{3}(y - \sqrt{3}) = 0}$.

Note that this is exactly the same problem as #1 from the worksheet "Tangent Planes and Linear Approximation"; this solution, however, is simpler than the one we came up with before.

4. *Let S be the surface $z = y \sin x$. Find the plane tangent to S at the point $\left(\frac{\pi}{6}, 2, 1\right)$.*

   Solution. If we let $f(x, y, z) = y \sin x - z$, then S is the level set $f(x, y, z) = 0$. Therefore, $\nabla f\left(\frac{\pi}{6}, 2, 1\right)$ will be a normal vector for the tangent plane we want.

   $\nabla f = \langle y \cos x, \sin x, -1\rangle$, so $\nabla f\left(\frac{\pi}{6}, 2, 1\right) = \left\langle \sqrt{3}, \frac{1}{2}, -1\right\rangle$. Since $\left(\frac{\pi}{6}, 2, 1\right)$ is a point on the tangent plane, the tangent plane has equation $\left\langle \sqrt{3}, \frac{1}{2}, -1\right\rangle \cdot \left\langle x - \frac{\pi}{6}, y - 2, z - 1\right\rangle = 0$, or $\boxed{\sqrt{3}\left(x - \frac{\pi}{6}\right) + \frac{1}{2}(y-2) - (z-1) = 0}$.

   Note that this is exactly the same problem as #2 from the worksheet "Tangent Planes and Linear Approximation"; this solution, however, is simpler than the one we came up with before.

5. Suppose that $3x + 4y - 5z = -4$ is the plane tangent to the graph of $f(x, y)$ at the point $(1, 2, 3)$.

   (a) *Find $\nabla f(1, 2)$.*

   Solution. We are told that $3x + 4y - 5z = -4$ is the plane tangent to the surface $z = f(x, y)$ at the point $(1, 2, 3)$.

   Let's think about how we normally find the tangent plane to a graph. We could use the result of #3 from the worksheet "Tangent Planes and Linear Approximation", or we could simply express the graph as a level surface. Let's take the second approach.

   To find the plane tangent to $z = f(x, y)$ at the point $(1, 2, 3)$, let's write $g(x, y, z) = f(x, y) - z$. Then, the surface $z = f(x, y)$ can also be described as the level set $g(x, y, z) = 0$. Therefore, $\nabla g(1, 2, 3)$ must be a normal vector for the tangent plane. From the equation we are given from the tangent plane, we can see that $\langle 3, 4, -5\rangle$ is also a normal vector for the tangent plane. Therefore, we know that $\nabla g(1, 2, 3)$ must be parallel to $\langle 3, 4, -5\rangle$; in other words, $\nabla g(1, 2, 3)$ must be a scalar multiple of $\langle 3, 4, -5\rangle$.

   Of course, we're interested in $\nabla f = \langle f_x, f_y\rangle$, and we had defined g in terms of f, so let's try to write ∇g in terms of f. Since $g(x, y, z) = f(x, y) - z$, $\nabla g = \langle f_x, f_y, -1\rangle$, and $\nabla g(1, 2, 3) = \langle f_x(1, 2), f_y(1, 2), -1\rangle$.

   So, we can conclude that $\langle f_x(1, 2), f_y(1, 2), -1\rangle$ is a scalar multiple of $\langle 3, 4, -5\rangle$. What scalar? Well, looking at the last component of each vector, we can see that the scalar must be $\frac{1}{5}$; that is, $\langle f_x(1, 2), f_y(1, 2), -1\rangle = \frac{1}{5}\langle 3, 4, -5\rangle$. So, $f_x(1, 2) = \frac{3}{5}$ and $f_y(1, 2) = \frac{4}{5}$, which means that $\nabla f(1, 2) = \boxed{\left\langle \frac{3}{5}, \frac{4}{5}\right\rangle}$.

   (b) *Use linear approximation to approximate $f(1.1, 1.9)$.*

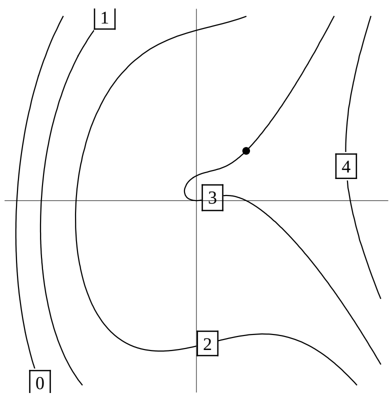
   Solution. Now that we know $f_x(1, 2)$ and $f_y(1, 2)$, we can write down the linearization $L(x, y)$ of $f(x, y)$ at $(1, 2)$.[(1)] However, there is an even simpler way to do this problem. Remember

(1) It is $L(x, y) = f(1, 2) + f_x(1, 2)(x - 1) + f_y(1, 2)(y - 2) = 3 + \frac{3}{5}(x - 1) + \frac{4}{5}(y - 2)$.

that the graph of $L(x, y)$ was exactly the tangent plane to f at $(1, 2)$. So, $z = L(x, y)$ is the same as $3x + 4y - 5z = -4$, or $z = \frac{3}{5}x + \frac{4}{5}y + \frac{4}{5}$. That is, $L(x, y) = \frac{3}{5}x + \frac{4}{5}y + \frac{4}{5}$. So, $f(1.1, 1.9) \approx L(1.1, 1.9) = \frac{3}{5}(1.1) + \frac{4}{5}(1.9) + \frac{4}{5} = \boxed{2.98}$.

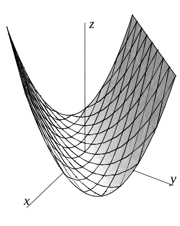
Directional Derivatives

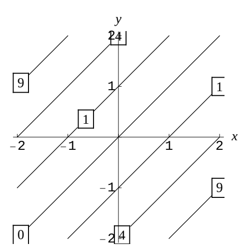
1. Here is the level set diagram of a function $f(x,y)$; the value of f on each level set is labeled. Imagine that $f(x,y)$ represents temperature on the blackboard, and an ant is standing at the point (a,b), which is marked on the diagram.



- What direction should the ant go to warm up most quickly? That is, in what direction should he go to experience the highest instantaneous rate of change of temperature (with respect to distance)?
- What direction should the ant go to cool down most quickly? That is, in what direction should he go to experience the lowest (most negative) instantaneous rate of change of temperature?

2. Let $f(x,y) = (x-y)^2 = x^2 - 2xy + y^2$. (The graph and level set diagram of f are shown.)





Calculate the following directional derivatives of f.

(a) $D_{\vec{u}}f(1,0)$ where $\vec{u} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$.

(c) $D_{\vec{u}}f(0,1)$ where $\vec{u} = \left\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle$.

(b) $D_{\vec{u}}f(1,0)$ where $\vec{u} = \left\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle$.

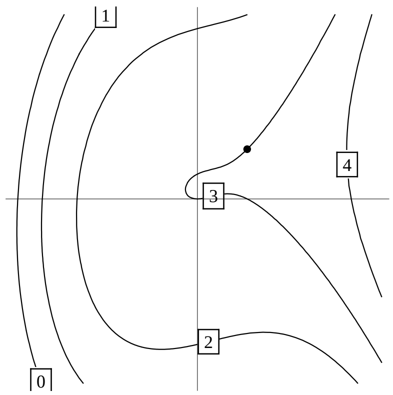
(d) $D_{\vec{u}}f(0,1)$ where $\vec{u} = \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$.

3. A fly is flying around a room in which the temperature is given by $T(x, y, z) = x^2 + y^4 + 2z^2$. The fly is at the point $(1, 1, 1)$ and realizes that he's cold. In what direction should he fly to warm up most quickly? If he flies in this direction, what will be the instantaneous rate of change of his temperature?

4. You're hiking a mountain which is the graph of $f(x, y) = 15 - x^2 - 2xy - 3y^2$. You're standing at $(1, 1, 9)$. You wish to head in a direction which will maintain your elevation (so you want the instantaneous change in your elevation to be 0). How many possible directions are there for you to head? What are they?

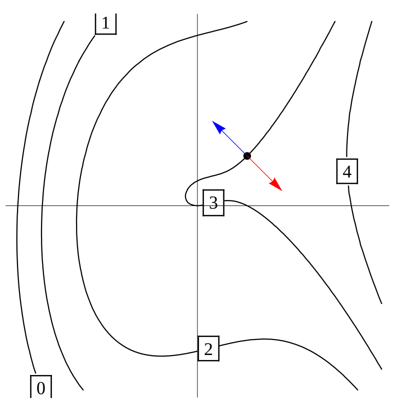
Directional Derivatives

1. Here is the level set diagram of a function $f(x, y)$; the value of f on each level set is labeled. Imagine that $f(x, y)$ represents temperature on the blackboard, and an ant is standing at the point (a, b), which is marked on the diagram.

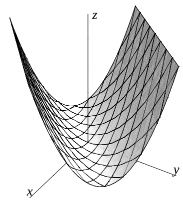


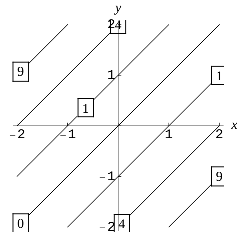
- *What direction should the ant go to warm up most quickly? That is, in what direction should he go to experience the highest instantaneous rate of change of temperature (with respect to distance)?*
- *What direction should the ant go to cool down most quickly? That is, in what direction should he go to experience the lowest (most negative) instantaneous rate of change of temperature?*

Solution. Using the same reasoning as in #3, the fly should fly in the direction of $\nabla f(a, b)$ to warm up most quickly (marked in red below) and the direction of $-\nabla f(a, b)$ to cool down most quickly (marked in blue below). Remember that we always want our direction vectors to be unit vectors, so the direction of $\nabla f(a, b)$ really means the vector $\frac{\nabla f(a,b)}{|\nabla f(a,b)|}$, and the direction of $-\nabla f(a, b)$ really means the vector $-\frac{\nabla f(a,b)}{|\nabla f(a,b)|}$.



2. Let $f(x, y) = (x - y)^2 = x^2 - 2xy + y^2$. (The graph and level set diagram of f are shown.)





Calculate the following directional derivatives of f.

(a) $D_{\vec{u}}f(1,0)$ *where* $\vec{u} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$.

Solution. We'll use the formula $D_{\vec{u}}f(1,0) = \nabla f(1,0) \cdot \vec{u}$. In this case, $\nabla f = \langle 2(x-y), -2(x-y) \rangle$, so $\nabla f(1,0) = \langle 2, -2 \rangle$ and $D_{\vec{u}}f(1,0) = \langle 2, -2 \rangle \cdot \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle = \boxed{0}$.

(b) $D_{\vec{u}}f(1,0)$ *where* $\vec{u} = \left\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle$.

Solution. $D_{\vec{u}}f(1,0) = \nabla f(1,0) \cdot \vec{u} = \langle 2, -2 \rangle \cdot \left\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle = \boxed{2\sqrt{2}}$.

(c) $D_{\vec{u}}f(0,1)$ *where* $\vec{u} = \left\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle$.

Solution. $D_{\vec{u}}f(0,1) = \nabla f(0,1) \cdot \vec{u} = \langle -2, 2 \rangle \cdot \left\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle = \boxed{-2\sqrt{2}}$.

(d) $D_{\vec{u}}f(0,1)$ *where* $\vec{u} = \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$.

Solution. $D_{\vec{u}}f(0,1) = \nabla f(0,1) \cdot \vec{u} = \langle -2, 2 \rangle \cdot \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle = \boxed{2\sqrt{2}}$.

3. *A fly is flying around a room in which the temperature is given by* $T(x,y,z) = x^2 + y^4 + 2z^2$. *The fly is at the point* $(1,1,1)$ *and realizes that he's cold. In what direction should he fly to warm up most quickly? If he flies in this direction, what will be the instantaneous rate of change of his temperature?*

Solution. To answer the first question, we want to find the unit vector $\vec{u}$ which maximizes the directional derivative $D_{\vec{u}}T(1,1,1)$. We know that this directional derivative is equal to $\nabla T(1,1,1) \cdot \vec{u}$, which is in turn equal to $|\nabla T(1,1,1)||\vec{u}| \cos\theta$, where θ is the angle between $\nabla T(1,1,1)$ and $\vec{u}$. Since we are looking for a unit vector $\vec{u}$, $|\vec{u}| = 1$, so the directional derivative will just be $|\nabla T(1,1,1)| \cos\theta$. To maximize this, we need to make $\cos\theta$ as large as possible. The largest that $\cos\theta$ can be is 1, and this happens when $\theta = 0$.

So, we want the angle θ between $\nabla T(1,1,1)$ and $\vec{u}$ to be 0, which means that we want $\vec{u}$ to go in the same direction as $\nabla T(1,1,1)$. We can calculate $\nabla T(1,1,1)$ easily:

$$\begin{aligned} \nabla T &= \langle 2x, 4y^3, 4z \rangle \\ \nabla T(1,1,1) &= \langle 2, 4, 4 \rangle \end{aligned}$$

We want $\vec{u}$ to be a unit vector going in the same direction, which means we simply divide this vector by its length (which is $\sqrt{2^2 + 4^2 + 4^2} = 6$), so $\vec{u} = \boxed{\left\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right\rangle}$.

The instantaneous rate of change of the fly's temperature when he flies in this direction is simply $D_{\vec{u}}T(1,1,1) = \nabla T(1,1,1) \cdot \vec{u} = \langle 2,4,4\rangle \cdot \left\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right\rangle = \boxed{6}$.

4. *You're hiking a mountain which is the graph of* $f(x,y) = 15 - x^2 - 2xy - 3y^2$. *You're standing at* $(1,1,9)$. *You wish to head in a direction which will maintain your elevation (so you want the instantaneous change in your elevation to be 0). How many possible directions are there for you to head? What are they?*

Solution. If you head in a direction given by a unit vector $\vec{u}$, then the directional derivative will be $\nabla f(1,1) \cdot \vec{u}$. You want this to be 0, so you want $\vec{u}$ to be perpendicular to $\nabla f(1,1)$. There are $\boxed{\text{two unit vectors}}$ in the plane which are perpendicular to $\nabla f(1,1)$.

We calculate $\nabla f = \langle -2x-2y, -2x-6y\rangle$, so $\nabla f(1,1) = \langle -4,-8\rangle$. The two unit vectors perpendicular to this are $\boxed{\langle \frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\rangle \text{ and } \langle -\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\rangle}$.[1]

[1] To find these, start with any vector that is perpendicular to $\langle -4,-8\rangle$; one example is $\langle 8,-4\rangle$ (we can tell it is perpendicular to $\langle -4,-8\rangle$ because if we dot it with $\langle -4,-8\rangle$, we get 0). To get a unit vector, divide this vector by its length ($4\sqrt{5}$) to get $\langle \frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\rangle$. The other unit vector which is perpendicular to $\langle -4,-8\rangle$ must be the negative of the one we have already found.

Maxima and Minima

1. Find all critical points of $f(x, y) = x^2 + y^2$.

2. Find all critical points of $f(x, y) = -x^2 - y^2 - 4xy$.

3. Let $f(x, y)$ be a function of two variables and $\vec{u} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$. Write $D_{\vec{u}}(D_{\vec{u}}f)$ in terms of f_{xx}, f_{xy}, and f_{yy}. (You may assume, as we do most of the time, that f and all of its derivatives are continuous.)

4. Find all critical points of $f(x, y) = xy^2 - x^2 - 2y^2$ and determine whether each is a local minimum, local maximum, or saddle point.

5. Find the absolute maximum and minimum values of $f(x, y) = y^2 - x^2$ on the square $|x| \leq 1, |y| \leq 1$.

Maxima and Minima

1. *Find all critical points of* $f(x, y) = x^2 + y^2$.

   Solution. We want $\nabla f = \vec{0}$. We calculate $\nabla f = \langle 2x, 2y\rangle$, and the only way that this can be equal to $\langle 0, 0\rangle$ is for x and y to both be 0. Thus, the only critical point is $(x, y) = \boxed{(0, 0)}$.

2. *Find all critical points of* $f(x, y) = -x^2 - y^2 - 4xy$.

   Solution. We want $\nabla f = \vec{0}$, and we can calculate $\nabla f = \langle -2x-4y, -2y-4x\rangle$. So, we want $-2x-4y = 0$ and $-2y - 4x = 0$. The first equation tells us that $x = -2y$; plugging this into the second, we get that $6y = 0$, so $y = 0$. Plugging this back into the first equation gives $x = 0$, so $\boxed{(0, 0)}$ is the only critical point.

3. *Let* $f(x, y)$ *be a function of two variables and* $\vec{u} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$. *Write* $D_{\vec{u}}(D_{\vec{u}} f)$ *in terms of* f_{xx}, f_{xy}, *and* f_{yy}. *(You may assume, as we do most of the time, that* f *and all of its derivatives are continuous.)*

   Solution. Let's write $g = D_{\vec{u}} f$. Then, we are looking for $D_{\vec{u}} g$. First, let's just figure out what $D_{\vec{u}} f$ is:

$$\begin{aligned} D_{\vec{u}} f &= \nabla f \cdot \vec{u} \\ &= \langle f_x, f_y \rangle \cdot \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle \\ &= \frac{3}{5} f_x + \frac{4}{5} f_y \end{aligned}$$

   We're looking for $D_{\vec{u}} g$, which (by the same reasoning) is $\frac{3}{5} g_x + \frac{4}{5} g_y$. Now, since $g = \frac{3}{5} f_x + \frac{4}{5} f_y$,

$$\begin{aligned} g_x &= \frac{3}{5} f_{xx} + \frac{4}{5} f_{yx} \\ g_y &= \frac{3}{5} f_{xy} + \frac{4}{5} f_{yy} \end{aligned}$$

   Therefore,

$$\begin{aligned} D_{\vec{u}}(D_{\vec{u}} f) &= D_{\vec{u}} g \\ &= \frac{3}{5} g_x + \frac{4}{5} g_y \\ &= \frac{3}{5}\left(\frac{3}{5} f_{xx} + \frac{4}{5} f_{yx}\right) + \frac{4}{5}\left(\frac{3}{5} f_{xy} + \frac{4}{5} f_{yy}\right) \\ &= \frac{9}{25} f_{xx} + \frac{12}{25} f_{yx} + \frac{12}{25} f_{xy} + \frac{16}{25} f_{yy} \\ &= \boxed{\frac{9}{25} f_{xx} + \frac{24}{25} f_{xy} + \frac{16}{25} f_{yy}} \text{ by Clairaut's Theorem (which says that } f_{xy} = f_{yx}) \end{aligned}$$

4. *Find all critical points of* $f(x, y) = xy^2 - x^2 - 2y^2$ *and determine whether each is a local minimum, local maximum, or saddle point.*

   Solution. We want $\nabla f = \vec{0}$, and we can calculate $\nabla f = \langle y^2 - 2x, 2xy - 4y\rangle$. So, we want $y^2 - 2x = 0$ and $2xy - 4y = 0$. The first equation tells us that $x = \frac{y^2}{2}$; plugging this into the second equation gives

$y^3 - 4y = 0$. We can factor this as $y(y-2)(y+2) = 0$, so $y = 0$, 2, or -2. Since $x = \frac{y^2}{2}$, this gives us three points, $(0,0)$ and $(2, \pm 2)$.

To classify our three critical points, we need to calculate the discriminant $f_{xx}f_{yy} - f_{xy}^2$ at each point. Let's first just calculate it in terms of x and y: $f_{xx} = -2$, $f_{xy} = 2y$, and $f_{yy} = 2x - 4$, so $f_{xx}f_{yy} - f_{xy}^2 = -4(x-2) - 4y^2$. So:

- At $(0,0)$, the discriminant is 8, which tells us that $(0,0)$ is either a local minimum or a local maximum. To decide which, we can use the fact that $f_{xx} = -2 < 0$, which tells us that $\boxed{(0,0) \text{ is a local maximum}}$.

- At $(2,2)$, the discriminant is -16, so $\boxed{(2,2) \text{ is a saddle point}}$.

- At $(2,-2)$, the discriminant is -16, so $\boxed{(2,-2) \text{ is a saddle point}}$.

5. *Find the absolute maximum and minimum values of $f(x,y) = y^2 - x^2$ on the square $|x| \le 1, |y| \le 1$.*

   Solution. Since $|x| \le 1$, $|y| \le 1$ is a closed bounded region in $\mathbb{R}^2$, we know that $f(x,y)$ must attain its maximum and minimum values on this region. Remember that the basic strategy is to check the critical points and boundary separately. That is, we'll find all the critical points, and then we'll find all the points on the boundary where the absolute minimum or maximum might occur; after we've done that, we'll plug each point into f to see which gives the highest value and which gives the lowest.

   The critical points are where $\nabla = \vec{0}$. Since $\nabla f = \langle -2x, 2y \rangle$, the only critical point is $(0,0)$.[(1)]

   Now, we look at the boundary. The boundary is composed of four separate pieces, and we'll look at each one individually:

   - Let's look at the piece where $x = 1$. Here, $f(x,y) = f(1,y) = y^2 - 1$, and we are concerned with y in the interval $[-1,1]$. So, we want to maximize and minimize $y^2 - 1$ on the interval $[-1,1]$. You can do this using calculus or just by looking at the graph of $y^2 - 1$; either way, you should find that it's biggest when $y = \pm 1$ and smallest when $y = 0$. This gives us three candidate points, $(1,1)$, $(1,-1)$, and $(1,0)$.

   - On the piece where $x = -1$, $f(x,y) = f(-1,y) = y^2 - 1$, and we again want to maximize and minimize $y^2 - 1$ on the interval $[-1,1]$. The maximum is when $y = \pm 1$ and the minimum is when $y = 0$. This gives us three more candidate points, $(-1,1)$, $(-1,-1)$, and $(-1,0)$.

   - On the piece $y = 1$, $f(x,y) = f(x,1) = 1 - x^2$, and we want to maximize and minimize this on the interval $[-1,1]$. The maximum occurs when $x = 0$, and the minimum occurs when $x = \pm 1$, so we get three more candidates, $(0,1)$, $(-1,1)$, and $(1,1)$.

   - On the piece $y = -1$, $f(x,y) = f(x,-1) = 1 - x^2$, and we get three more candidates, $(0,-1)$, $(-1,-1)$, and $(1,-1)$.

   So, we now have a big list of points where the absolute minimum and maximum might occur: $(0,0)$, $(1,1)$, $(1,-1)$, $(1,0)$, $(-1,1)$, $(-1,-1)$, $(-1,0)$, $(0,1)$, $(-1,1)$, $(1,1)$, $(0,-1)$, $(-1,-1)$, and $(1,-1)$. We evaluate f at each of these points, and we find that the absolute maximum occurs at $(0,\pm 1)$, where the function is equal to 1. The absolute minimum occurs at $(\pm 1, 0)$, where the function is equal to -1.

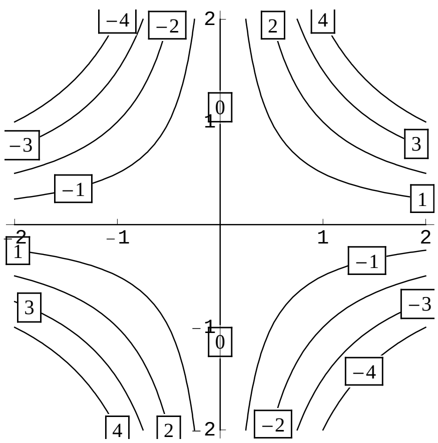
(1) You can use the Second Derivative Test to see what type of critical point $(0,0)$ is, but it's not necessary to do so since we are really just compiling a list of candidate points that we will check later.

Lagrange Multipliers

Here are some examples of problems that can be solved using Lagrange multipliers:

- The equation $g(x, y) = c$ defines a curve in the plane. Find the point(s) on the curve closest to the origin.
- The temperature in a room is given by $T(x, y, z) = 100x + xy + 5yz^2$. A bug walks on a spherical balloon which is given by the equation $x^2 + y^2 + z^2 = 3$. What is the warmest point the bug can reach?

1. Here is the level set diagram of $f(x, y) = 2xy$.



(a) Estimate the maximum and minimum values of f on the ellipse $\dfrac{x^2}{4} + y^2 = 1$.

(b) Find the maximum and minimum values of f on the ellipse $\dfrac{x^2}{4} + y^2 = 1$.

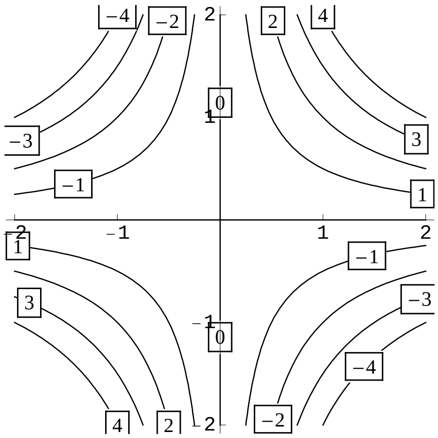
2. Minimize $2x + 4y + 6z$ if $x^2 + y^2 + z^2 = 14$.

3. Minimize $x^2 + y^2 + z^2$ subject to the constraints $x + y + z = 6$ and $x + 2y - 3z = 14$.

4. Maximize and minimize $f(x, y, z) = xyz$ subject to the constraint that $x^2 + y^2 + z^2 = 1$.

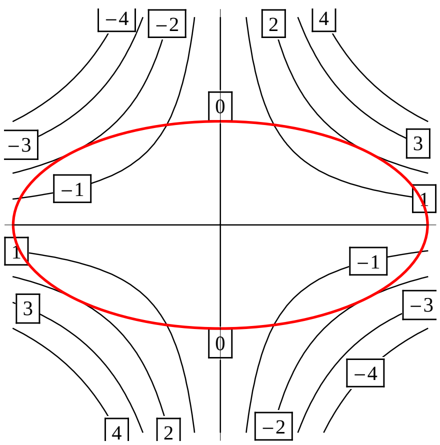
Lagrange Multipliers

1. Here is the level set diagram of $f(x, y) = 2xy$.



(a) *Estimate the maximum and minimum values of f on the ellipse* $\dfrac{x^2}{4} + y^2 = 1$.

Solution. Let's draw in the ellipse on the level set diagram:



From this picture, we can see that the highest value f achieves on the ellipse is 2, and the lowest value is -2.

(b) *Find the maximum and minimum values of f on the ellipse* $\dfrac{x^2}{4} + y^2 = 1$.

Solution. Let's write $g(x, y) = \frac{x^2}{4} + y^2$. Then, we are trying to extremize $f(x, y) = 2xy$ subject to the constraint that $g(x, y) = 1$. Therefore, the method of Lagrange multipliers says that we should look for points where $\nabla g = \vec{0}$ or $\nabla f = \lambda \nabla g$.

$\nabla g = \left\langle \frac{x}{2}, 2y \right\rangle$, so ∇g can only be $\vec{0}$ at $(x, y) = (0, 0)$. However, the point $(0, 0)$ does not lie on the ellipse $\frac{x^2}{4} + y^2 = 1$, so we can ignore it.

So, we need $\nabla f = \lambda \nabla g$, or $\langle 2y, 2x \rangle = \lambda \left\langle \frac{x}{2}, 2y \right\rangle$. We also need to make sure we are on the ellipse, so we are really trying to solve three equations simultaneously:

$$2y = \lambda\left(\frac{x}{2}\right) \qquad (1)$$

$$2x = \lambda(2y) \qquad (2)$$

$$\frac{x^2}{4} + y^2 = 1 \qquad (3)$$

(2) tells us that $x = \lambda y$; if we plug this into (1), we get that $2y = \frac{\lambda^2 y}{2}$, or $4y = \lambda^2 y$. Rewriting this as $(4-\lambda^2)y = 0$, we find that either $\lambda^2 = 4$ (so $\lambda = \pm 2$) or $y = 0$. Let's look at these possibilities separately:

- If $y = 0$, then the equation $x = \lambda y$ tells us that $x = 0$. But $(0,0)$ does not lie on the ellipse, so this does not satisfy all three equations.
- If $\lambda = 2$, then $x = 2y$. Plugging this into (3), $2y^2 = 1$, so $y = \pm\frac{1}{\sqrt{2}}$. Since $x = 2y$, this gives us two candidate points $\left(\frac{2}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{2}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$. (We can check that all three equations are satisfied.)
- If $\lambda = -2$, then $x = -2y$. Plugging this into (3), $2y^2 = 1$, so $y = \pm\frac{1}{\sqrt{2}}$. Since $x = -2y$, this gives us two candidate points $\left(-\frac{2}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(\frac{2}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$.

We have four candidate points, and we just evaluate f at each of them to figure out the maximum and minimum values of f on the ellipse:

$$\begin{aligned} f\left(\frac{2}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) &= 2 \\ f\left(-\frac{2}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) &= 2 \\ f\left(-\frac{2}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) &= -2 \\ f\left(\frac{2}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) &= -2 \end{aligned}$$

So, the maximum value of f on the ellipse is $\boxed{2}$, and the minimum value is $\boxed{\text{-2}}$.

2. *Minimize* $2x + 4y + 6z$ *if* $x^2 + y^2 + z^2 = 14$.

Solution. Let $f(x,y,z) = 2x + 4y + 6z$ and $g(x,y,z) = x^2 + y^2 + z^2$. Then, $\nabla f = \langle 2,4,6\rangle$ and $\nabla g = \langle 2x, 2y, 2z\rangle$.

We want $\nabla g = \vec{0}$ or $\nabla f = \lambda \nabla g$. The former happens when $x = y = z = 0$, but this point does not satisfy our constraint $x^2 + y^2 + z^2 = 14$.

To have $\nabla f = \lambda\nabla g$, we need $1 = \lambda x$, $2 = \lambda y$, and $3 = \lambda z$, so $x = \frac{1}{\lambda}$, $y = \frac{2}{\lambda}$, and $z = \frac{3}{\lambda}$. To find λ, we use the fact that $x^2 + y^2 + z^2 = 14$: we need $\frac{1}{\lambda^2} + \frac{4}{\lambda^2} + \frac{9}{\lambda^2} = 14$, so $\lambda = \pm 1$.

Therefore, our candidate points are $(1,2,3)$ and $(-1,-2,-3)$. Evaluating f at each point, we have

$$\begin{aligned} f(1,2,3) &= 28 \\ f(-1,-2,-3) &= -28 \end{aligned}$$

So, the minimum value is $f(-1,-2,-3) = -28$.

3. *Minimize* $x^2 + y^2 + z^2$ *subject to the constraints* $x + y + z = 6$ *and* $x + 2y - 3z = 14$.

Solution. Let $f(x, y, z) = x^2 + y^2 + z^2$, $g(x, y, z) = x + y + z$, and $h(x, y, z) = x + 2y - 3z$. Then,

$$\begin{aligned}\nabla f(x, y, z) &= \langle 2x, 2y, 2z\rangle \\ \nabla g(x, y, z) &= \langle 1, 1, 1\rangle \\ \nabla h(x, y, z) &= \langle 1, 2, -3\rangle\end{aligned}$$

Since ∇g and ∇h are not parallel, we just need to solve $\nabla f = \lambda \nabla g + \mu \nabla h$. That is, we need

$$\begin{aligned}2x &= \lambda + \mu \\ 2y &= \lambda + 2\mu \\ 2z &= \lambda - 3\mu\end{aligned}$$

So,

$$\begin{aligned}x &= \frac{\lambda + \mu}{2} \\ y &= \frac{\lambda + 2\mu}{2} \\ z &= \frac{\lambda - 3\mu}{2}\end{aligned}$$

Now we use our constraints: $x + y + z = 6$ and $x + 2y - 3z = 14$. Plugging in our expressions for x, y, and z, we need $\frac{3\lambda}{2} = 6$ and $7\mu = 14$. The first condition tells us that $\lambda = 4$, and the second tells us that $\mu = 2$. Therefore, $x = 3$, $y = 4$, and $z = -1$.

The minimum is therefore $f(3, 4, -1) = 26$.

4. *Maximize and minimize* $f(x, y, z) = xyz$ *subject to the constraint that* $x^2 + y^2 + z^2 = 1$.

Solution. Let $g(x, y, z) = x^2 + y^2 + z^2$, so that we can write our constraint as $g(x, y, z) = 1$. Then, we want to solve $\nabla g = \vec{0}$ or $\nabla f = \lambda \nabla g$.

Since $\nabla g = \langle 2x, 2y, 2z\rangle$, $\nabla g = \vec{0}$ only when $(x, y, z) = (0, 0, 0)$, but this point does not satisfy the constraint.

So, let's focus on $\nabla f = \lambda \nabla g$. Since $\nabla f = \langle yz, xz, xy\rangle$, we have three equations:

$$\begin{aligned}yz &= 2\lambda x \\ xz &= 2\lambda y \\ xy &= 2\lambda z\end{aligned}$$

We also have a fourth equation, the constraint $x^2 + y^2 + z^2 = 1$. There are a number of ways to tackle solving these equations. Here, notice that if we multiply the first equation by x, the second by y, and the third by z, then we end up with three equations with xyz on the left side:

$$\begin{aligned}xyz &= 2\lambda x^2 \\ xyz &= 2\lambda y^2 \\ xyz &= 2\lambda z^2\end{aligned}$$

So, $2\lambda x^2 = 2\lambda y^2 = 2\lambda z^2$. There are two ways this could happen: either $\lambda = 0$ or $x^2 = y^2 = z^2$.

- If $\lambda = 0$, then the three equations each say $xyz = 0$. We could try to figure out what values of x, y, and z work out with this, but we actually don't really need to: no matter what points we find, we already know that $xyz = 0$, so $f(x, y, z) = xyz$ will be 0.

- If $x^2 = y^2 = z^2$, then the constraint $x^2 + y^2 + z^2 = 1$ tells us that each of x^2, y^2, and z^2 must be $\frac{1}{3}$. So, x, y, and z are each $\pm\frac{1}{\sqrt{3}}$. Therefore, $f(x, y, z) = xyz$ will be $\pm\frac{1}{3\sqrt{3}}$.

So, the minimum value of $f(x, y, z)$ on $x^2 + y^2 + z^2 = 1$ is $\boxed{-\dfrac{1}{3\sqrt{3}}}$, and the maximum value is $\boxed{\dfrac{1}{3\sqrt{3}}}$.

Notice that we did not need to completely solve for x, y, z, and λ to answer the question.

More Extremal Problems

1. (a) Use Lagrange multipliers to find the absolute minimum and maximum values of $f(x, y) = x^2 + 4y^2$ subject to the constraint $y = x^2 - 2$, if they exist.

   (b) Sketch the level set diagram of $f(x, y) = x^2 + 4y^2$ and the constraint curve $y = x^2 - 2$. Where are the candidate points that the method of Lagrange multipliers finds?

2. Decide whether each statement is true or false. (If true, explain what strategy you would use to find the absolute minimum and maximum values.)

   (a) Every continuous function $f(x, y)$ must attain an absolute minimum and absolute maximum value on $x^2 + 4y^2 < 1$.

   (b) Every continuous function $f(x, y)$ must attain an absolute minimum and absolute maximum value on $x^2 + 4y^2 = 1$.

   (c) Every continuous function $f(x, y)$ must attain an absolute minimum and absolute maximum value on $x^2 - 4y^2 = 1$.

   (d) Every continuous function $f(x, y)$ must attain an absolute minimum and absolute maximum value on $x^2 + 4y^2 \leq 1$, $y \geq 0$.

   (e) Every continuous function $f(x, y, z)$ must attain an absolute minimum and absolute maximum value on $x^2 + 4y^2 + z^2 = 1$.

   (f) Every continuous function $f(x, y, z)$ must attain an absolute minimum and absolute maximum value on $x^2 + 4y^2 = 1$.

   (g) Every continuous function $f(x, y, z)$ must attain an absolute minimum and absolute maximum value on the intersection of $x^2 + 4y^2 + z^2 = 1$ and $x + y + z = 1$.

3. Find the absolute maximum and minimum values of xyz^2 on the solid $x^2 + 4y^2 + z^2 \leq 16$, if they exist.

More Extremal Problems

1. (a) *Use Lagrange multipliers to find the absolute minimum and maximum values of $f(x, y) = x^2 + 4y^2$ subject to the constraint $y = x^2 - 2$, if they exist.*

Solution. Since the constraint curve $y = x^2 - 2$ goes on forever, there is no reason to expect that the function $f(x, y)$ will actually attain an absolute minimum or absolute maximum. However, if we sketch the level set diagram (see (b)), we can see that $f(x, y)$ should have a minimum on $y = x^2 - 2$ but $\boxed{\text{no maximum}}$.

To find the absolute minimum value, we'll use Lagrange multipliers. Our constraint is $y = x^2 - 2$, or $x^2 - y = 2$. Let $g(x, y) = x^2 - y$. Then, we want to solve $\nabla f = \lambda \nabla g$ or $\nabla g = \vec{0}$. Since $\nabla g = \langle 2x, -1 \rangle$, ∇g can never be $\vec{0}$ (the second component of ∇g is always -1).

So, we focus on $\nabla f = \lambda \nabla g$, or $\langle 2x, 8y \rangle = \lambda \langle 2x, -1 \rangle$. Together with our constraint, we have three equations:

$$2x = \lambda(2x) \quad (1)$$
$$8y = -\lambda \quad (2)$$
$$x^2 - y = 2 \quad (3)$$

(1) can be rewritten as $2x(1 - \lambda) = 0$, so either $x = 0$ or $\lambda = 1$.[(1)] Let's look at these possibilities separately.

- If $x = 0$, then (3) tells us that $y = -2$, so we have the point $(0, -2)$. (In this case, (2) tells us that $\lambda = 16$, but we don't really need that information.)
- If $\lambda = 1$, then (2) tells us that $y = -\frac{1}{8}$, and (3) then says that $x = \pm\sqrt{\frac{15}{8}}$, so we get two possible points, $\left(\pm\sqrt{\frac{15}{8}}, -\frac{1}{8}\right)$.

Now, we evaluate f at our three candidate points:

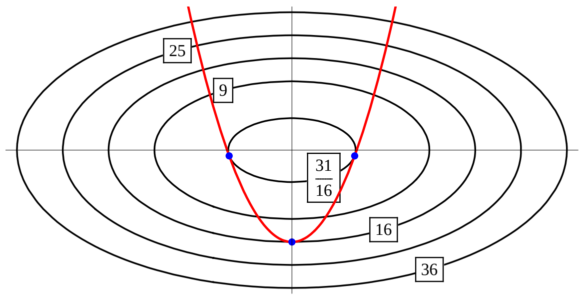
$$f(0, -2) = 16$$
$$f\left(\sqrt{\frac{15}{8}}, -\frac{1}{8}\right) = \frac{31}{16}$$
$$f\left(-\sqrt{\frac{15}{8}}, -\frac{1}{8}\right) = \frac{31}{16}$$

Since we've already decided that f achieves a minimum but no maximum on the constraint curve, we can see that $\boxed{\text{the minimum value of } f \text{ on the constraint curve is } \frac{31}{16}}$.

(b) *Sketch the level set diagram of $f(x, y) = x^2 + 4y^2$ and the constraint curve $y = x^2 - 2$. Where are the candidate points that the method of Lagrange multipliers finds?*

Solution. Here is the diagram, with the constraint curve shown in red and the points shown in blue:

(1) Be very careful: it's temping to see the equation $2x = \lambda(2x)$ and cancel $2x$ from both sides to get $\lambda = 1$, but then you lose the possibility that $x = 0$.



Note that the constraint curve is tangent to the level sets of f at the blue points; this is exactly what the Lagrange multiplier equation looks for.

2. Decide whether each statement is true or false. (If true, explain what strategy you would use to find the absolute minimum and maximum values.)

   (a) *Every continuous function $f(x, y)$ must attain an absolute minimum and absolute maximum value on $x^2 + 4y^2 < 1$.*

   Solution. False; this region is not closed (it doesn't include its boundary).

   (b) *Every continuous function $f(x, y)$ must attain an absolute minimum and absolute maximum value on $x^2 + 4y^2 = 1$.*

   Solution. True. To find the absolute minimum and maximum, use Lagrange multipliers (find the maximum and minimum of $f(x, y)$ subject to the constraint that $x^2 + 4y^2 = 1$).

   (c) *Every continuous function $f(x, y)$ must attain an absolute minimum and absolute maximum value on $x^2 - 4y^2 = 1$.*

   Solution. False. $x^2 - 4y^2 = 1$ describes a hyperbola in the plane, which goes on forever.

   (d) *Every continuous function $f(x, y)$ must attain an absolute minimum and absolute maximum value on $x^2 + 4y^2 \leq 1$, $y \geq 0$.*

   Solution. True. To find the absolute minimum and maximum, first find the value of f on all critical points of f which satisfy $x^2 + 4y^2 < 1$ and $y > 0$ (that is, critical points inside the boundary), and then find the maximum and minimum values of f on the boundary. The largest of these values is the maximum; the smallest is the minimum.

   Notice that the boundary really has two pieces: $y = 0$ (with $-1 \leq x \leq 1$) and $x^2 + 4y^2 = 1$ (with $x \geq 0$). To deal with the first, just substitute $y = 0$ into the expression for f to get a single variable problem. To deal with the second, you could use Lagrange multipliers (be sure to discard any points you find with $y < 0$).

   (e) *Every continuous function $f(x, y, z)$ must attain an absolute minimum and absolute maximum value on $x^2 + 4y^2 + z^2 = 1$.*

   Solution. True, as this is an ellipsoid; use Lagrange multipliers to find the minimum and maximum.

   (f) *Every continuous function $f(x, y, z)$ must attain an absolute minimum and absolute maximum value on $x^2 + 4y^2 = 1$.*

   Solution. False; since f is a function of three variables, we must think about the domain as living in $\mathbb{R}^3$ (space). In $\mathbb{R}^3$, the equation $x^2 + 4y^2 = 1$ describes a cylinder which is infinitely long.

(g) *Every continuous function $f(x, y, z)$ must attain an absolute minimum and absolute maximum value on the intersection of $x^2 + 4y^2 + z^2 = 1$ and $x + y + z = 1$.*

Solution. True, as we are talking about the intersection of an ellipsoid and a plane, which is a closed loop. To find the minimum and maximum, use Lagrange multipliers (with two constraints).

3. *Find the absolute maximum and minimum values of xyz^2 on the solid $x^2+4y^2+z^2 \leq 16$, if they exist.*

Solution. This is a homework problem!

Double Integrals

1. Write a double integral $\iint_{\mathcal{R}} f(x,y)\, dA$ which gives the volume of the top half of a solid ball of radius 5. (You need to specify a function $f(x,y)$ as well as a region $\mathcal{R}$.)

2. (a) If $\mathcal{R}$ is any region in the plane ($\mathbb{R}^2$), what does the double integral $\iint_{\mathcal{R}} 1\, dA$ represent? Why?

   (b) Suppose the shape of a flat plate is described as a region $\mathcal{R}$ in the plane, and $f(x,y)$ gives the density of the plate at the point (x,y) in kilograms per square meter. What does the double integral $\iint_{\mathcal{R}} f(x,y)\, dA$ represent? Why?

3. If $\mathcal{R}$ is the rectangle $[1,2] \times [3,4]$, compute the double integral $\iint_{\mathcal{R}} 6x^2 y\, dA$.

4. If $\mathcal{R}$ is the rectangle $[0,1] \times [-1,2]$, compute the double integral $\displaystyle\iint_{\mathcal{R}} 2ye^x \; dA$.

5. Find the volume of the solid that lies under $z = x^2 + y^2$ and above the square $0 \leq x \leq 2$, $-1 \leq y \leq 1$.

6. Find the volume of the solid enclosed by the surfaces $z = 4 - x^2 - y^2$, $z = x^2 + 2y^2 - 2$, $x = -1$, $x = 1$, $y = -1$, and $y = 1$.

Double Integrals

1. *Write a double integral* $\iint_{\mathcal{R}} f(x,y)\, dA$ *which gives the volume of the top half of a solid ball of radius 5. (You need to specify a function* $f(x,y)$ *as well as a region* $\mathcal{R}$*.)*

   Solution. We know that $\iint_{\mathcal{R}} f(x,y)$ can be interpreted as the volume of the solid under $z = f(x,y)$ over the region $\mathcal{R}$. So, we'd like to think of a function $f(x,y)$ and a region $\mathcal{R}$ so that the solid under $z = f(x,y)$ over $\mathcal{R}$ is the top half of a solid ball of radius 5.

   Let's look at the sphere of radius 5 centered at the origin. We know that it has equation $x^2 + y^2 + z^2 = 25$, so the top half of this sphere can be described by $z = \sqrt{25 - x^2 - y^2}$. The solid under $z = \sqrt{25 - x^2 - y^2}$ and over the region $x^2 + y^2 \leq 25$ is therefore half of a solid ball of radius 5. So, the double integral $\boxed{\iint_{\mathcal{R}} \sqrt{25 - x^2 - y^2}\, dA \text{ where } \mathcal{R} \text{ is the disk } x^2 + y^2 \leq 25}$ gives the volume of the top half of a solid ball of radius 5.

2. (a) *If* $\mathcal{R}$ *is any region in the plane (*$\mathbb{R}^2$*), what does the double integral* $\iint_{\mathcal{R}} 1\, dA$ *represent? Why?*

   Solution. Remember that we are thinking of the double integral $\iint_{\mathcal{R}} f(x,y)\, dA$ as a limit of Riemann sums, obtained from the following process:

   1. Slice the region $\mathcal{R}$ into small pieces.
   2. In each piece, the value of f will be approximately constant, so multiply the value of f at any point by the area ΔA of the piece.[(1)]
   3. Add up all of these products. (This is a Riemann sum.)
   4. Take the limit of the Riemann sums as the area of the pieces tends to 0.

   Now, if f is just the function $f(x,y) = 1$, then in Step 2, we end up simply multiplying 1 by the area of the piece, which gives us the area of the piece. So, in Step 3, when we add all of these products up, we are just adding up the area of all the small pieces, which gives the area of the whole region.

   So, $\iint_{\mathcal{R}} 1\, dA$ represents the $\boxed{\text{area of the region } \mathcal{R}}$.

   (b) *Suppose the shape of a flat plate is described as a region* $\mathcal{R}$ *in the plane, and* $f(x,y)$ *gives the density of the plate at the point* (x,y) *in kilograms per square meter. What does the double integral* $\iint_{\mathcal{R}} f(x,y)\, dA$ *represent? Why?*

   Solution. Following the process described in (a), in Step 2, we multiply the approximate density of each piece by the area of that piece, which gives the approximate mass of that piece. Adding those up gives the approximate mass of the entire plate, and taking the limit gives us the exact $\boxed{\text{mass of the plate}}$.

[(1)] Actually, it's also fine to just approximate the area of the piece.

3. *If $\mathcal{R}$ is the rectangle $[1,2] \times [3,4]$, compute the double integral* $\iint_{\mathcal{R}} 6x^2y\ dA$.

Solution. We can rewrite this as an integrated integral in two ways: $\int_1^2 \left(\int_3^4 6x^2y\ dy\right) dx$ or $\int_3^4 \left(\int_1^2 6x^2y\ dx\right) dy$. These will give the same answer (that's what Fubini's Theorem says), so let's just use the first. We need to first do the inner integral, which is $\int_3^4 6x^2y\ dy$. When we do this integral, we treat x as a constant. So, this integral is equal to $3x^2y^2\Big|_{y=3}^{y=4} = 3x^2(16-9) = 21x^2$. So, our iterated integral becomes $\int_1^2 21x^2\ dx = 7x^3\Big|_{x=1}^{x=2} = \boxed{49}$.

4. *If $\mathcal{R}$ is the rectangle $[0,1] \times [-1,2]$, compute the double integral* $\iint_{\mathcal{R}} 2ye^x\ dA$.

Solution. We can rewrite the double integral as an iterated integral in two ways: $\int_0^1 \int_{-1}^2 2ye^x\ dy\ dx$ or $\int_{-1}^2 \int_0^1 2ye^x\ dx\ dy$. Let's use the first to compute.

$$\begin{aligned}
\int_0^1 \int_{-1}^2 2ye^x\ dy\ dx &= \int_0^1 \left(y^2e^x\Big|_{y=-1}^{y=2} \right)\ dx \\
&= \int_0^1 3e^x\ dx \\
&= 3e^x\Big|_{x=0}^{x=1} \\
&= \boxed{3e-3}
\end{aligned}$$

5. *Find the volume of the solid that lies under $z = x^2+y^2$ and above the square $0 \le x \le 2$, $-1 \le y \le 1$.*

Solution. We know that the volume of the solid lying under a surface $z = f(x,y)$ and above a region $\mathcal{R}$ in the plane is given by the double integral $\iint_{\mathcal{R}} f(x,y)\ dA$, so the volume we want in this problem is given by the double integral $\iint_{\mathcal{R}} (x^2+y^2)\ dA$ where $\mathcal{R}$ is the square $[0,2] \times [-1,1]$. We know that this double integral is equal to the iterated integrals $\int_0^2 \int_{-1}^1 (x^2+y^2)\ dy\ dx$ and $\int_{-1}^1 \int_0^2 (x^2+y^2)\ dx\ dy$.

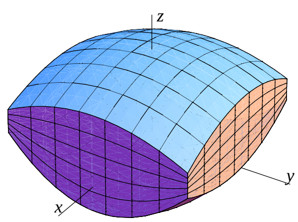
Let's use the first iterated integral:

$$\begin{aligned}\int_0^2 \int_{-1}^1 (x^2+y^2)\, dy\, dx &= \int_0^2 \left[\left(x^2 y + \frac{y^3}{3} \right)\Big|_{y=-1}^{y=1} \right] dx \\ &= \int_0^2 \left(2x^2 + \frac{2}{3} \right) dx \\ &= \frac{2x^3}{3} + \frac{2}{3}x \Big|_{x=0}^{x=2} \\ &= \boxed{\frac{20}{3}}\end{aligned}$$

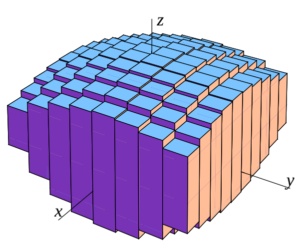
6. *Find the volume of the solid enclosed by the surfaces $z = 4 - x^2 - y^2$, $z = x^2 + 2y^2 - 2$, $x = -1$, $x = 1$, $y = -1$, and $y = 1$.*

Solution. When we study triple integrals, we'll see another way to do this problem.

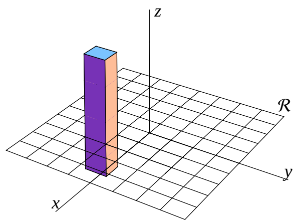
First, let's figure out what this solid looks like. The surface $z = 4 - x^2 - y^2$ is a paraboloid which opens downward, with its highest point at $(0, 0, 4)$. The surface $z = x^2 + 2y^2 - 2$ is a paraboloid which opens upward, with its lowest point at $(0, 0, -2)$. So, here is a picture of the solid:



Here, the top surface is $z = 4 - x^2 - y^2$, and the bottom is $z = x^2 + 2y^2 - 2$. To find the volume of the solid, let's imagine approximating it using boxes:



Basically, what we are doing is chopping the rectangle $\mathcal{R} = [-1, 1] \times [-1, 1]$ into lots of small rectangles, each of area ΔA. Then we look at a particular box:

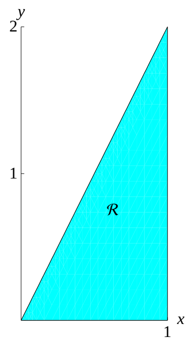


Its volume is the area ΔA multiplied by the height of the box. The height of the box is the difference between the z-value at the top (on the surface $z = 4 - x^2 - y^2$) and the bottom (on the surface $z = x^2 + 2y^2 - 2$). So, its volume is approximately $[(4 - x^2 - y^2) - (x^2 + 2y^2 - 2)]\Delta A = (6 - 2x^2 - 3y^2)\Delta A$. If we add all of these up and take the limit as $\Delta A \to 0$, we get the double integral $\iint_{\mathcal{R}} (6 - 2x^2 - 3y^2)\, dA$, which we compute by converting to an iterated integral:

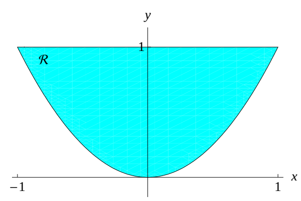
$$\begin{aligned}
\iint_{\mathcal{R}} (6 - 2x^2 - 3y^2)\, dA &= \int_{-1}^{1} \int_{-1}^{1} (6 - 2x^2 - 3y^2)\, dy\, dx \\
&= \int_{-1}^{1} \left(6y - 2x^2 y - y^3 \Big|_{y=-1}^{y=1} \right) dx \\
&= \int_{-1}^{1} (10 - 4x^2)\, dx \\
&= 10x - \frac{4}{3}x^3 \Big|_{x=-1}^{x=1} \\
&= \boxed{\frac{52}{3}}
\end{aligned}$$

Double Integrals over General Regions

1. Let $\mathcal{R}$ be the region in the plane bounded by the lines $y = 0$, $x = 1$, and $y = 2x$. Evaluate the double integral $\iint_{\mathcal{R}} 2xy \; dA$.

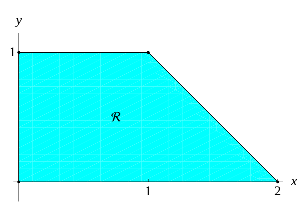


2. Let $\mathcal{R}$ be the region bounded by $y = x^2$ and $y = 1$. Write the double integral $\iint_{\mathcal{R}} f(x, y) \; dA$ as an iterated integral in both possible orders.



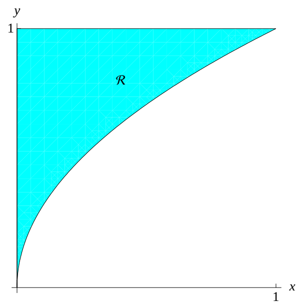
3. *For many regions, one order of integration will be simpler to deal with than the other. That is the case in this problem: use the shape of the region to decide which order of integration to use. Why is the other order more difficult?*

   Let $\mathcal{R}$ be the trapezoid with vertices $(0, 0)$, $(2, 0)$, $(1, 1)$, and $(0, 1)$. Write the double integral $\iint_{\mathcal{R}} f(x, y) \; dA$ as an iterated integral.



4. *Sometimes, when converting a double integral to an iterated integral, we decide the order of integration based on the integrand, rather than the shape of the region — some integrands are easy to integrate with respect to one variable and much harder (or even impossible) to integrate with respect to the other. That is the case in this problem.*

   Evaluate the double integral $\iint_{\mathcal{R}} \sqrt{y^3+1}\, dA$ where $\mathcal{R}$ is the region in the first quadrant bounded by $x = 0$, $y = 1$, and $y = \sqrt{x}$. (To decide the order of integration, first think about whether it's easier to integrate the integrand with respect to x or with respect to y.)



5. In each part, you are given an iterated integral. Sketch the region of integration, and then change the order of integration.

   (a) $\int_0^4 \int_0^x f(x,y)\, dy\, dx.$

   (b) $\int_0^4 \int_0^{\sqrt{y}} f(x,y)\, dx\, dy.$

   (c) $\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)\, dx\, dy.$

6. Let a be a constant between 0 and 4. Let $\mathcal{R}$ be the region bounded by $y = x^2 + a$ and $y = 4$. Write the double integral $\iint_{\mathcal{R}} f(x, y)\, dA$ as an iterated integral in both possible orders.

7. Evaluate the iterated integral $\displaystyle\int_0^1 \int_{-\sqrt{1-x^2}}^{0} 2x \cos\left(y - \frac{y^3}{3}\right)\, dy\, dx.$

More problems on the other side!

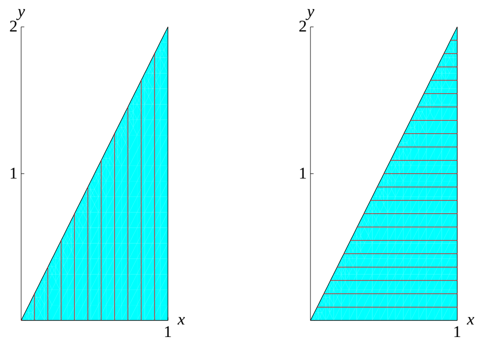
8. A flat plate is in the shape of the region in the first quadrant bounded by $x = 0$, $y = 0$, $y = \ln x$ and $y = 2$. If the density of the plate at point (x, y) is xe^y grams per cm^2, find the mass of the plate. (Suppose the x- and y-axes are marked in cm.)

9. Let $\mathcal{U}$ be the solid above $z = 0$, below $z = 4 - y^2$, and between the surfaces $x = \sin y - 1$ and $x = \sin y + 1$. Find the volume of $\mathcal{U}$.

Double Integrals over General Regions

1. *Let $\mathcal{R}$ be the region in the plane bounded by the lines $y = 0$, $x = 1$, and $y = 2x$. Evaluate the double integral* $\iint_{\mathcal{R}} 2xy \; dA$.

   Solution. We can either slice the region $\mathcal{R}$ vertically or horizontally.[1]



- **Slicing vertically**:

  Slicing vertically amounts to slicing the interval $[0, 1]$ on the x-axis, so our outer integral will be $\int_0^1$ something dx. To figure out the inner integral, we look at a general slice. Remember that, on a single slice, x is (roughly) constant, and we want to describe what y does. The bottom of each slice is on the line $y = 0$, and the top is on the line $y = 2x$, so the inner integral has endpoints of integration 0 and $2x$. Therefore, our iterated integral is

$$\boxed{\int_0^1 \int_0^{2x} 2xy \; dy \; dx} = \int_0^1 \left(xy^2 \Big|_{y=0}^{y=2x} \right) dx = \int_0^1 4x^3 \; dx = x^4 \Big|_{x=0}^{x=1} = \boxed{1}$$

- **Slicing horizontally**:

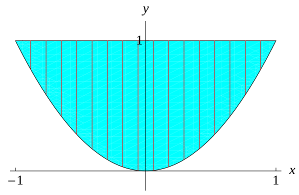
  Slicing horizontally amounts to slicing the interval $[0, 2]$ on the y-axis, so our outer integral will be $\int_0^2$ something dy. To figure out the inner integral, we look at a general slice. The left end of each slice is on the line $y = 2x$, and the right end is on the line $x = 1$. Since we are describing

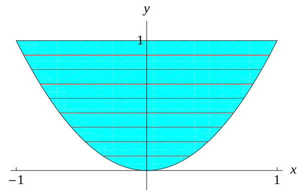
[1] Remember that this is a streamlined version of the real process. Really, to get a Riemann sum approximation, we chop the region $\mathcal{R}$ into lots of small rectangles, each of width Δx and height Δy. The area of each piece is then $\Delta A = \Delta x \Delta y$. We have one product "$f(x, y)\Delta x \Delta y$" per little rectangle, and we need to add these all up to get a Riemann sum. (See #2 of the worksheet "Double Integrals" for more details.) When converting to an iterated integral, we're really deciding whether we want to add up in rows or columns first. If we add up in rows, we visualize adding up in a horizontal slice first and getting one sum per horizontal slice (then we add up all of those sums, one per slice). Similarly, if we add up in columns, we visualize adding up in a vertical slice first and then adding up all those sums, one per vertical slice. So, when we say "slice horizontally," we *really* mean we're going to add up in rows first.

a horizontal slice, we want to describe how x varies, so x goes from $\frac{y}{2}$ to 1. Thus, the iterated integral is $\boxed{\int_0^2 \int_{y/2}^1 2xy\ dx\ dy}$, which is of course also equal to 1.

2. *Let $\mathcal{R}$ be the region bounded by $y = x^2$ and $y = 1$. Write the double integral $\iint_{\mathcal{R}} f(x,y)\ dA$ as an iterated integral in both possible orders.*

Solution. Again, we think of slicing either vertically or horizontally.





- **Slicing vertically**:

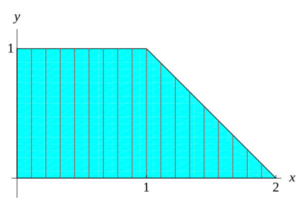
  Slicing vertically amounts to slicing the interval $[-1, 1]$ on the x-axis, so the outer integral will be $\int_{-1}^1$ something dx. To write the inner integral, we want to describe what y does within a single slice (thinking of x as being constant). The bottom of each slice lies on $y = x^2$, and the top lies on $y = 1$, so the iterated integral is $\boxed{\int_{-1}^1 \int_{x^2}^1 f(x,y)\ dy\ dx}$.

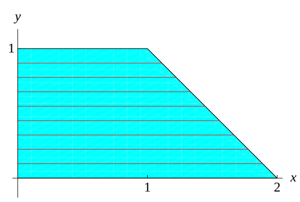
- **Slicing horizontally**:

  Slicing horizontally amounts to slicing the interval $[0, 1]$ on the y-axis, so the outer integral will be $\int_0^1$ something dy. The left side of each slice lies on $y = x^2$, and the right side of each slice also lies on $y = x^2$. Remember, though, that we are trying to describe how x varies in a slice (and we think of y as being constant), so x goes from the left half of $y = x^2$, where $x = -\sqrt{y}$, to the right half, where $x = \sqrt{y}$. Thus, the iterated integral is $\boxed{\int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} f(x,y)\ dx\ dy}$.

3. *Let $\mathcal{R}$ be the trapezoid with vertices $(0,0)$, $(2,0)$, $(1,1)$, and $(0,1)$. Write the double integral $\iint_{\mathcal{R}} f(x,y)\ dA$ as an iterated integral.*

Solution. Let's compare slicing vertically with slicing horizontally:



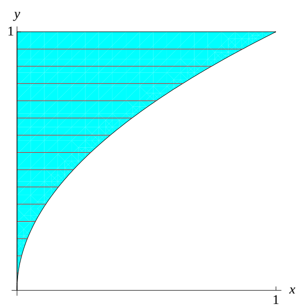


Notice that, if we slice vertically, there are two "types" of slices. The slices to the left of $x = 1$ go from $y = 0$ to $y = 1$, whereas the slices to the right go from $y = 0$ to the diagonal side of the trapezoid.

In contrast, if we slice horizontally, all of the slices have the same description: they go from $x = 0$ to the diagonal side. This seems simpler, so let's go with this method. When we slice horizontally, we are slicing the interval $[0,1]$ on the y-axis, so our outer integral will be $\int_0^1$ something dy. Each slice goes from $x = 0$ to the diagonal side. The diagonal side is $y = 2 - x$ (we know it's a line containing the points $(2,0)$ and $(1,1)$). We want to describe how x varies in each slice, so x goes from 0 to $2 - y$. So, the iterated integral is $\boxed{\int_0^1 \int_0^{2-y} f(x,y)\ dx\ dy}$.[2]

4. *Evaluate the double integral* $\iint_{\mathcal{R}} \sqrt{y^3+1}\ dA$ *where* $\mathcal{R}$ *is the region in the first quadrant bounded by* $x = 0$, $y = 1$, *and* $y = \sqrt{x}$. *(To decide the order of integration, first think about whether it's easier to integrate the integrand with respect to* x *or with respect to* y*.)*

   Solution. The integrand is much easier to integrate with respect to x than with respect to y. Therefore, we should try to rewrite the double integral as an iterated integral where the inner integral is with respect to x. This means our outer integral will be with respect to y, which corresponds in our strategy to slicing the region horizontally.



This amounts to slicing the interval $[0,1]$ on the y-axis, so the outer integral will be $\int_0^1$ something dy. Each slice has its left end on $x = 0$ and its right end on $y = \sqrt{x}$. We want to describe how x varies within a slice, so we rewrite $y = \sqrt{x}$ as $x = y^2$. This gives the iterated integral

$$\begin{aligned}\int_0^1 \int_0^{y^2} \sqrt{y^3+1}\ dx\ dy &= \int_0^1 \left(x\sqrt{y^3+1}\Big|_{x=0}^{x=y^2} \right) dy \\ &= \int_0^1 y^2\sqrt{y^3+1}\ dy\end{aligned}$$

We can evaluate this integral using substitution: if we let $u = y^3 + 1$, then $du = 3y^2\ dy$, and we can rewrite the integral as

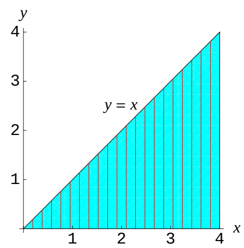
$$\begin{aligned}\int_1^2 \frac{1}{3}\sqrt{u}\ du &= \frac{2}{9}u^{3/2}\Big|_{u=1}^{u=2} \\ &= \boxed{\frac{2}{9}\left(2^{3/2} - 1\right)}\end{aligned}$$

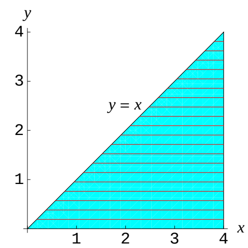
[2] If you used the other order of integration, you should have a sum of iterated integrals $\int_0^1 \int_0^1 f(x,y)\ dy\ dx + \int_1^2 \int_0^{2-x} f(x,y)\ dy\ dx$.

5. In each part, you are given an iterated integral. Sketch the region of integration, and then change the order of integration.

(a) $\int_0^4 \int_0^x f(x,y)\ dy\ dx$.

Solution. Let's just think of our strategy in reverse. The fact that the outer integral is $\int_0^4$ something dx tells us that we are slicing the interval $[0,4]$ on the x-axis, so we are making vertical slices from $x = 0$ to $x = 4$. The inner integral tells us that the bottom of each slice is on $y = 0$, and the top of each slice is on $y = x$. So, the region of integration (with vertical slices) looks like the picture on the left:

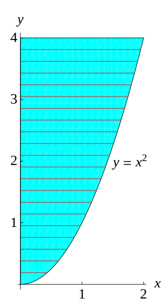


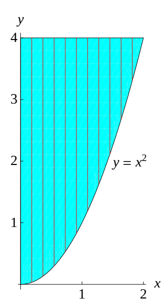


To change the order of integration, we want to instead use horizontal slices (the picture on the right). Now, we are slicing the interval $[0,4]$ on the y-axis, so the outer integral is $\int_0^4$ something dy. Each slice has its left edge on $y = x$ (or $x = y$, since we really want to describe x in terms of y) and its right edge on $x = 4$, so we can rewrite the iterated integral as $\boxed{\int_0^4 \int_y^4 f(x,y)\ dx\ dy}$.

(b) $\int_0^4 \int_0^{\sqrt{y}} f(x,y)\ dx\ dy$.

Solution. The fact that the outer integral is $\int_0^4$ something dy tells us that we are slicing the interval $[0,4]$ on the y-axis, so we are making horizontal slices from $y = 0$ to $y = 4$. The inner integral tells us that the left side of each slice is on $x = 0$ and the right side is on $x = \sqrt{y}$ (or $y = x^2$). So, the region of integration looks like this:

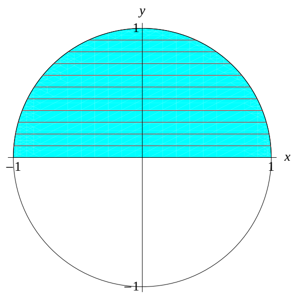


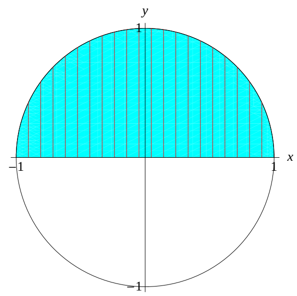


To change the order of integration, we use vertical slices. Now, we are slicing the interval $[0,2]$ on the x-axis. The bottom of each slice is on $y = x^2$, and the top of each slice is on $y = 4$, so we can rewrite the integral as $\boxed{\int_0^2 \int_{x^2}^4 f(x,y)\ dy\ dx}$.

(c) $\displaystyle\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)\;dx\;dy.$

Solution. The fact that the outer integral is $\displaystyle\int_0^1$ something dy tells us that we are slicing the interval $[0,1]$ on the y-axis, so we are making horizontal slices from $y=0$ to $y=1$. The inner integral tells us that the left side of each slice is on $x=-\sqrt{1-y^2}$ and the right side of each slice is on $x=\sqrt{1-y^2}$. $x=-\sqrt{1-y^2}$ describes the left half of the circle $x^2+y^2=1$, and $x=\sqrt{1-y^2}$ describes the right half, so the region of integration looks like this:

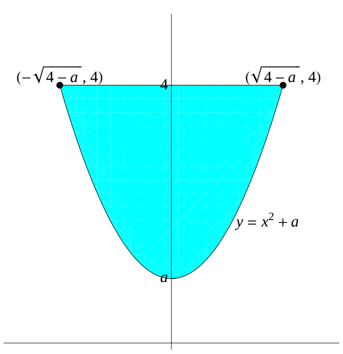




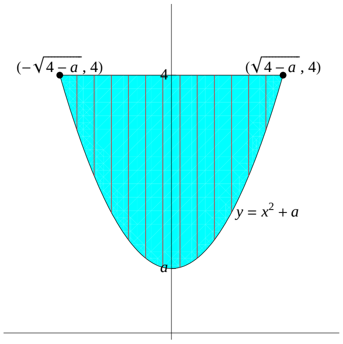
To change the order of integration, we use vertical slices. Now, we are slicing the interval $[-1,1]$ on the x-axis, so the outer integral is $\displaystyle\int_{-1}^1$ something dx. Each slice has its bottom edge on $y=0$ and its top edge on the top half of the circle $x^2+y^2=1$ (or $y=\sqrt{1-x^2}$), so we can rewrite the iterated integral as $\boxed{\displaystyle\int_{-1}^1 \int_0^{\sqrt{1-x^2}} f(x,y)\;dy\;dx}$.

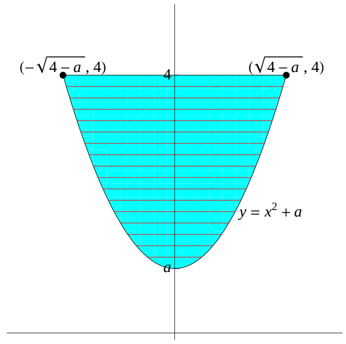
6. *Let a be a constant between 0 and 4. Let $\mathcal{R}$ be the region bounded by $y=x^2+a$ and $y=4$. Write the double integral $\iint_{\mathcal{R}} f(x,y)\;dA$ as an iterated integral in both possible orders.*

Solution. The curves $y=x^2+a$ and $y=4$ intersect where $x^2=4-a$, so $x=\pm\sqrt{4-a}$. So, the region $\mathcal{R}$ looks like this:



To write the double integral as an iterated integral, we think of slicing either vertically or horizontally.





- **Slicing vertically**:

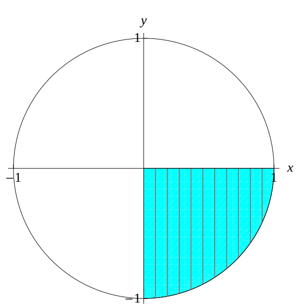
  Slicing vertically corresponds to slicing the interval $[-\sqrt{4-a}, \sqrt{4-a}]$ on the x-axis, so the outer integral will be $\int_{-\sqrt{4-a}}^{\sqrt{4-a}}$ something dx. Each slice has its bottom edge on $y = x^2 + a$ and its top edge on $y = 4$, so the iterated integral is $\boxed{\int_{-\sqrt{4-a}}^{\sqrt{4-a}} \int_{x^2+a}^{4} f(x,y)\, dy\, dx}$. Remember that a is a constant, so it's fine to have it in the outer integral.

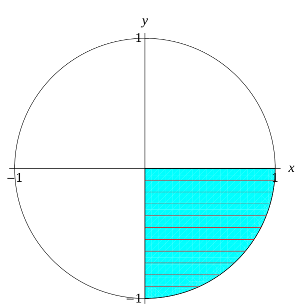
- **Slicing horizontally**:

  Slicing horizontally corresponds to slicing the interval $[a, 4]$ on the y-axis, so the outer integral will be $\int_a^4$ something dy. Each slice has its left edge on $y = x^2 + a$ (so $x = -\sqrt{y-a}$) and its right edge on $y = x^2 + a$ (so $x = \sqrt{y-a}$). Thus, the iterated integral is $\boxed{\int_a^4 \int_{-\sqrt{y-a}}^{\sqrt{y-a}} f(x,y)\, dx\, dy}$.

7. *Evaluate the iterated integral* $\int_0^1 \int_{-\sqrt{1-x^2}}^{0} 2x \cos\left(y - \frac{y^3}{3}\right)\, dy\, dx$.

Solution. We don't know how to integrate the integrand with respect to y, but we can integrate it with respect to x. This suggests that we should change the order of integration, as in #5. First, let's figure out what the region looks like. The fact that the outer integral is $\int_0^1$ something dx tells us that we are slicing the interval $[0, 1]$ on the x-axis, so we are making vertical slices from $x = 0$ to $x = 1$. The inner integral tells us that the bottom of each slice is on $y = -\sqrt{1-x^2}$ (the bottom half of the circle $x^2 + y^2 = 1$) and the top of each slice is on $y = 0$. So, the region of integration looks like this:





To change the order of integration, we switch to using horizontal slices. Now, we are slicing the interval $[-1,0]$ on the y-axis, so our outer integral will be $\int_{-1}^{0}$ something dy. Each slice has its left edge on $x=0$ and its right edge on the right half of the circle $x^2+y^2=1$ (so $x=\sqrt{1-y^2}$). Therefore, we can rewrite the given integral as

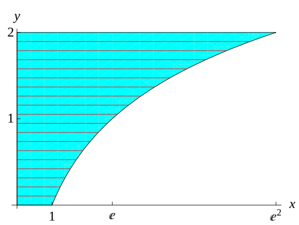
$$\begin{aligned}\int_{-1}^{0}\int_{0}^{\sqrt{1-y^2}} 2x\cos\left(y-\frac{y^3}{3}\right)\,dx\,dy &= \int_{-1}^{0}\left[x^2\cos\left(y-\frac{y^3}{3}\right)\Big|_0^{\sqrt{1-y^2}}\right]dy \\ &= \int_{-1}^{0}(1-y^2)\cos\left(y-\frac{y^3}{3}\right)dy\end{aligned}$$

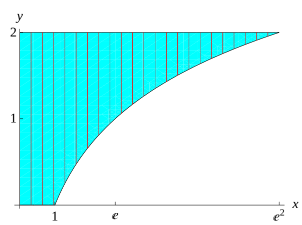
We can use substitution to evaluate this integral: let $u=y-\frac{y^3}{3}$; then, $du=(1-y^2)dy$, so the integral becomes

$$\begin{aligned}\int_{-2/3}^{0}\cos u\,du &= \sin u\Big|_{u=-2/3}^{u=0} \\ &= \boxed{-\sin\left(-\frac{2}{3}\right)}\end{aligned}$$

8. *A flat plate is in the shape of the region in the first quadrant bounded by $x=0$, $y=0$, $y=\ln x$ and $y=2$. If the density of the plate at point (x,y) is xe^y grams per cm^2, find the mass of the plate. (Suppose the x- and y-axes are marked in cm.)*

Solution. As we learned in #2(b) of the worksheet "Double Integrals", we can find the mass of the plate by taking the double integral of the density, where the region of integration is the plate. In this case, the integrand xe^y is easy to integrate with respect to x and with respect to y, so we will pick an order of integration based on the shape of the region. We can either slice horizontally or vertically:





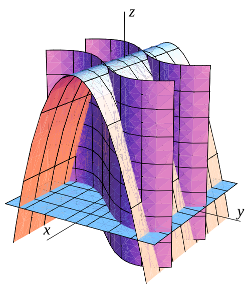
As in #3, this region is simpler to describe using horizontal slices: with vertical slices, there are two "types" of slices, but with horizontal slices, there is only one.

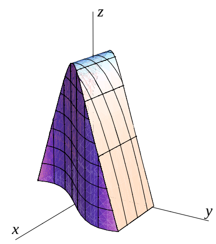
If we use horizontal slices, we are slicing the interval $[0,2]$ on the y-axis. Each slice goes from $x=0$ to $y=\ln x$ (or $x=e^y$), so the iterated integral is

$$\begin{aligned}\int_0^2\int_0^{e^y} xe^y\,dx\,dy &= \int_0^2\left(\frac{1}{2}x^2e^y\Big|_{x=0}^{x=e^y}\right)dy \\ &= \int_0^2\frac{1}{2}e^{3y}\,dy \\ &= \frac{1}{6}e^{3y}\Big|_{y=0}^{y=2} \\ &= \boxed{\frac{1}{6}\left(e^6-1\right)}\end{aligned}$$

9. *Let $\mathcal{U}$ be the solid above $z = 0$, below $z = 4 - y^2$, and between the surfaces $x = \sin y - 1$ and $x = \sin y + 1$. Find the volume of $\mathcal{U}$.*

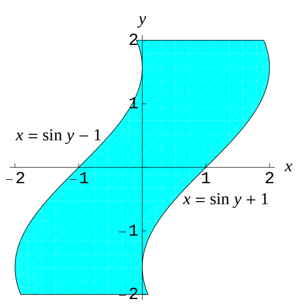
Solution. The picture on the left shows the four surfaces $z = 0$, $z = 4 - y^2$, $x = \sin y - 1$, and $x = \sin y + 1$. The picture on the right shows just the solid $\mathcal{U}$.



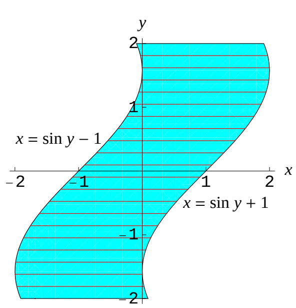


This solid can be described as the solid under $z = 4 - y^2$ over the region $\mathcal{R}$, where $\mathcal{R}$ is where the solid meets the xy-plane. So, its volume will just be $\iint_{\mathcal{R}} (4 - y^2)\ dA$.

To calculate this double integral, we need to describe $\mathcal{R}$ and convert the double integral to an iterated integral. The surface $z = 4 - y^2$ intersects the xy-plane $z = 0$ where $4 - y^2 = 0$, or $y = \pm 2$, so $y = 2$ and $y = -2$ are 2 boundaries of the region $\mathcal{R}$. The other two are $x = \sin y - 1$ and $x = \sin y + 1$. So, $\mathcal{R}$ looks like this:



It's easier to slice this region horizontally:

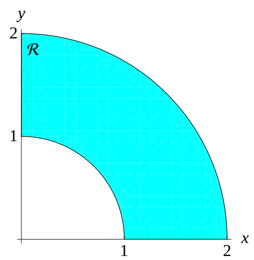


This amounts to slicing the interval $[-2, 2]$ on the y-axis, so the outer integral will be $\int_{-2}^{2}$ something dy. The left side of each slice is on $x = \sin y - 1$, and the right side is on $x = \sin y + 1$, so we can rewrite the double integral as an iterated integral

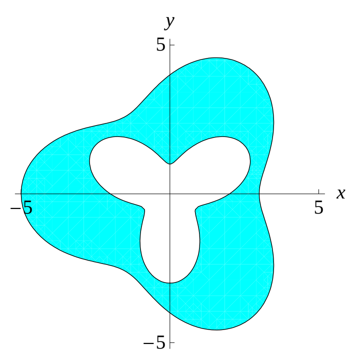
$$\begin{aligned}
\int_{-2}^{2}\int_{\sin y-1}^{\sin y+1}(4-y^2)\ dx\ dy &= \int_{-2}^{2}\left[x(4-y^2)\Big|_{x=\sin y-1}^{x=\sin y+1}\right]dy \\
&= \int_{-2}^{2} 2(4-y^2)dy \\
&= 8y - \frac{2y^3}{3}\Big|_{y=-2}^{y=2} \\
&= \boxed{\frac{64}{3}}
\end{aligned}$$

Double Integrals in Polar Coordinates

1. A flat plate is in the shape of the region $\mathcal{R}$ in the first quadrant lying between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$. The density of the plate at point (x, y) is $x + y$ kilograms per square meter (suppose the axes are marked in meters). Find the mass of the plate.

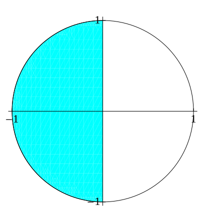


2. Find the area of the region $\mathcal{R}$ lying between the curves $r = 2 + \sin 3\theta$ and $r = 4 - \cos 3\theta$. (You may leave your answer as an iterated integral in polar coordinates.)

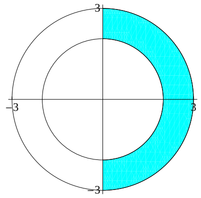


3. In each part, rewrite the double integral as an iterated integral in polar coordinates. (Do not evaluate.)

   (a) $\iint_{\mathcal{R}} \sqrt{1 - x^2 - y^2}\, dA$ where $\mathcal{R}$ is the left half of the unit disk.

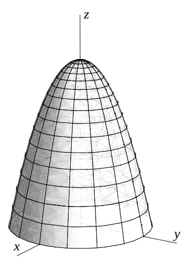


(b) $\iint_{\mathcal{R}} x^2 \, dA$ where $\mathcal{R}$ is the right half of the ring $4 \leq x^2 + y^2 \leq 9$.

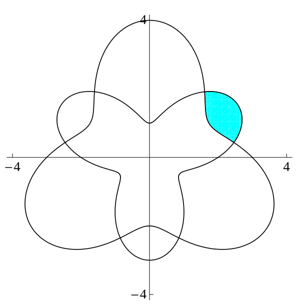


4. Rewrite the iterated integral in Cartesian coordinates $\int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} xy \, dx \, dy$ as an iterated integral in polar coordinates. (Try to draw the region of integration.) You need not evaluate.

5. Find the volume of the solid enclosed by the xy-plane and the paraboloid $z = 9 - x^2 - y^2$. (You may leave your answer as an iterated integral in polar coordinates.)



6. The region inside the curve $r = 2 + \sin 3\theta$ and outside the curve $r = 3 - \sin 3\theta$ consists of three pieces. Find the area of one of these pieces. (You may leave your answer as an iterated integral in polar coordinates.)



When doing integrals in polar coordinates, you often need to integrate trigonometric functions. The **double-angle formulas** are very useful for this. (For instance, they are helpful for the integral in #2.)

The double-angle formulas are easily derived from the fact

$$e^{it} = \cos t + i \sin t \tag{1}$$

If θ is any angle, then

$$e^{i\theta} e^{i\theta} = e^{2i\theta}.$$

Using (1) with $t = \theta$ on the left and $t = 2\theta$ on the right, this becomes

$$\begin{aligned} (\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta) &= \cos 2\theta + i \sin 2\theta \\ \cos^2\theta - \sin^2\theta + 2i\sin\theta\cos\theta &= \cos 2\theta + i\sin 2\theta \end{aligned}$$

Equating the real parts of both sides, $\boxed{\cos^2\theta - \sin^2\theta = \cos 2\theta}$. Equating the imaginary parts, $\boxed{2\sin\theta\cos\theta = \sin 2\theta}$.

The formula $\cos 2\theta = \cos^2\theta - \sin^2\theta$ also leads to useful identities for $\cos^2\theta$ and $\sin^2\theta$:

$$\begin{aligned} \cos 2\theta &= \cos^2\theta - \sin^2\theta \\ &= \cos^2\theta - (1 - \cos^2\theta) \\ &= 2\cos^2\theta - 1 \end{aligned}$$

$$\boxed{\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)}$$

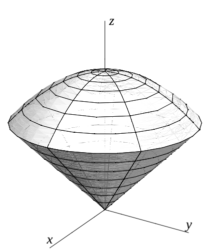
$$\begin{aligned} \cos 2\theta &= \cos^2\theta - \sin^2\theta \\ &= (1 - \sin^2\theta) - \sin^2\theta \\ &= 1 - 2\sin^2\theta \end{aligned}$$

$$\boxed{\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)}$$

These two identities make it easy to integrate $\sin^2\theta$ and $\cos^2\theta$.

For the remaining problems, use polar coordinates or Cartesian coordinates, whichever seems easier.

7. Find the volume of the "ice cream cone" bounded by the single cone $z = \sqrt{x^2 + y^2}$ and the paraboloid $z = 3 - \frac{x^2}{4} - \frac{y^2}{4}$.

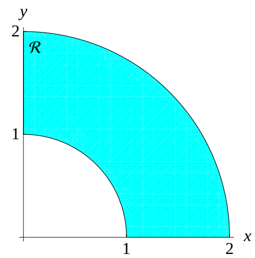


8. A flat plate is in the shape of the region $\mathcal{R}$ defined by the inequalities $x^2 + y^2 \leq 4$, $0 \leq y \leq 1$, $x \leq 0$. The density of the plate at the point (x, y) is $-xy$. Find the mass of the plate.

9. Find the area of the region which lies inside the circle $x^2+(y-1)^2 = 1$ but outside the circle $x^2+y^2 = 1$.

Double Integrals in Polar Coordinates

1. *A flat plate is in the shape of the region $\mathcal{R}$ in the first quadrant lying between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$. The density of the plate at point (x, y) is $x + y$ kilograms per square meter (suppose the axes are marked in meters). Find the mass of the plate.*



Solution. As we saw in #2(b) of the worksheet "Double Integrals", the mass is the double integral of density. That is, the mass is $\iint_{\mathcal{R}} (x + y)\ dA$.

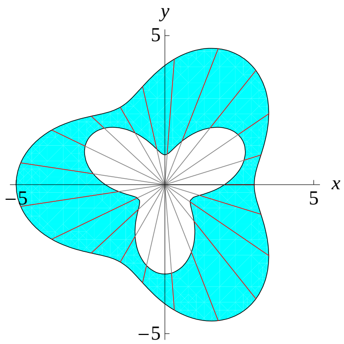
To compute double integrals, we always convert them to iterated integrals. In this case, we'll use a double integral in polar coordinates. The region $\mathcal{R}$ is the polar rectangle $0 \le \theta \le \frac{\pi}{2}$, $1 \le r \le 2$, so we can rewrite the double integral as an iterated integral in polar coordinates:

$$\begin{aligned}
\iint_{\mathcal{R}} (x+y)\ dA &= \int_0^{\pi/2} \int_1^2 (r\cos\theta + r\sin\theta)(r\ dr\ d\theta) \\
&= \int_0^{\pi/2} \int_1^2 r^2(\cos\theta + \sin\theta)\ dr\ d\theta \\
&= \int_0^{\pi/2} \left(\frac{1}{3} r^3 (\cos\theta + \sin\theta) \Big|_{r=1}^{r=2} \right) d\theta \\
&= \int_0^{\pi/2} \frac{7}{3}(\cos\theta + \sin\theta)\ d\theta \\
&= \frac{7}{3}(\sin\theta - \cos\theta) \Big|_{\theta=0}^{\theta=\pi/2} \\
&= \boxed{\frac{14}{3}}
\end{aligned}$$

2. *Find the area of the region $\mathcal{R}$ lying between the curves $r = 2 + \sin 3\theta$ and $r = 4 - \cos 3\theta$. (You may leave your answer as an iterated integral in polar coordinates.)*

Solution. As we saw in #2(a) of the worksheet "Double Integrals", the area of the region $\mathcal{R}$ is equal to the double integral $\iint_{\mathcal{R}} 1\ dA$. To compute the value of this double integral, we will convert it to an iterated integral.

This region is not a polar rectangle, so we'll think about slicing. Let's make slices where θ is constant:



Our slices go all the way around the origin, so the outer integral will have θ going from 0 to 2π. Along each slice, r goes from the inner curve ($r = 2 + \sin 3\theta$) to the outer curve ($r = 4 - \cos 3\theta$). So, the iterated integral is

$$\boxed{\int_0^{2\pi} \int_{2+\sin 3\theta}^{4-\cos 3\theta} 1 \cdot r \, dr \, d\theta} = \int_0^{2\pi} \left(\frac{1}{2} r^2 \Big|_{r=2+\sin 3\theta}^{r=4-\cos 3\theta} \right) d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} \left[(4 - \cos 3\theta)^2 - (2 + \sin 3\theta)^2 \right] \, d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} \left(16 - 8\cos 3\theta + \cos^2 3\theta - 4 - 4\sin 3\theta - \sin^2 3\theta \right) \, d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} \left(12 - 8\cos 3\theta + \cos 6\theta - 4\sin 3\theta \right) \, d\theta$$

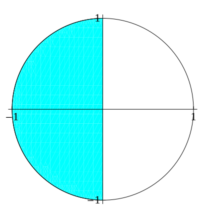
by the double angle identity $\cos 2t = \cos^2 t - \sin^2 t$

$$= \frac{1}{2} \left(12\theta - \frac{8}{3} \sin 3\theta + \frac{1}{6} \sin 6\theta + \frac{4}{3} \cos 3\theta \right) \Big|_{\theta=0}^{\theta=2\pi}$$

$$= \boxed{12\pi}$$

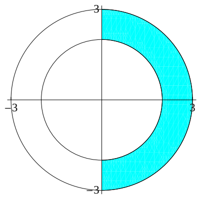
3. In each part, rewrite the double integral as an iterated integral in polar coordinates. (Do not evaluate.)

(a) $\iint_{\mathcal{R}} \sqrt{1 - x^2 - y^2} \, dA$ *where* $\mathcal{R}$ *is the left half of the unit disk.*



Solution. The region $\mathcal{R}$ is the polar rectangle $\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$, $0 \leq r \leq 1$. In polar coordinates, the integrand is $\sqrt{1 - r^2}$. So, we can rewrite the double integral as an iterated integral $\boxed{\int_{\pi/2}^{3\pi/2} \int_0^1 \sqrt{1 - r^2} \cdot r \, dr \, d\theta}$.

(b) $\iint_{\mathcal{R}} x^2 \, dA$ *where* $\mathcal{R}$ *is the right half of the ring* $4 \le x^2 + y^2 \le 9$.



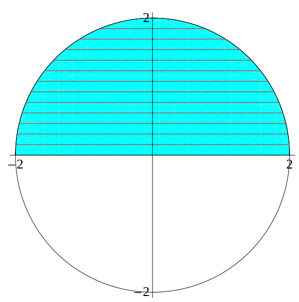
Solution. The region $\mathcal{R}$ is the polar rectangle $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, $2 \le r \le 3$.(1) In polar coordinates, the integrand is $(r\cos\theta)^2$. So, we can rewrite the double integral as an iterated integral

$$\boxed{\int_{-\pi/2}^{\pi/2} \int_2^3 r^2 \cos^2\theta \cdot r \; dr \; d\theta.}$$

4. *Rewrite the iterated integral in Cartesian coordinates* $\int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} xy \; dx \; dy$ *as an iterated integral in polar coordinates. (Try to draw the region of integration.) You need not evaluate.*

Solution. Let's first write the integrand in polar coordinates. Since $x = r\cos\theta$ and $y = r\sin\theta$, the integrand can be written as $r^2 \sin\theta\cos\theta$.

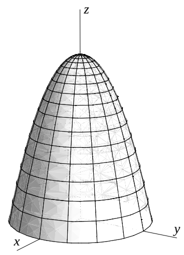
Next, let's figure out the region of integration. Since the outer integral is $\int_0^2$ something dy, we are slicing the interval $[0, 2]$ on the y-axis, so we are making horizontal slices from $y = 0$ to $y = 2$. The inner integral tells us that the left side of each slice is on $x = -\sqrt{4-y^2}$ and the right side of each slice is on $x = \sqrt{4-y^2}$. $x = -\sqrt{4-y^2}$ is the left half of the circle $x^2 + y^2 = 4$, and $x = \sqrt{4-y^2}$ is the right half of the circle $x^2 + y^2 = 4$, so our region of integration (with horizontal slices) looks like this:



This region is the polar rectangle $0 \le \theta \le \pi$, $0 \le r \le 2$. So, the integral in polar coordinates is $\boxed{\int_0^\pi \int_0^2 r^2 \sin\theta\cos\theta \cdot r \; dr \; d\theta}$.

5. *Find the volume of the solid enclosed by the xy-plane and the paraboloid* $z = 9 - x^2 - y^2$. *(You may leave your answer as an iterated integral in polar coordinates.)*

(1) Normally, we want θ to be between 0 and 2π. However, if it's more convenient for a polar integral, we relax this restriction.



Solution. Let's break this down into two steps:

1. First, we'll write a double integral expressing the volume.
2. Then, we'll convert the double integral to an iterated integral.

Notice that the solid can be described as the solid under $z = 9 - x^2 - y^2$ over the region $\mathcal{R}$, where $\mathcal{R}$ is where the solid meets the xy-plane. So, its volume will be $\iint_{\mathcal{R}} (9 - x^2 - y^2)\ dA$. Let's describe $\mathcal{R}$ in more detail. The surface $z = 9 - x^2 - y^2$ intersects the xy-plane $z = 0$ where $x^2 + y^2 = 9$, so the region $\mathcal{R}$ is the disk $x^2 + y^2 \leq 9$.

Now, we'll convert this double integral to an iterated integral. The region $\mathcal{R}$ is the polar rectangle $0 \leq \theta \leq 2\pi, 0 \leq r \leq 3$, so we can rewrite the double integral as

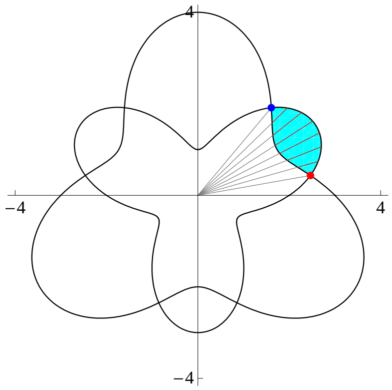
$$\begin{aligned}
\iint_{\mathcal{R}} (9 - x^2 - y^2)\ dA &= \boxed{\int_0^{2\pi} \int_0^3 (9 - r^2) r\ dr\ d\theta} \\
&= \int_0^{2\pi} \int_0^3 (9r - r^3)\ dr\ d\theta \\
&= \int_0^{2\pi} \int_0^3 \left(\frac{9}{2} r^2 - \frac{r^4}{4} \Big|_{r=0}^{r=3} \right)\ d\theta \\
&= \int_0^{2\pi} \frac{81}{4}\ d\theta \\
&= \frac{81}{4}\theta \Big|_{\theta=0}^{\theta=2\pi} \\
&= \boxed{\frac{81\pi}{2}}
\end{aligned}$$

6. *The region inside the curve $r = 2 + \sin 3\theta$ and outside the curve $r = 3 - \sin 3\theta$ consists of three pieces. Find the area of one of these pieces. (You may leave your answer as an iterated integral in polar coordinates.)*

Solution. Since we are finding area, our integral will be $\iint_{\mathcal{R}} 1\ dA$, where $\mathcal{R}$ is the region of integration. As always, to evaluate the double integral, we need to rewrite it as an iterated integral (this time, in polar coordinates).

Let's make slices where $\theta =$ constant.[2]

[2] When we're dealing with regions that aren't polar rectangles, it's almost always easier to slice where $\theta =$ constant.

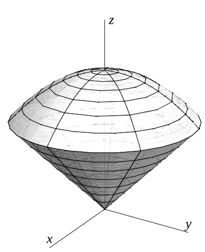


We are slicing from the θ of the red point to the θ of the blue point. Let's find these. The red point and blue point are points where the curves $r = 2 + \sin 3\theta$ and $r = 3 - \sin 3\theta$ intersect, so let's solve $2 + \sin 3\theta = 3 - \sin 3\theta$. This happens when $\sin 3\theta = \frac{1}{2}$, or $3\theta = \frac{\pi}{6}, \frac{5\pi}{6}$. So, the red point has $\theta = \frac{\pi}{18}$, the blue point has $\theta = \frac{5\pi}{18}$, and our outer integral will have θ going from $\frac{\pi}{18}$ to $\frac{5\pi}{18}$.

Along a slice, r goes from the inner curve ($r = 3 - \sin 3\theta$) to the outer curve ($r = 2 + \sin 3\theta$), so we can rewrite our double integral as

$$\begin{aligned}
\int_{\pi/18}^{5\pi/18} \int_{3-\sin 3\theta}^{2+\sin 3\theta} 1 \cdot r \; dr \; d\theta &= \int_{\pi/18}^{5\pi/18} \left(\frac{1}{2} r^2 \Big|_{r=3-\sin 3\theta}^{r=2+\sin 3\theta} \right) d\theta \\
&= \frac{1}{2} \int_{\pi/18}^{5\pi/18} \left[(2+\sin 3\theta)^2 - (3-\sin 3\theta)^2 \right] \; d\theta \\
&= \frac{1}{2} \int_{\pi/18}^{5\pi/18} (-5 + 10 \sin 3\theta) \; d\theta \\
&= \frac{1}{2} \left(-5\theta - \frac{10}{3} \cos 3\theta \right) \Big|_{\theta=\pi/18}^{\theta=5\pi/18} \\
&= \boxed{\frac{5}{\sqrt{3}} - \frac{5\pi}{9}}
\end{aligned}$$

7. *Find the volume of the "ice cream cone" bounded by the single cone $z = \sqrt{x^2 + y^2}$ and the paraboloid $z = 3 - \frac{x^2}{4} - \frac{y^2}{4}$.*

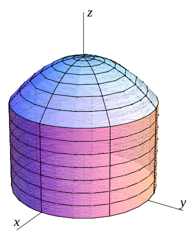
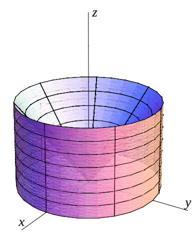
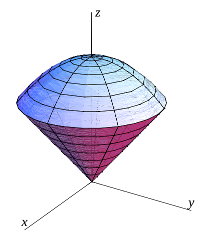


Solution. Let $\mathcal{R}$ be the projection of the solid onto the xy-plane; that is, let $\mathcal{R}$ be the region we see if we look down on the solid from above. This will be a disk, so let's do the integral in polar coordinates.

First, we'll rewrite everything in terms of polar coordinates. The cone $z = \sqrt{x^2 + y^2}$ can be rewritten as $z = r$, and the paraboloid $z = 3 - \frac{x^2}{4} - \frac{y^2}{4}$ can be rewritten as $z = 3 - \frac{r^2}{4}$.

To find the disk $\mathcal{R}$, notice that, if we look at the solid from above, the disk we see is the size of the circle where the two surfaces intersect. The surfaces intersect where $r = 3 - \frac{r^2}{4}$; this can be rewritten as $r^2 + 4r - 12 = 0$, or $(r+6)(r-2) = 0$. Since $r \geq 0$, the intersection is $r = 2$. So, the region $\mathcal{R}$ is a disk centered at the origin with radius 2. This is a polar rectangle with $0 \leq r \leq 2$, $0 \leq \theta < 2\pi$.

One way to find the volume of the solid is to find the volume under the paraboloid over $\mathcal{R}$, find the volume under the cone over $\mathcal{R}$, and subtract the latter from the former.(3) That is:

volume under $z = 3 - \frac{r^2}{4}$ over $\mathcal{R}$	minus	volume under $z = r$ over $\mathcal{R}$	equals	volume we want
				

So, the iterated integral in polar coordinates is

$$\begin{aligned}
\int_0^{2\pi} \int_0^2 \left(3 - \frac{r^2}{4}\right) r\ dr\ d\theta - \int_0^{2\pi} \int_0^2 r \cdot r\ dr\ d\theta &= \int_0^{2\pi} \int_0^2 \left(3 - \frac{r^2}{4} - r\right) r\ dr\ d\theta \\
&= \int_0^{2\pi} \int_0^2 \left(3r - \frac{r^3}{4} - r^2\right)\ dr\ d\theta \\
&= \int_0^{2\pi} \left(\frac{3r^2}{2} - \frac{r^4}{16} - \frac{r^3}{3} \Big|_{r=0}^{r=2} \right) d\theta \\
&= \int_0^{2\pi} \frac{7}{3}\ d\theta \\
&= \boxed{\frac{14\pi}{3}}
\end{aligned}$$

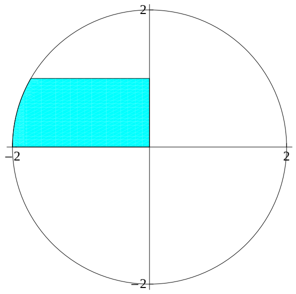
Notice that we end up simply integrating the difference between $3 - \frac{r^2}{4}$ and r; this is really the height of the solid above the point (r, θ). For an explanation of why this works in terms of Riemann sums, see #6 of the worksheet "Double Integrals".

8. *A flat plate is in the shape of the region $\mathcal{R}$ defined by the inequalities $x^2 + y^2 \leq 4$, $0 \leq y \leq 1$, $x \leq 0$. The density of the plate at the point (x, y) is $-xy$. Find the mass of the plate.*

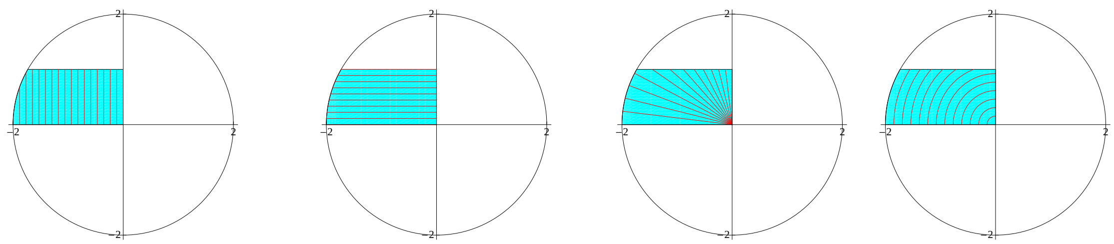
   Solution. As we saw in #2(b) of the worksheet "Double Integrals", the mass is the double integral of density. That is, the mass is $\iint_{\mathcal{R}} -xy\ dA$.

   Here is a picture of the region:

(3) This is similar to what you were asked to do in the homework problem §12.3, #30.



There are four ways we could slice: two in Cartesian (vertically or horizontally) and two in polar (where θ is constant or where r is constant). Here are pictures of all four:



vertical (x = constant) horizontal (y = constant) θ = constant r = constant

When slicing vertically, along θ = constant, or along r = constant, there are multiple "types" of slices. However, if we slice horizontally, there is only one "type" of slice. This suggests that we should go with horizontal slices. Slicing horizontally amounts to slicing the interval $[0,1]$ on the y-axis, so the outer integral will be $\int_0^1$ something dy. Each slice has its left end on the left edge of the circle $x^2+y^2=4$ (so where $x=-\sqrt{4-y^2}$) and its right end on $x=0$, so we can rewrite the double integral as

$$\boxed{\int_0^1 \int_{-\sqrt{4-y^2}}^0 -xy\ dx\ dy} = \int_0^1 \left(-\frac{1}{2}x^2y \Big|_{x=-\sqrt{4-y^2}}^{x=0} \right) dy$$

$$= \int_0^1 \frac{1}{2}(4-y^2)y\ dy$$

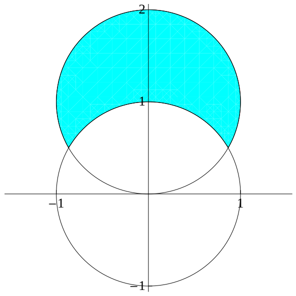
$$= \frac{1}{2}\int_0^1 (4y-y^3)\ dy$$

$$= \frac{1}{2}\left(2y^2 - \frac{y^4}{4}\right)\Big|_{y=0}^{y=1}$$

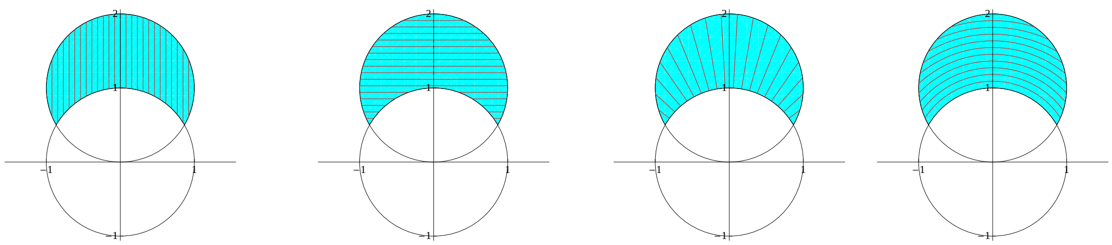
$$= \boxed{\frac{7}{8}}$$

9. *Find the area of the region which lies inside the circle* $x^2+(y-1)^2=1$ *but outside the circle* $x^2+y^2=1$.

Solution. Here is a picture of the region, which we'll call $\mathcal{R}$:



There are four ways we could slice: two in Cartesian (vertically or horizontally) and two in polar (where θ is constant or where r is constant). Here are pictures of all four:



vertical (x = constant) horizontal (y = constant) θ = constant r = constant

When slicing vertically or horizontally, we can see that there are multiple "types" of slices. When slicing where θ = constant or r = constant, there is only one type of slice. So, let's do this in polar coordinates.

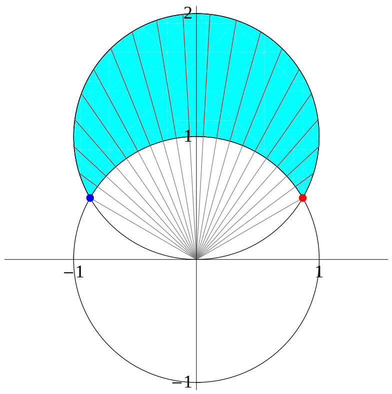
First, let's write the equations of the two circles in polar coordinates. The circle $x^2 + y^2 = 1$ is just $r = 1$. The circle $x^2 + (y-1)^2 = 1$ is more complicated:

$$\begin{aligned} x^2 + (y-1)^2 &= 1 \\ (r\cos\theta)^2 + (r\sin\theta - 1)^2 &= 1 \\ r^2\cos^2\theta + r^2\sin^2\theta - 2r\sin\theta + \cancel{1} &= \cancel{1} \\ r^2(\cos^2\theta + \sin^2\theta) - 2r\sin\theta &= 0 \\ r^2 &= 2r\sin\theta \\ r &= 2\sin\theta \end{aligned}$$

(In the last step, we've divided both sides by r; this is fine since $r > 0$ on the circle.[4])

We'll use the third picture, where we slice along θ = constant. (We can use the fourth as well, but we're more used to doing polar integrals by slicing where θ = constant.) Here's a picture with more detail.

[4] Actually, $r = 0$ at the very bottom of the circle, but as it's just one point, it doesn't really matter.



We are slicing from the θ of the red point to the θ of the blue point. Let's find these values. The red point and blue point are points where the curves $r = 1$ and $r = 2\sin\theta$ intersect, so let's solve $1 = 2\sin\theta$. This happens when $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$. So, the red point has $\theta = \frac{\pi}{6}$, the blue point has $\theta = \frac{5\pi}{6}$, and our outer integral will have θ going from $\frac{\pi}{6}$ to $\frac{5\pi}{6}$.

Along each slice, r goes from the lower circle ($r = 1$) to the upper circle ($r = 2\sin\theta$), so the inner integral will have r going from 1 to $2\sin\theta$. So, we can rewrite our double integral as

$$\boxed{\int_{\pi/6}^{5\pi/6} \int_{1}^{2\sin\theta} 1 \cdot r \; dr \; d\theta} = \int_{\pi/6}^{5\pi/6} \left(\frac{1}{2} r^2 \Big|_{r=1}^{r=2\sin\theta} \right) d\theta$$

$$= \int_{\pi/6}^{5\pi/6} \left(2\sin^2\theta - \frac{1}{2} \right) d\theta$$

$$= \int_{\pi/6}^{5\pi/6} \left(\frac{1}{2} - \cos 2\theta \right) d\theta$$

$$\text{by the identity } \sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$$

$$= \frac{\theta}{2} - \frac{1}{2}\sin 2\theta \Big|_{\theta=\pi/6}^{\theta=5\pi/6}$$

$$= \boxed{\frac{\pi}{3} + \frac{\sqrt{3}}{2}}$$

Applications of Double Integrals: Center of Mass and Surface Area

1. A flat plate ("lamina") is described by the region $\mathcal{R}$ bounded by $y = 0$, $x = 1$, and $y = 2x$. The density of the plate at the point (x, y) is given by the function $f(x, y)$.

   (a) Write double integrals giving the first moment of the plate about the x-axis and the first moment of the plate about the y-axis. (You need not convert to iterated integrals.)

   (b) The <u>center of mass</u> of the plate is defined to be the point $(\overline{x}, \overline{y})$ where

   $$\overline{x} = \frac{\text{first moment of plate about } y\text{-axis}}{\text{mass of plate}} \qquad \text{and} \qquad \overline{y} = \frac{\text{first moment of plate about } x\text{-axis}}{\text{mass of plate}}.$$

   Write expressions for $\overline{x}$ and $\overline{y}$ in terms of iterated integrals.

2. In this problem, we will look at the portion of the paraboloid $z = x^2 + y^2 + 1$ with $z < 10$. Let's call this surface $\mathcal{S}$.

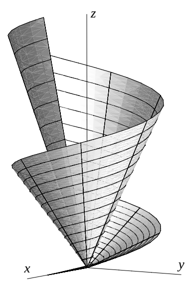
   (a) Parameterize the surface $\mathcal{S}$.[1] Describe any restrictions on the parameters.

   (b) Find the surface area of $\mathcal{S}$.

[1] Remember that this basically means we want to describe the surface using two variables — those are the parameters. Although we may use cylindrical or spherical coordinates to come up with a parameterization, our final parameterization should always describe the surface in Cartesian coordinates.

3. In each part, write a double integral that expresses the surface area of the given surface $\mathcal{S}$. Sketch the region of integration of your double integral. (You do not need to convert the double integral to an iterated integral or evaluate it.)

   (a) $\mathcal{S}$ is parameterized by $\vec{r}(u, v) = \langle u \cos v, u \sin v, uv \rangle$, $0 \le u \le 2\pi$, $0 \le v \le 4\pi$.



   (b) $\mathcal{S}$ is the part of the surface from (a) under the plane $z = 20$.

4. Find the surface area of the following surfaces.

   (a) $\mathcal{S}$ is the portion of the plane $3x - 3y + z = 12$ which lies inside the cylinder $x^2 + y^2 = 1$.

   (b) $\mathcal{S}$ is the portion of the plane $3x - 3y + z = 12$ which lies inside the cylinder $y^2 + z^2 = 1$.

   (c) $\mathcal{S}$ is a sphere of radius 1.

Applications of Double Integrals: Center of Mass and Surface Area

1. A flat plate ("lamina") is described by the region $\mathcal{R}$ bounded by $y = 0$, $x = 1$, and $y = 2x$. The density of the plate at the point (x, y) is given by the function $f(x, y)$.

   (a) *Write double integrals giving the first moment of the plate about the x-axis and the first moment of the plate about the y-axis. (You need not convert to iterated integrals.)*

   Solution. The first moment about the x-axis is $\iint_{\mathcal{R}} yf(x,y)\;dA$, and the first moment about the y-axis is $\iint_{\mathcal{R}} xf(x,y)\;dA$.

   (b) *The center of mass of the plate is defined to be the point $(\overline{x}, \overline{y})$ where*

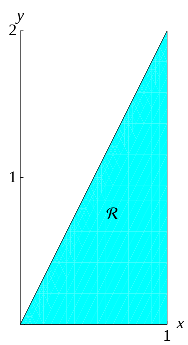
   $$\overline{x} = \frac{\textit{first moment of plate about } y\textit{-axis}}{\textit{mass of plate}} \qquad \textit{and} \qquad \overline{y} = \frac{\textit{first moment of plate about } x\textit{-axis}}{\textit{mass of plate}}.$$

   Write expressions for $\overline{x}$ and $\overline{y}$ in terms of iterated integrals.

   Solution. We know that the mass of the plate is obtained by integrating the density, so the mass is equal to $\iint_{\mathcal{R}} f(x,y)\;dA$. So, in terms of double integrals,

   $$\overline{x} = \frac{\iint_{\mathcal{R}} xf(x,y)\;dA}{\iint_{\mathcal{R}} f(x,y)\;dA} \qquad \text{and} \qquad \overline{y} = \frac{\iint_{\mathcal{R}} yf(x,y)\;dA}{\iint_{\mathcal{R}} f(x,y)\;dA}.$$

   Since we are asked to write this in terms of iterated integrals, we need to actually look at the region $\mathcal{R}$. It looks like this:

   Let's slice vertically. Then, we are slicing the interval $[0, 1]$ on the x-axis, so the outer integral will be $\int_0^1$ something dx. Each slice has its bottom end on $y = 0$ and its top end on $y = 2x$, so the inner integral has y going from 0 to $2x$. This is true for all of the integrals we have, so

   $$\boxed{\overline{x} = \frac{\int_0^1 \int_0^{2x} xf(x,y)\;dy\;dx}{\int_0^1 \int_0^{2x} f(x,y)\;dy\;dx} \qquad \text{and} \qquad \overline{y} = \frac{\int_0^1 \int_0^{2x} yf(x,y)\;dy\;dx}{\int_0^1 \int_0^{2x} f(x,y)\;dy\;dx}}$$

2. In this problem, we will look at the portion of the paraboloid $z = x^2 + y^2 + 1$ with $z < 10$. Let's call this surface $\mathcal{S}$.

(a) *Parameterize the surface $\mathcal{S}$. Describe any restrictions on the parameters.*

Solution. This is the same problem as #1 on the worksheet "Parametric Surfaces". There, we came up with three possible parameterizations:

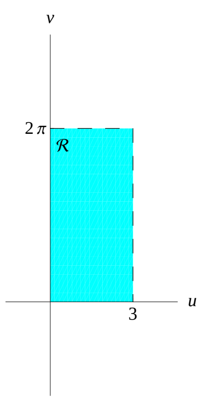
i. $\vec{r}(u, v) = \langle u, v, u^2 + v^2 + 1\rangle$ with $u^2 + v^2 < 9$.

ii. $\vec{r}(u, v) = \langle u \cos v, u \sin v, u^2 + 1\rangle$ with $0 \le u < 3$, $0 \le v < 2\pi$.

iii. $\vec{r}(u, v) = \langle \sqrt{u - 1} \cos v, \sqrt{u - 1} \sin v, u\rangle$ with $1 \le u < 10$, $0 \le v < 2\pi$.

(b) *Find the surface area of $\mathcal{S}$.*

Solution. We can do this using any of the parameterizations from (a). Let's use the second, $\vec{r}(u, v) = \langle u \cos v, u \sin v, u^2 + 1\rangle$ with $0 \le u < 3$, $0 \le v < 2\pi$. The region $\mathcal{R}$ in the uv-plane described by the restrictions $0 \le u < 3$, $0 \le v < 2\pi$ is a rectangle:



We know that the surface area is given by the double integral $\iint_{\mathcal{R}} |\vec{r}_u \times \vec{r}_v| \, dA$. Let's first calculate $|\vec{r}_u \times \vec{r}_v|$:

$$
\begin{aligned}
\vec{r}_u &= \langle \cos v, \sin v, 2u\rangle \\
\vec{r}_v &= \langle -u \sin v, u \cos v, 0\rangle \\
\vec{r}_u \times \vec{r}_v &= \langle -2u^2 \cos v, -2u^2 \sin v, u\rangle \\
|\vec{r}_u \times \vec{r}_v| &= \sqrt{4u^4 + u^2} \\
&= \sqrt{u^2(4u^2 + 1)} \\
&= |u|\sqrt{4u^2 + 1} \\
&= u\sqrt{4u^2 + 1} \text{ since } u \ge 0
\end{aligned}
$$

So, the double integral expressing the surface area is $\boxed{\iint_{\mathcal{R}} u\sqrt{4u^2 + 1} \, dA}$.

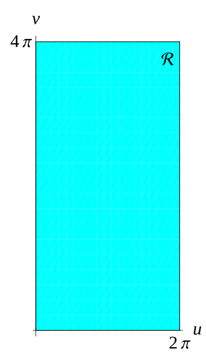
As always, we evaluate double integrals by converting them to iterated integrals. In this case, our region of integration is a rectangle, so it makes sense to do this in Cartesian coordinates (as opposed to polar coordinates). The double integral becomes the iterated integral

$$\begin{aligned}\int_0^3 \int_0^{2\pi} u\sqrt{4u^2+1}\, dv\, du &= \int_0^3 2\pi u\sqrt{4u^2+1}\, du \\ &= \frac{\pi}{6}(4u^2+1)^{3/2}\Big|_{u=0}^{u=3} \\ &= \boxed{\frac{\pi}{6}\left(37^{3/2}-1\right)}\end{aligned}$$

3. In each part, write a double integral that expresses the surface area of the given surface $\mathcal{S}$. Sketch the region of integration of your double integral. (You do not need to convert the double integral to an iterated integral or evaluate it.)

(a) *$\mathcal{S}$ is parameterized by $\vec{r}(u,v) = \langle u\cos v, u\sin v, uv\rangle$, $0 \le u \le 2\pi$, $0 \le v \le 4\pi$.*

Solution. Since we are given a parameterization of $\mathcal{S}$, we can just write down the double integral: it is $\iint_{\mathcal{R}} |\vec{r}_u \times \vec{r}_v|\, dA$, where $\mathcal{R}$ is the region in the uv-plane which describes the possible (u,v). The restrictions $0 \le u \le 2\pi$, $0 \le v \le 4\pi$ define a rectangle in the uv-plane:



Let's compute the integrand $|\vec{r}_u \times \vec{r}_v|$:

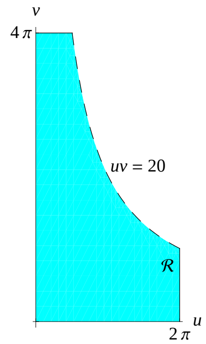
$$\begin{aligned}\vec{r}_u &= \langle \cos v, \sin v, v\rangle \\ \vec{r}_v &= \langle -u\sin v, u\cos v, u\rangle \\ \vec{r}_u \times \vec{r}_v &= \langle u\sin v - uv\cos v, -u\cos v - uv\sin v, u\rangle \\ |\vec{r}_u \times \vec{r}_v| &= u\sqrt{2+v^2} \text{ since } u \ge 0\end{aligned}$$

So, a double integral which gives the surface area of $\mathcal{S}$ is $\iint_{\mathcal{R}} u\sqrt{2+v^2}\, dA$, where $\mathcal{R}$ is the region shown (the rectangle $0 \le u \le 2\pi$, $0 \le v \le 4\pi$).

(b) *$\mathcal{S}$ is the part of the surface from (a) under the plane $z = 20$.*

Solution. We can use the same parameterization as in (a), so the integrand $|\vec{r}_u \times \vec{r}_v|$ for the double integral will not change. What *will* change is the region of integration: the restriction $z < 20$ imposes restrictions on u and v.

In our parameterization $\vec{r}(u,v) = \langle u\cos v, u\sin v, uv\rangle$, $z = uv$, so the restriction $z < 20$ means $uv < 20$. So, the region of integration $\mathcal{R}$ now consists of points (u,v) satisfying $0 \le u \le 2\pi$, $0 \le v \le 4\pi$, $uv < 20$. The region looks like this:



So, a double integral expressing the surface area is $\iint_{\mathcal{R}} u\sqrt{2+v^2}\ dA$, where $\mathcal{R}$ is the region shown.

4. Find the surface area of the following surfaces.

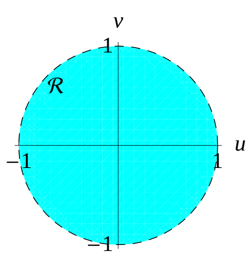
   (a) *$\mathcal{S}$ is the portion of the plane $3x - 3y + z = 12$ which lies inside the cylinder $x^2 + y^2 = 1$.*

   Solution. First, we need to parameterize the surface $\mathcal{S}$.

   Since our surface is part of a plane, let's first parameterize the plane. Then, since we only want the part of the plane inside the cylinder $x^2 + y^2 = 1$, we'll use the inequality $x^2 + y^2 < 1$ to figure out restrictions on our parameters.

   Remember that parameterizing a surface amounts to describing each point (x, y, z) on the surface using just two variables. In this case, we can easily write z in terms of x and y: $z = 12 - 3x + 3y$, so let's use x and y as our parameters. To avoid confusion, we'll rename them u and v, so our parameterization is $x = u$, $y = v$, $z = 12 - 3u + 3v$. We can write this as a parametric vector function $\vec{r}(u, v) = \langle u, v, 12 - 3u + 3v\rangle$.

   Since we want $x^2 + y^2 < 1$ and $x = u$, $y = v$, the restriction on parameters that we have is $u^2 + v^2 < 1$. This describes a unit disk in the uv-plane, which we'll call $\mathcal{R}$:



   We know that the double integral expressing the surface area is $\iint_{\mathcal{R}} |\vec{r}_u \times \vec{r}_v|\ dA$, so let's calculate the integrand:

$$\begin{aligned}\vec{r}_u &= \langle 1, 0, -3\rangle \\ \vec{r}_v &= \langle 0, 1, 3\rangle \\ \vec{r}_u \times \vec{r}_v &= \langle 3, -3, 1\rangle \\ |\vec{r}_u \times \vec{r}_v| &= \sqrt{19}\end{aligned}$$

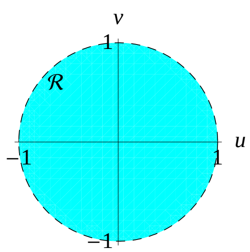
So, the double integral expressing the surface area is $\boxed{\iint_{\mathcal{R}} \sqrt{19}\, dA}$.

Since the region $\mathcal{R}$ is a disk, we could do this integral in polar coordinates. However, since the integrand is just a constant, there's an even easier way: we can pull the constant out to get $\iint_{\mathcal{R}} \sqrt{19}\, dA = \sqrt{19} \iint_{\mathcal{R}} 1\, dA$, and we know that $\iint_{\mathcal{R}} 1\, dA$ is the area of $\mathcal{R}$. In this case, $\mathcal{R}$ is a unit disk, so its area is π. Therefore, the value of the double integral is $\boxed{\sqrt{19}\pi}$.

(b) *$\mathcal{S}$ is the portion of the plane $3x - 3y + z = 12$ which lies inside the cylinder $y^2 + z^2 = 1$.*

Solution. Since we are talking about the same plane as in (a), we might think to parameterize the surface the same way. However, then the part of the plane lying inside the cylinder is described by $v^2 + (12 - 3u + 3v)^2 < 1$, which is a hard region to describe in the uv-plane. So, let's try a different parameterization.

The plane is described by $3x - 3y + z = 12$, which means we can easily describe any of the three variables x, y, and z in terms of the other two. Since we are restricting ourselves to points where $y^2 + z^2 < 1$, let's use $y = u$ and $z = v$; then, the region in the uv-plane is easy to describe: it's just the disk $u^2 + v^2 < 1$. Since $3x - 3y + z = 12$, $x = y - \frac{z}{3} + 4 = u - \frac{v}{3} + 4$. So, we have the parameterization $\vec{r}(u, v) = \left\langle u - \frac{v}{3} + 4, u, v \right\rangle$ with $u^2 + v^2 < 1$. If we let $\mathcal{R}$ denote the disk $u^2 + v^2 < 1$, then a double integral giving the surface area is $\iint_{\mathcal{R}} |\vec{r}_u \times \vec{r}_v|\, dA$. Here's a picture of the region:



Let's write out the integrand:

$$\begin{aligned}
\vec{r}_u &= \langle 1, 1, 0 \rangle \\
\vec{r}_v &= \left\langle -\frac{1}{3}, 0, 1 \right\rangle \\
\vec{r}_u \times \vec{r}_v &= \left\langle 1, -1, \frac{1}{3} \right\rangle \\
|\vec{r}_u \times \vec{r}_v| &= \frac{\sqrt{19}}{3}
\end{aligned}$$

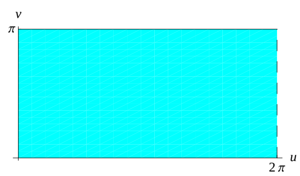
So, the double integral we want to compute is $\iint_{\mathcal{R}} \frac{\sqrt{19}}{3}\, dA = \frac{\sqrt{19}}{3} \iint_{\mathcal{R}} 1\, dA$, which is $\frac{\sqrt{19}}{3}$ times the area of $\mathcal{R}$. Since $\mathcal{R}$ is a disk of radius 1, its area is π. So, the surface area of $\mathcal{S}$ is $\boxed{\frac{\sqrt{19}}{3}\pi}$.

(c) *$\mathcal{S}$ is a sphere of radius 1.*

Solution. We can position our sphere anywhere we want, so let's put it with its center at the origin. Then, the sphere can be described very simply in spherical coordinates as the surface $\rho = 1$, so it makes sense to use θ and ϕ from spherical coordinates as our parameters. The point $(1, \theta, \phi)$ in spherical coordinates is $x = \sin\phi\cos\theta$, $y = \sin\phi\sin\theta$, $z = \cos\phi$. If we write $u = \theta$ and $v = \phi$, this gives us the parameterization $\vec{r}(u, v) = \langle \sin v \cos u, \sin v \sin u, \cos v\rangle$, and $0 \leq u < 2\pi$, $0 \leq v \leq \pi$. So, if $\mathcal{R}$ is the rectangle $0 \leq u < 2\pi$, $0 \leq v \leq \pi$, then the surface area is the double integral $\iint_{\mathcal{R}} |\vec{r}_u \times \vec{r}_v| \; dA$. Let's compute the integrand:

$$\begin{aligned}
\vec{r}_u &= \langle -\sin v \sin u, \sin v \cos u, 0\rangle \\
\vec{r}_v &= \langle \cos v \cos u, \cos v \sin u, -\sin v\rangle \\
\vec{r}_u \times \vec{r}_v &= \langle -\sin^2 v \cos u, -\sin^2 v \sin u, -\sin v \cos v\rangle \\
|\vec{r}_u \times \vec{r}_v| &= \sqrt{\sin^4 v \cos^2 u + \sin^4 v \sin^2 u + \sin^2 v \cos^2 v} \\
&= \sqrt{\sin^4 v + \sin^2 v \cos^2 v} \\
&= \sqrt{\sin^2 u(\sin^2 v + \cos^2 v)} \\
&= \sqrt{\sin^2 v} \\
&= |\sin v|
\end{aligned}$$

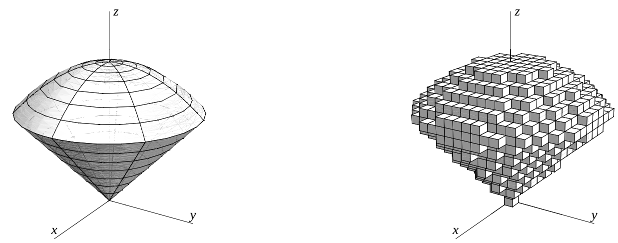
Since $0 \leq v \leq \pi$, $\sin v \geq 0$, so $|\sin v| = \sin v$. Therefore, an integral giving the surface area of the sphere is $\boxed{\iint_{\mathcal{R}} \sin v \; dA \text{ where } \mathcal{R} \text{ is the rectangle } 0 \leq u < 2\pi,\ 0 \leq v \leq \pi}$. Here is a sketch of the region:



To compute, we need to convert this to an iterated integral. Since the region is a rectangle, it's easiest to do this in Cartesian coordinates, and we get

$$\begin{aligned}
\int_0^{2\pi} \int_0^{\pi} \sin v \; dv \; du &= \int_0^{2\pi} \left(-\cos v \Big|_{v=0}^{v=\pi} \right) \; du \\
&= \int_0^{2\pi} 2 \; du \\
&= \boxed{4\pi}
\end{aligned}$$

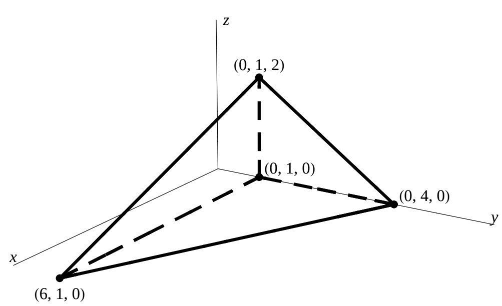
Triple Integrals



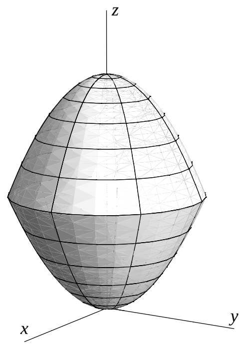
1. (a) If $\mathcal{U}$ is any solid (in space), what does the triple integral $\iiint_{\mathcal{U}} 1\ dV$ represent? Why?

   (b) Suppose the shape of a solid object is described by the solid $\mathcal{U}$, and $f(x, y, z)$ gives the density of the object at the point (x, y, z) in kilograms per cubic meter. What does the triple integral $\iiint_{\mathcal{U}} f(x, y, z)\ dV$ represent? Why?

2. Let $\mathcal{U}$ be the solid tetrahedron bounded by the planes $x = 0$, $y = 1$, $z = 0$, and $x + 2y + 3z = 8$. (The vertices of this tetrahedron are $(0, 1, 0)$, $(0, 1, 2)$, $(6, 1, 0)$, and $(0, 4, 0)$). Write the triple integral $\iiint_{\mathcal{U}} f(x, y, z)\ dV$ as an iterated integral.



3. Let $\mathcal{U}$ be the solid enclosed by the paraboloids $z = x^2+y^2$ and $z = 8-(x^2+y^2)$. (Note: The paraboloids intersect where $z = 4$.) Write $\iiint_{\mathcal{U}} f(x,y,z)\ dV$ as an iterated integral in the order $dz\ dy\ dx$.



4. In this problem, we'll look at the iterated integral $\int_0^1 \int_0^z \int_{y^2}^1 f(x,y,z)\ dx\ dy\ dz$.

   (a) Rewrite the iterated integral in the order $dx\ dz\ dy$.

   (b) Rewrite the iterated integral in the order $dz\ dy\ dx$.

5. Let $\mathcal{U}$ be the solid contained in $x^2+y^2-z^2 = 16$ and lying between the planes $z = -3$ and $z = 3$. Sketch $\mathcal{U}$ and write an iterated integral which expresses its volume. In which orders of integration can you write just a single iterated integral (as opposed to a sum of iterated integrals)?

Triple Integrals

1. (a) *If $\mathcal{U}$ is any solid (in space), what does the triple integral* $\iiint_{\mathcal{U}} 1\ dV$ *represent? Why?*

   Solution. Remember that we are thinking of the triple integral $\iiint_{\mathcal{U}} f(x,y,z)\ dV$ as a limit of Riemann sums, obtained from the following process:

   1. Slice the solid $\mathcal{U}$ into small pieces.
   2. In each piece, the value of f will be approximately constant, so multiply the value of f at any point by the volume ΔV of the piece. (It's okay to approximate the volume ΔV.)
   3. Add up all of these products. (This is a Riemann sum.)
   4. Take the limit of the Riemann sums as the volume of the pieces tends to 0.

   Now, if f is just the function $f(x,y,z) = 1$, then in Step 2, we end up simply multiplying 1 by the volume of the piece, which gives us the volume of the piece. So, in Step 3, when we add all of these products up, we are just adding up the volume of all the small pieces, which gives the volume of the whole solid.

   So, $\iiint_{\mathcal{U}} 1\ dV$ represents the $\boxed{\text{volume of the solid } \mathcal{U}}$.

   (b) *Suppose the shape of a solid object is described by the solid $\mathcal{U}$, and $f(x,y,z)$ gives the density of the object at the point (x,y,z) in kilograms per cubic meter. What does the triple integral* $\iiint_{\mathcal{U}} f(x,y,z)\ dV$ *represent? Why?*

   Solution. Following the process described in (a), in Step 2, we multiply the approximate density of each piece by the volume of that piece, which gives the approximate mass of that piece. Adding those up gives the approximate mass of the entire solid object, and taking the limit gives us the exact $\boxed{\text{mass of the solid object}}$.

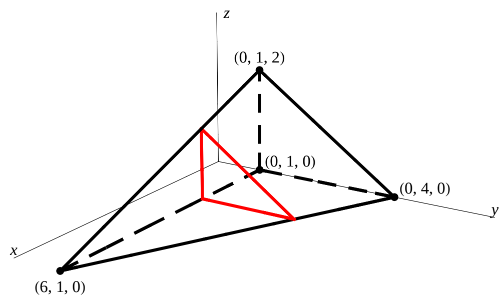
2. *Let $\mathcal{U}$ be the solid tetrahedron bounded by the planes $x = 0$, $y = 1$, $z = 0$, and $x + 2y + 3z = 8$. (The vertices of this tetrahedron are $(0,1,0)$, $(0,1,2)$, $(6,1,0)$, and $(0,4,0)$). Write the triple integral* $\iiint_{\mathcal{U}} f(x,y,z)\ dV$ *as an iterated integral.*

   Solution. We'll do this in all 6 possible orders. Let's do it by writing the outer integral first, which means we think of slicing. There are three possible ways to slice: parallel to the yz-plane, parallel to the xz-plane, and parallel to the xy-plane.

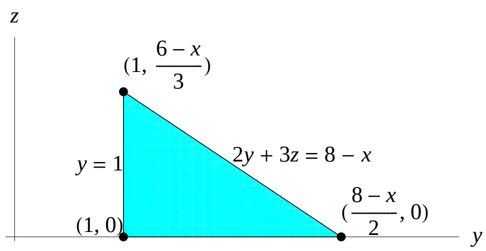
   (a) **Slice parallel to the yz-plane.**

   If we do this, we are slicing the interval $[0,6]$ on the x-axis, so the outer (single) integral will be $\int_0^6$ something dx.

   To write the inner two integrals, we look at a typical slice and describe it. When we do this, we think of x as being constant (since, within a slice, x is constant). Here is a typical slice:



Each slice is a triangle, with one edge on the plane $y = 1$, one edge on the plane $z = 0$, and one edge on the plane $x+2y+3z = 8$. (Since we are thinking of x as being constant, we might rewrite this last equation as $2y + 3z = 8 - x$.) Here's another picture of the slice, in 2D:



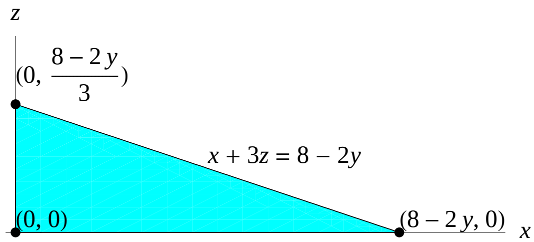
Now, we write a (double) iterated integral that describes this region. This will make up the inner two integrals of our final answer. There are two ways to do this:

i. If we slice vertically, we are slicing the interval $\left[1, \frac{8-x}{2}\right]$ on the y-axis, so the outer integral (of the two we are working on) will be $\displaystyle\int_1^{(8-x)/2}$ something dy. Each slice goes from $z = 0$ to the line $2y + 3z = 8 - x$ (since we're trying to describe z within a vertical slice, we'll rewrite this as $z = \frac{8-x-2y}{3}$), so the inner integral will be $\displaystyle\int_0^{(8-x-2y)/3} f(x, y, z)\ dz$. This gives us the iterated integral $\boxed{\displaystyle\int_0^6 \int_1^{(8-x)/2} \int_0^{(8-x-2y)/3} f(x, y, z)\ dz\ dy\ dx}$.

ii. If we slice horizontally, we are slicing the interval $\left[0, \frac{6-x}{3}\right]$ on the z-axis, so the outer integral (of the two we are working on) will be $\displaystyle\int_0^{(6-x)/3}$ something dz. Each slice goes from $y = 1$ to the line $2y + 3z = 8 - x$ (since we are trying to describe y in a horizontal slice, we'll rewrite this as $y = \frac{8-x-3z}{2}$), so the inner integral will be $\displaystyle\int_1^{(8-x-3z)/2} f(x, y, z)\ dy$. This gives the final answer $\boxed{\displaystyle\int_0^6 \int_0^{(6-x)/3} \int_1^{(8-x-3z)/2} f(x, y, z)\ dy\ dz\ dx}$.

(b) **Slice parallel to the xz-plane.**

If we do this, we are slicing the interval $[1, 4]$ on the y-axis. So, our outer (single) integral will be $\displaystyle\int_1^4$ something dy. Each slice is a triangle with edges on the planes $x = 0$, $z = 0$, and $x+2y+3z = 8$ (or $x + 3z = 8 - 2y$). Within a slice, y is constant, so we can just look at x and z:



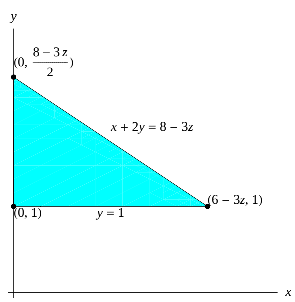
Our inner two integrals will describe this region.

i. If we slice vertically, we are slicing the interval $[0, 8-2y]$ on the x-axis, so the outer integral (of the two we're working on) will be $\int_0^{8-2y}$ something dx. Each slice goes from $z = 0$ to $z = \frac{8-2y-x}{3}$, which gives the iterated integral $\boxed{\int_1^4 \int_0^{8-2y} \int_0^{(8-2y-x)/3} f(x,y,z)\ dz\ dx\ dy}$.

ii. If we slice horizontally, we are slicing the interval $\left[0, \frac{8-2y}{3}\right]$ on the z-axis, so the outer integral (of the two we're working on) will be $\int_0^{(8-2y)/3}$ something dz. Each slice goes from $x = 0$ to $x = 8-2y-3z$, which gives the iterated integral $\boxed{\int_1^4 \int_0^{(8-2y)/3} \int_0^{8-2y-3z} f(x,y,z)\ dx\ dz\ dy}$.

(c) **Slice parallel to the xy-plane.**

If we do this, we are slicing the interval $[0, 2]$ on the z-axis, so the outer (single) integral will be $\int_0^2$ something dz. Each slice is a triangle with edges on the planes $x = 0$, $y = 1$, and $x+2y+3z = 8$ (or $x + 2y = 8 - 3z$). Within a slice, z is constant, so we can just look at x and y:

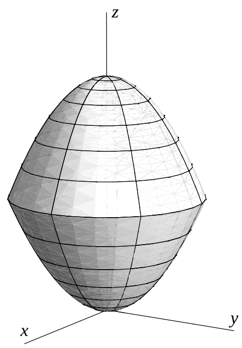


Our inner two integrals will describe this region.

i. If we slice vertically, we are slicing the interval $[0, 6-3z]$ on the x-axis, so the outer integral (of the two we're working on) will be $\int_0^{6-3z}$ something dx. Each slice will go from $y = 1$ to the line $x + 2y = 8 - 3z$ (which we write as $y = \frac{8-3z-x}{2}$ since we're trying to describe y), which gives us the final integral $\boxed{\int_0^2 \int_0^{6-3z} \int_1^{(8-3z-x)/2} f(x,y,z)\ dy\ dx\ dz}$.

ii. If we slice horizontally, we are slicing the interval $\left[1, \frac{8-3z}{2}\right]$ on the y-axis, so the outer integral (of the two we're working on) will be $\int_1^{(8-3z)/2}$ something dy. Each slice will go from $x = 0$ to $x + 2y = 8 - 3z$ (which we write as $x = 8 - 3z - 2y$ since we're trying to describe x), which gives us the answer $\boxed{\int_0^2 \int_1^{(8-3z)/2} \int_0^{8-3z-2y} f(x, y, z)\ dx\ dy\ dz}$.

3. *Let $\mathcal{U}$ be the solid enclosed by the paraboloids $z = x^2 + y^2$ and $z = 8 - (x^2 + y^2)$. (Note: The paraboloids intersect where $z = 4$.) Write $\iiint_{\mathcal{U}} f(x, y, z)\ dV$ as an iterated integral in the order $dz\ dy\ dx$.*

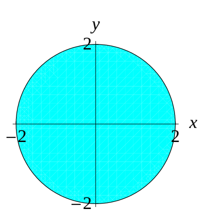


Solution. We can either do this by writing the inner integral first or by writing the outer integral first. In this case, it's probably easier to write the inner integral first, but we'll show both methods.

- **Writing the inner integral first:**

  We are asked to have our inner integral be with respect to z, so we want to describe how z varies along a vertical line (where x and y are fixed) to write the inner integral. We can see that, along any vertical line through the solid, z goes from the bottom paraboloid ($z = x^2 + y^2$) to the top paraboloid ($z = 8 - (x^2 + y^2)$), so the inner integral will be $\int_{x^2+y^2}^{8-(x^2+y^2)} f(x, y, z)\ dz$.

  To write the outer two integrals, we want to describe the projection of the region onto the xy-plane. (A good way to think about this is, if we moved our vertical line around to go through the whole solid, what x and y would we hit? Alternatively, if we could stand at the "top" of the z-axis and look "down" at the solid, what region would we see?) In this case, the widest part of the solid is where the two paraboloids intersect, which is $z = 4$ and $x^2 + y^2 = 4$. Therefore, the projection is the region $x^2 + y^2 \leq 4$, a disk in the xy-plane:

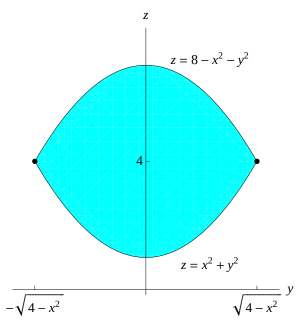
  We want to write an iterated integral in the order $dy\ dx$ to describe this region, which means we should slice vertically. We slice $[-2, 2]$ on the x-axis, so the outer integral will be $\int_{-2}^{2}$ something dx.

Along each slice, y goes from the bottom of the circle ($y = -\sqrt{4-x^2}$) to the top ($y = \sqrt{4-x^2}$), so we get the iterated integral $\boxed{\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{x^2+y^2}^{8-(x^2+y^2)} f(x,y,z)\ dz\ dy\ dx}$.

- **Writing the outer integral first:**

  We are asked to have our outer integral be with respect to x, so we want to make slices parallel to the yz-plane. This amounts to slicing the interval $[-2, 2]$ on the x-axis, so the outer integral will be $\int_{-2}^{2}$ something dx.

  Each slice is a region bounded below by $z = x^2 + y^2$ and above by $z = (8 - x^2) - y^2$. (Remember that, within a slice, x is constant.) Note that these curves intersect where $x^2 + y^2 = (8 - x^2) - y^2$, or $2y^2 = 8 - 2x^2$. This happens at $y = \pm\sqrt{4-x^2}$. At either of these y-values, $z = 4$. So, here is a picture of the region:



  The two integrals describing this region are supposed to be in the order $dz\ dy$, which means we are slicing vertically. Slicing vertically amounts to slicing the interval $[-\sqrt{4-x^2}, \sqrt{4-x^2}]$ on the y-axis, so the outer integral (of these two integrals) will be $\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}$ something dy. Along each vertical slice, z goes from $x^2 + y^2$ to $8 - (x^2 + y^2)$, so we get the final iterated integral $\boxed{\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{x^2+y^2}^{8-(x^2+y^2)} f(x,y,z)\ dz\ dy\ dx}$.

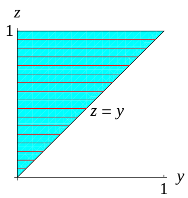
4. In this problem, we'll look at the iterated integral $\int_0^1 \int_0^z \int_{y^2}^1 f(x,y,z)\ dx\ dy\ dz$.

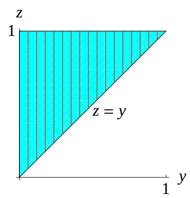
   (a) *Rewrite the iterated integral in the order $dx\ dz\ dy$.*

   Solution. One strategy is to draw the solid region of integration and then write the integral as we did in #3. However, drawing the solid region of integration is rather challenging, so here's another approach.

   Remember that we can think of a triple integral as either a single integral of a double integral or a double integral of a single integral, and we know how to change the order of integration in a double integral. (See, for instance, #5 on the worksheet "Double Integrals over General Regions".) This effectively means that we can change the order of the inner two integrals by thinking of them together as a double integral, or we can change the order of the outer two integrals by thinking of them together as a double integral.

For this question, we just need to change the order of the outer two integrals, so we focus on those. They are $\int_0^1 \int_0^z \text{stuff } dy\, dz$.[(1)] Since this integral is $dy\, dz$, we should visualize the yz-plane. The fact that the outer integral is $\int_0^1 \text{something } dz$ tells us that we are slicing the interval $[0, 1]$ on the z-axis. The fact that the inner integral is $\int_0^z \text{stuff } dy$ tells us that each slice starts at $y = 0$ and goes to $y = z$. So, our region (with horizontal slices) looks like the picture on the left:





To change the order of integration, we want to use vertical slices (the picture on the right). Now, we are slicing the interval $[0, 1]$ on the y-axis, so the outer integral will be $\int_0^1 \text{something } dy$. Each slice has its bottom edge on $z = y$ and its top edge on $z = 1$, so we rewrite $\int_0^1 \int_0^z \text{stuff } dy\, dz$ as $\int_0^1 \int_y^1 \text{stuff } dz\, dy$. Remembering that "stuff" was the inner integral $\int_{y^2}^1 f(x, y, z)\, dx$, this gives us the iterated integral $\boxed{\int_0^1 \int_y^1 \int_{y^2}^1 f(x, y, z)\, dx\, dz\, dy}$.

(b) *Rewrite the iterated integral in the order $dz\ dy\ dx$.*

Solution. Let's continue from (a). As explained there, we can change the order of the outer two integrals or of the inner two integrals. From (a), we have our iterated integral in the order $dx\ dz\ dy$. If we change the order of the inner two integrals, then we'll have our iterated integral in the order $dz\ dx\ dy$. If we then change the order of the outer two integrals of this, we'll get it into the order $dz\ dy\ dx$. So, we really have two steps.

- **Step 1: Change the order of the inner double integral from (a).**

  We had $\int_0^1 \int_y^1 \int_{y^2}^1 f(x, y, z)\, dx\, dz\, dy$, so we are going to focus on the inner double integral $\int_y^1 \int_{y^2}^1 f(x, y, z)\, dx\, dz$. Remember that, since this is the inner double integral and y is the outer variable, we now think of y as being a constant.[(2)] Then, the region of integration of the integral $\int_y^1 \int_{y^2} 1 f(x, y, z)\, dz\, dz$ is just a rectangle (sliced horizontally):

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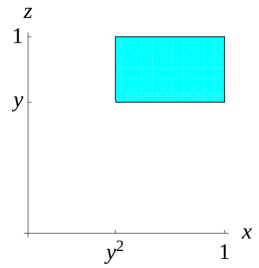
(1) Here, "stuff" is the inner integral $\int_{y^2}^1 f(x, y, z)\, dx$.

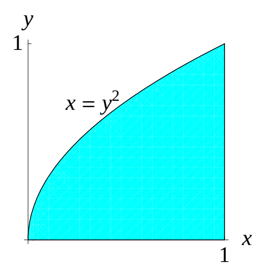
(2) In fact, we should think of y as being a constant between 0 and 1, since the outer integral has y going from 0 to 1.



If we change to slicing horizontally, we rewrite this as $\int_{y^2}^{1}\int_{y}^{1} f(x,y,z)\ dz\ dx$.[(3)] Putting the outer integral back, we get the iterated integral $\int_0^1\int_{y^2}^{1}\int_{y}^{1} f(x,y,z)\ dz\ dx\ dy$.

- **Step 2: Change the order of the outer double integral.**

  Now, we're working with $\int_0^1\int_{y^2}^{1}\int_{y}^{1} f(x,y,z)\ dz\ dx\ dy$, and we want to change the order of the outer double integral, which is $\int_0^1\int_{y^2}^{1}$ stuff $dx\ dy$ with "stuff" being the inner integral $\int_y^1 f(x,y,z)\ dz$. The region of integration of $\int_0^1\int_{y^2}^{1}$ stuff $dx\ dy$ looks like this (with horizontal slices):

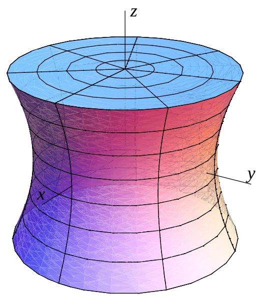


  If we change to slicing vertically, then the integral becomes $\int_0^1\int_0^{\sqrt{x}}$ stuff $dy\ dx$. Remembering that "stuff" was the inner integral, we get our final answer $\boxed{\int_0^1\int_0^{\sqrt{x}}\int_y^1 f(x,y,z)\ dz\ dy\ dx}$.

5. *Let $\mathcal{U}$ be the solid contained in $x^2+y^2-z^2=16$ and lying between the planes $z=-3$ and $z=3$. Sketch $\mathcal{U}$ and write an iterated integral which expresses its volume. In which orders of integration can you write just a single iterated integral (as opposed to a sum of iterated integrals)?*

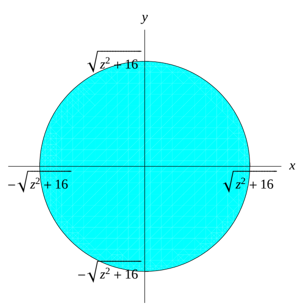
   Solution. Here is a picture of $\mathcal{U}$:[(4)]

(3) Another way of thinking about it is that we're using Fubini's Theorem.
(4) To remember how to sketch things like this, look back at #3 of the worksheet "Quadric Surfaces".



By #1(a), we know that a triple integral expressing the volume of $\mathcal{U}$ is $\iiint_{\mathcal{U}} 1 \ dV$. We are asked to rewrite this as an iterated integral. Let's think about slicing the solid (which means we'll write the outer integral first). If we slice parallel to the xy-plane, then we are really slicing $[-3, 3]$ on the z-axis, and the outer integral is $\int_{-3}^{3}$ something dz.

We use our inner two integrals to describe a typical slice. Each slice is just the disk enclosed by the circle $x^2 + y^2 = z^2 + 16$, which is a circle of radius $\sqrt{z^2 + 16}$:

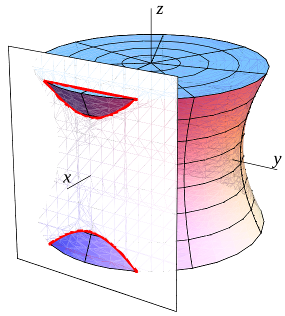


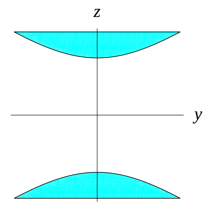
We can slice this region vertically or horizontally; let's do it vertically. This amounts to slicing $\left[-\sqrt{z^2+16}, \sqrt{z^2+16}\right]$ on the x-axis, so the outer integral is $\int_{-\sqrt{z^2+16}}^{\sqrt{z^2+16}}$ something dx. Along each slice, y goes from the bottom of the circle ($y = -\sqrt{z^2 + 16 - x^2}$) to the top of the circle ($y = \sqrt{z^2 + 16 - x^2}$). So, the inner integral is $\int_{-\sqrt{z^2+16-x^2}}^{\sqrt{z^2+16-x^2}} f(x, y, z) \ dy$.

Putting this all together, we get the iterated integral $\boxed{\int_{-3}^{3} \int_{-\sqrt{z^2+16}}^{\sqrt{z^2+16}} \int_{-\sqrt{z^2+16-x^2}}^{\sqrt{z^2+16-x^2}} 1 \ dy \ dx \ dz}$.

We are also asked in which orders we can write just a single iterated integral. Clearly, we've done so with the order $dy \ dx \ dz$. We also could have with $dx \ dy \ dz$, since that would just be slicing the same disk horizontally.

If we had dx or dy as our outer integral, then we would need to write multiple integrals. For instance, if we slice the hyperboloid parallel to the yz-plane, some slices would look like this:

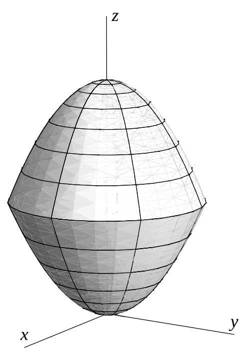




We would need to write a sum of integrals to describe such a slice. So, we can write a single iterated integral only in the orders $\boxed{dy\ dx\ dz \text{ and } dx\ dy\ dz}$.

Triple Integrals in Cylindrical or Spherical Coordinates

1. Let $\mathcal{U}$ be the solid enclosed by the paraboloids $z = x^2+y^2$ and $z = 8-(x^2+y^2)$. (Note: The paraboloids intersect where $z = 4$.) Write $\iiint_{\mathcal{U}} xyz\ dV$ as an iterated integral in cylindrical coordinates.

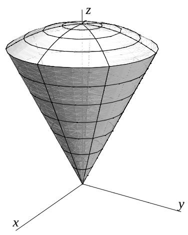


2. Find the volume of the solid ball $x^2 + y^2 + z^2 \leq 1$.

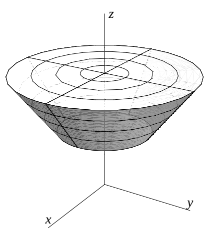
3. Let $\mathcal{U}$ be the solid inside both the cone $z = \sqrt{x^2 + y^2}$ and the sphere $x^2+y^2+z^2 = 1$. Write the triple integral $\iiint_{\mathcal{U}} z\ dV$ as an iterated integral in spherical coordinates.

For the remaining problems, use the coordinate system (Cartesian, cylindrical, or spherical) that seems easiest.

4. Let $\mathcal{U}$ be the "ice cream cone" bounded below by $z = \sqrt{3(x^2+y^2)}$ and above by $x^2+y^2+z^2 = 4$. Write an iterated integral which gives the volume of $\mathcal{U}$. (You need not evaluate.)



5. Write an iterated integral which gives the volume of the solid enclosed by $z^2 = x^2+y^2$, $z = 1$, and $z = 2$. (You need not evaluate.)

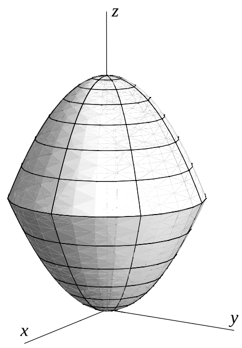


6. Let $\mathcal{U}$ be the solid enclosed by $z = x^2+y^2$ and $z = 9$. Rewrite the triple integral $\iiint_{\mathcal{U}} x\, dV$ as an iterated integral. (You need not evaluate, but can you guess what the answer is?)

7. The iterated integral in spherical coordinates $\int_{\pi/2}^{\pi} \int_0^{\pi/2} \int_1^2 \rho^3 \sin^3\phi \, d\rho\, d\phi\, d\theta$ computes the mass of a solid. Describe the solid (its shape and its density at any point).

Triple Integrals in Cylindrical or Spherical Coordinates

1. *Let $\mathcal{U}$ be the solid enclosed by the paraboloids $z = x^2 + y^2$ and $z = 8 - (x^2 + y^2)$. (Note: The paraboloids intersect where $z = 4$.) Write $\iiint_{\mathcal{U}} xyz\ dV$ as an iterated integral in cylindrical coordinates.*



Solution. This is the same problem as #3 on the worksheet "Triple Integrals", except that we are now given a specific integrand. It makes sense to do the problem in cylindrical coordinates since the solid is symmetric about the z-axis. In cylindrical coordinates, the two paraboloids have equations $z = r^2$ and $z = 8 - r^2$. In addition, the integrand xyz is equal to $(r\cos\theta)(r\sin\theta)z$.

Let's write the inner integral first. If we imagine sticking vertical lines through the solid, we can see that, along any vertical line, z goes from the bottom paraboloid $z = r^2$ to the top paraboloid $z = 8-r^2$. So, our inner integral will be $\int_{r^2}^{8-r^2} (r\cos\theta)(r\sin\theta)z\ dz$.

To write the outer two integrals, we want to describe the projection of the solid onto the xy-plane. As we had figured out last time, the projection was the disk $x^2 + y^2 \leq 4$. We can write an iterated integral in polar coordinates to describe this disk: the disk is $0 \leq r \leq 2$, $0 \leq \theta < 2\pi$, so our iterated integral will just be $\int_0^{2\pi} \int_0^2 (\text{inner integral}) \cdot r\ dr\ d\theta$. Therefore, our final answer is $\boxed{\int_0^{2\pi} \int_0^2 \int_{r^2}^{8-r^2} (r\cos\theta)(r\sin\theta)z \cdot r\ dz\ dr\ d\theta}$.

2. *Find the volume of the solid ball $x^2 + y^2 + z^2 \leq 1$.*

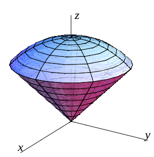
Solution. Let $\mathcal{U}$ be the ball. We know by #1(a) of the worksheet "Triple Integrals" that the volume of $\mathcal{U}$ is given by the triple integral $\iiint_{\mathcal{U}} 1\ dV$. To compute this, we need to convert the triple integral to an iterated integral.

The given ball can be described easily in spherical coordinates by the inequalities $0 \leq \rho \leq 1$, $0 \leq \phi \leq \pi$, $0 \leq \theta < 2\pi$, so we can rewrite the triple integral $\iiint_{\mathcal{U}} 1\ dV$ as an iterated integral in spherical coordinates

$$\boxed{\int_0^{2\pi}\int_0^{\pi}\int_0^1 1\cdot\rho^2\sin\phi\;d\rho\;d\phi\;d\theta} = \int_0^{2\pi}\int_0^{\pi}\left(\frac{\rho^3}{3}\sin\phi\Big|_{\rho=0}^{\rho=1}\right)\;d\phi\;d\theta$$
$$= \int_0^{2\pi}\int_0^{\pi}\frac{1}{3}\sin\phi\;d\phi\;d\theta$$
$$= \int_0^{2\pi}\left(-\frac{1}{3}\cos\phi\Big|_{\phi=0}^{\phi=\pi}\right)\;d\theta$$
$$= \int_0^{2\pi}\frac{2}{3}\;d\theta$$
$$= \boxed{\frac{4}{3}\pi}$$

3. *Let $\mathcal{U}$ be the solid inside both the cone $z=\sqrt{x^2+y^2}$ and the sphere $x^2+y^2+z^2=1$. Write the triple integral $\iiint_{\mathcal{U}} z\;dV$ as an iterated integral in spherical coordinates.*

   Solution. Here is a picture of the solid:

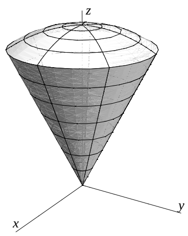
   We have to write both the integrand (z) and the solid of integration in spherical coordinates. We know that z in Cartesian coordinates is the same as $\rho\cos\phi$ in spherical coordinates, so the function we're integrating is $\rho\cos\phi$.

   The cone $z=\sqrt{x^2+y^2}$ is the same as $\phi=\frac{\pi}{4}$ in spherical coordinates.[1] The sphere $x^2+y^2+z^2=1$ is $\rho=1$ in spherical coordinates. So, the solid can be described in spherical coordinates as $0\le\rho\le 1$, $0\le\phi\le\frac{\pi}{4}$, $0\le\theta\le 2\pi$. This means that the iterated integral is $\boxed{\int_0^{2\pi}\int_0^{\pi/4}\int_0^1(\rho\cos\phi)\rho^2\sin\phi\;d\rho\;d\phi\;d\theta}$.

For the remaining problems, use the coordinate system (Cartesian, cylindrical, or spherical) that seems easiest.

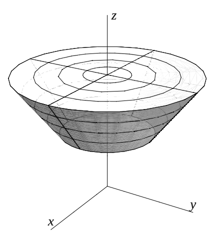
4. *Let $\mathcal{U}$ be the "ice cream cone" bounded below by $z=\sqrt{3(x^2+y^2)}$ and above by $x^2+y^2+z^2=4$. Write an iterated integral which gives the volume of $\mathcal{U}$. (You need not evaluate.)*

[1] Why? We could first rewrite $z=\sqrt{x^2+y^2}$ in cylindrical coordinates: it's $z=r$. In terms of spherical coordinates, this says that $\rho\cos\phi=\rho\sin\phi$, so $\cos\phi=\sin\phi$. That's the same as saying that $\tan\phi=1$, or $\phi=\frac{\pi}{4}$.



Solution. We know by #1(a) of the worksheet "Triple Integrals" that the volume of $\mathcal{U}$ is given by the triple integral $\iiint_{\mathcal{U}} 1\ dV$. The solid $\mathcal{U}$ has a simple description in spherical coordinates, so we will use spherical coordinates to rewrite the triple integral as an iterated integral. The sphere $x^2 + y^2 + z^2 = 4$ is the same as $\rho = 2$. The cone $z = \sqrt{3(x^2 + y^2)}$ can be written as $\phi = \frac{\pi}{6}$.[(2)] So, the volume is $\boxed{\int_0^{2\pi} \int_0^{\pi/6} \int_0^2 1 \cdot \rho^2 \sin\phi\ d\rho\ d\phi\ d\theta}$.

5. *Write an iterated integral which gives the volume of the solid enclosed by $z^2 = x^2 + y^2$, $z = 1$, and $z = 2$. (You need not evaluate.)*

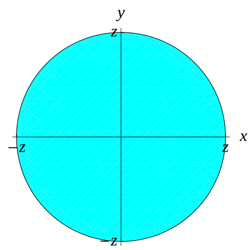


Solution. We know by #1(a) of the worksheet "Triple Integrals" that the volume of $\mathcal{U}$ is given by the triple integral $\iiint_{\mathcal{U}} 1\ dV$. To compute this, we need to convert the triple integral to an iterated integral. Since the solid is symmetric about the z-axis but doesn't seem to have a simple description in terms of spherical coordinates, we'll use cylindrical coordinates.

Let's think of slicing the solid, using slices parallel to the xy-plane. This means we'll write the outer integral first. We're slicing $[1, 2]$ on the z-axis, so our outer integral will be $\int_1^2$ something dz.

To write the inner double integral, we want to describe each slice (and, within a slice, we can think of z as being a constant). Each slice is just the disk enclosed by the circle $x^2 + y^2 = z^2$, which is a circle of radius z:

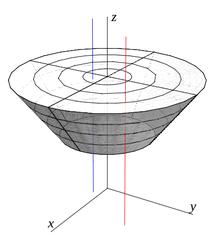
(2) This is true because $z = \sqrt{3(x^2 + y^2)}$ can be written in cylindrical coordinates as $z = r\sqrt{3}$. In terms of spherical coordinates, this says that $\rho\cos\phi = \sqrt{3}\rho\sin\phi$. That's the same as saying $\tan\phi = \frac{1}{\sqrt{3}}$, or $\phi = \frac{\pi}{6}$.



We'll use polar coordinates to write the iterated (double) integral describing this slice. The circle can be described as $0 \le \theta < 2\pi$ and $0 \le r \le z$ (and remember that we are still thinking of z as a constant), so the appropriate integral is $\int_0^{2\pi} \int_0^z 1 \cdot r \; dr \; d\theta$.

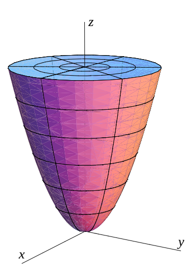
Putting this into our outer integral, we get the iterated integral $\boxed{\int_1^2 \int_0^{2\pi} \int_0^z 1 \cdot r \; dr \; d\theta \; dz}$.

Note: For this problem, writing the inner integral first doesn't work as well, at least not if we want to write the integral with dz as the inner integral. Why? Well, if we try to write the integral with dz as the inner integral, we imagine sticking vertical lines through the solid. The problem is that there are different "types" of vertical lines. For instance, along the red line in the picture below, z goes from the cone ($z = \sqrt{x^2+y^2}$ or $z = r$) to $z = 2$ (in the solid). But, along the blue line, z goes from $z = 1$ to $z = 2$. So, we'd have to write two separate integrals to deal with these two different situations.



6. *Let $\mathcal{U}$ be the solid enclosed by $z = x^2 + y^2$ and $z = 9$. Rewrite the triple integral $\iiint_{\mathcal{U}} x \; dV$ as an iterated integral. (You need not evaluate, but can you guess what the answer is?)*

Solution. $z = x^2 + y^2$ describes a paraboloid, so the solid looks like this:



Since the solid is symmetric about the z-axis, a good guess is that cylindrical coordinates will make things easier. In cylindrical coordinates, the integrand x is equal to $r \cos\theta$.

Let's think of slicing the solid, which means we'll write our outer integral first. If we slice parallel to the xy-plane, then we are slicing the interval $[0, 9]$ on the z-axis, so our outer integral is $\int_0^9$ something dz.

We use the inner two integrals to describe a typical slice; within a slice, z is constant. Each slice is a disk enclosed by the circle $x^2 + y^2 = z$ (which has radius $\sqrt{z}$). We know that we can describe this in polar coordinates as $0 \le r \le \sqrt{z}$, $0 \le \theta < 2\pi$. So, the inner two integrals will be $\int_0^{2\pi} \int_0^{\sqrt{z}} (r\cos\theta) \cdot r\ dr\ d\theta$. Therefore, the given triple integral is equal to the iterated integral

$$\boxed{\int_0^9 \int_0^{2\pi} \int_0^{\sqrt{z}} r\cos\theta \cdot r\ dr\ d\theta\ dz} = \int_0^9 \int_0^{2\pi} \left(\frac{1}{3} r^3 \cos\theta \Big|_{r=0}^{r=\sqrt{z}} \right) dr\ d\theta\ dz$$

$$= \int_0^9 \int_0^{2\pi} \frac{1}{3} z^{3/2} \cos\theta\ d\theta\ dz$$

$$= \int_0^9 \left(\frac{1}{3} z^{3/2} \sin\theta \Big|_{\theta=0}^{\theta=2\pi} \right) dz$$

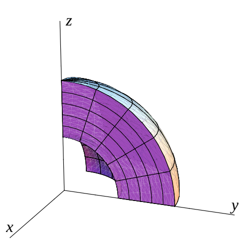
$$= \boxed{0}$$

That the answer is 0 should not be surprising because the integrand $f(x, y, z) = x$ is anti-symmetric about the plane $x = 0$ (this is sort of like saying the function is odd: $f(-x, y, z) = -f(x, y, z)$), but the solid is symmetric about the plane $x = 0$.

Note: If you decided to do the inner integral first, you probably ended up with dz as your inner integral. In this case, a valid iterated integral is $\boxed{\int_0^{2\pi} \int_0^3 \int_{r^2}^9 r\cos\theta \cdot r\ dz\ dr\ d\theta}$.

7. *The iterated integral in spherical coordinates* $\int_{\pi/2}^{\pi} \int_0^{\pi/2} \int_1^2 \rho^3 \sin^3\phi\ d\rho\ d\phi\ d\theta$ *computes the mass of a solid. Describe the solid (its shape and its density at any point).*

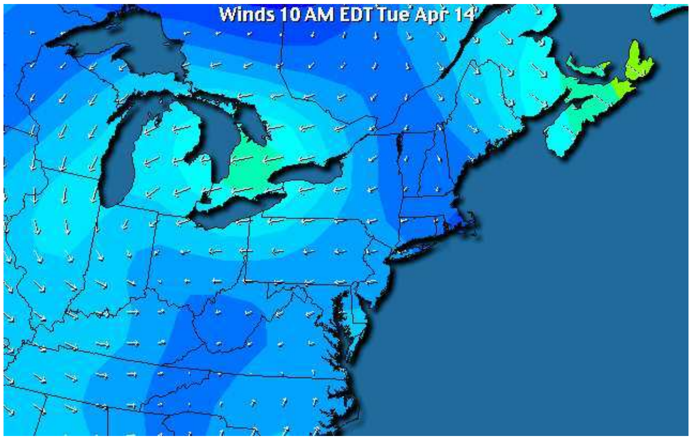
Solution. The shape of the solid is described by the region of integration. We can read this off from the bounds of integration: it is $\frac{\pi}{2} \le \theta \le \pi$, $0 \le \phi \le \frac{\pi}{2}$, $1 \le \rho \le 2$. We can visualize $1 \le \rho \le 2$ by imagining a solid ball of radius 2 with a solid ball of radius 1 taken out of the middle. $0 \le \phi \le \frac{\pi}{2}$ tells us we'll only have the top half of that, and $\frac{\pi}{2} \le \theta \le \pi$ tells us that we'll only be looking at one octant: the one with x negative and y positive:



To figure out the density, remember that we get mass by integrating the density. If we call this solid $\mathcal{U}$, then the iterated integral in the problem is the same as the triple integral $\iiint_{\mathcal{U}} \rho \sin^2\phi\ dV$ since dV is $\rho^2 \sin\phi\ d\rho\ d\phi\ d\theta$. So, the density of the solid at a point (ρ, ϕ, θ) is $\boxed{\rho \sin^2\phi}$.

Vector Fields and Line Integrals

Here is a weather map showing the wind velocity at various points in the Northeastern United States at 10 am on April 14. This is an example of a vector field (representing velocity). If we wanted to write it using mathematical notation, we could let $\vec{F}(x,y)$ be the velocity of the wind at a point (x,y) on the map.



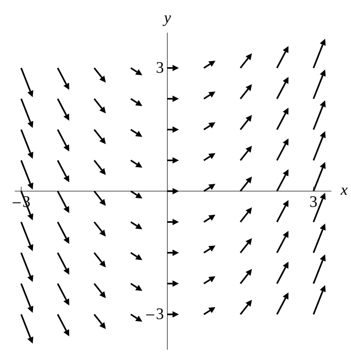
1. Match the following vector fields on $\mathbb{R}^2$ with their plots.

   (a) $\vec{F}(x,y) = \langle x, 1\rangle$.

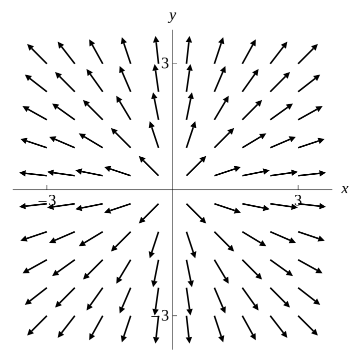
   (b) $\vec{F}(x,y) = \langle 1, x\rangle$.

   (c) $\vec{F} = \nabla f$, where f is the scalar-valued function $f(x,y) = x^2 + y^2$.

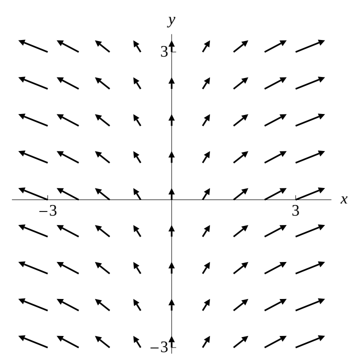
   (d) $\vec{F}(x,y) = \left\langle \dfrac{x}{\sqrt{x^2+y^2}}, \dfrac{y}{\sqrt{x^2+y^2}} \right\rangle$.



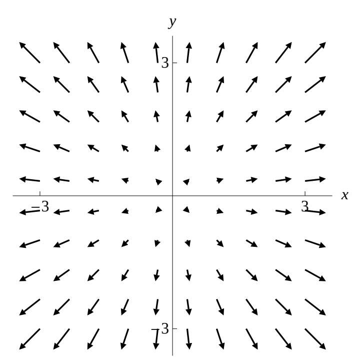
(I)



(II)

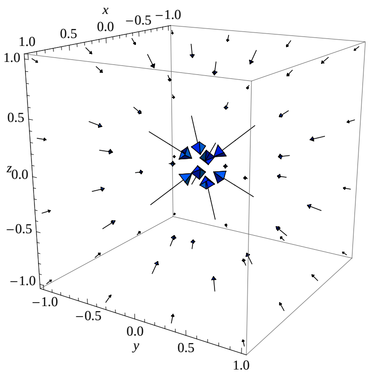


(III)

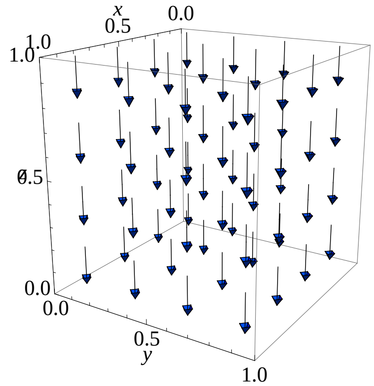


(IV)

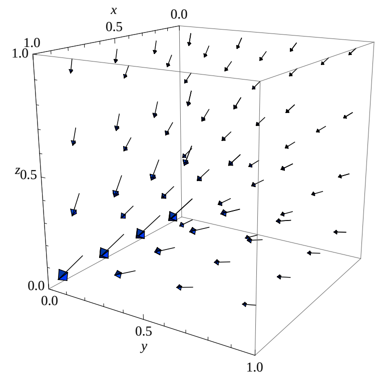
2. Match the following vector fields on $\mathbb{R}^3$ with their plots.



(I)



(II)



(III)

(a) $\vec{F}(x,y,z) = \langle 0, 0, -1\rangle$.

(b) $\vec{F}(x,y,z) = \left\langle 0, -\frac{y}{y^2+z^2}, -\frac{z}{y^2+z^2}\right\rangle$.

(c) $\vec{F}(x,y,z) = \left\langle -\frac{x}{(x^2+y^2+z^2)^{3/2}}, -\frac{y}{(x^2+y^2+z^2)^{3/2}}, -\frac{z}{(x^2+y^2+z^2)^{3/2}}\right\rangle$.

3. Vector fields are used to model various things. For each of the following descriptions, decide which of the vector field plots in #2 (I, II, or III) gives the most appropriate model.

(a) Force of gravity experienced by a fly in a room. More precisely, $\vec{F}(x,y,z)$ is the force due to gravity experienced by a fly located at point (x,y,z) in a room. (Remember that force is a vector.)

(b) Force of Earth's gravity experienced by a space shuttle. More precisely, $\vec{F}(x,y,z)$ is the force that Earth's gravitational field exerts on a space shuttle located at the point (x,y,z). In the picture you've chosen, where is the Earth?

(c) ∇f, where $f(x,y,z)$ is the temperature in a room in which there is a heater along one edge of the floor. In the picture you've chosen, where is the heater? (Hint: The gradient of a function f always points in the direction in which f is ______________?)

4. Let $\vec{F}$ be the vector field on $\mathbb{R}^2$ defined by $\vec{F}(x,y) = \langle 1, x \rangle$. (We saw this vector field already in #1.)

(a) Let C be the bottom half of the unit circle $x^2 + y^2 = 1$ (in $\mathbb{R}^2$), traversed counter-clockwise. Evaluate $\int_C \vec{F} \cdot d\vec{r}$.

Note: This line integral can also be written as $\int_C 1\ dx + x\ dy$.

(b) We write $-C$ to mean the same curve as C (in this case, the bottom half of the unit circle) but oriented in the opposite direction (so clockwise instead of counter-clockwise). What is $\int_{-C} \vec{F} \cdot d\vec{r}$?

(c) Now, let C be the line segment from $(0,0)$ to $(0,1)$. Looking at the picture of $\vec{F}$ (in #1), do you think $\int_C \vec{F} \cdot d\vec{r}$ is positive, negative, or zero? Why?

(d) What if C is instead the line segment from $(0,0)$ to $(1,1)$? Is the line integral $\int_C \vec{F} \cdot d\vec{r}$ positive, negative, or zero?

5. Let $f(x,y) = e^x + xy$ and $\vec{F} = \nabla f$, a vector field on $\mathbb{R}^2$. Let C be the curve in $\mathbb{R}^2$ parameterized by $\vec{r}(t) = \langle t, t^2 \rangle$, $0 \le t \le 1$.

   (a) Compute the line integral $\displaystyle\int_C \vec{F} \cdot d\vec{r}$.

   (b) What is $f(\vec{r}(t))$? Did you use this anywhere when you computed the line integral in (a)? Can you explain why this happened?

   (c) Suppose we want to look at a new curve C, parameterized by $\vec{r}(t) = \left\langle (\sin t)e^{\cos t^2 + \sqrt{t}}, \sin t + \cos t \right\rangle$ with $0 \le t \le \pi$. Find $\displaystyle\int_C \vec{F} \cdot d\vec{r}$.

Vector Fields and Line Integrals

1. Match the following vector fields on $\mathbb{R}^2$ with their plots.

   (a) $\vec{F}(x,y) = \langle x, 1\rangle$.

   Solution. Any vector $\langle x, 1\rangle$ points up, and the only plot that matches this is $\boxed{\text{(III)}}$.

   (b) $\vec{F}(x,y) = \langle 1, x\rangle$.

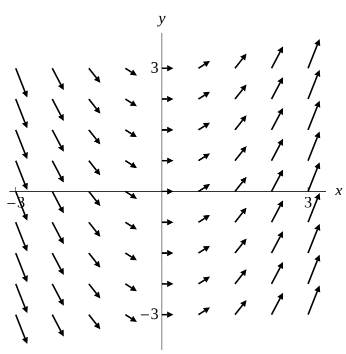
   Solution. Any vector $\langle 1, x\rangle$ points up, and the only plot that matches this is $\boxed{\text{(I)}}$.

   (c) $\vec{F} = \nabla f$, *where* f *is the scalar-valued function* $f(x,y) = x^2 + y^2$.

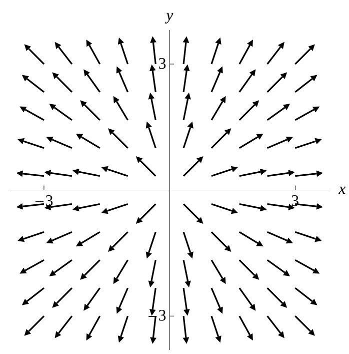
   Solution. $\nabla f(x,y) = \langle 2x, 2y\rangle$. If we draw the vector $\langle 2x, 2y\rangle$ with its tail at (x,y), then it points away from the origin. In addition, as x and y get bigger, the vectors $\langle 2x, 2y\rangle$ get longer, which describes plot $\boxed{\text{(IV)}}$.

   (d) $\vec{F}(x,y) = \left\langle \dfrac{x}{\sqrt{x^2+y^2}}, \dfrac{y}{\sqrt{x^2+y^2}} \right\rangle$.

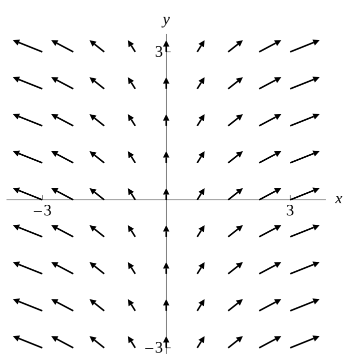
   Solution. The only choice left is $\boxed{\text{(II)}}$. The vectors there all appear to be the same length, and indeed $\left|\left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right\rangle\right| = 1$ for all (x,y) (except for $(0,0)$, since $\vec{F}(0,0)$ is undefined).



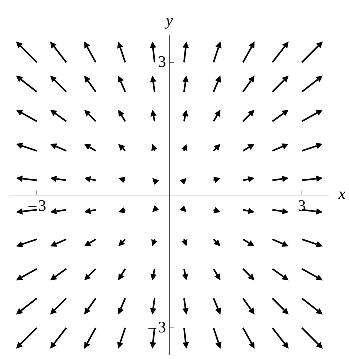
(I)



(II)

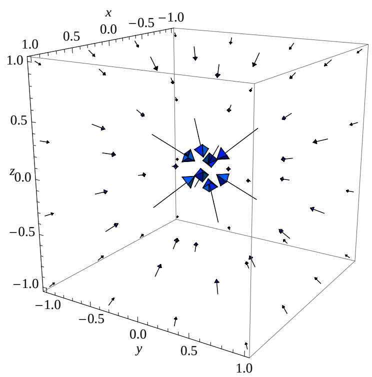


(III)

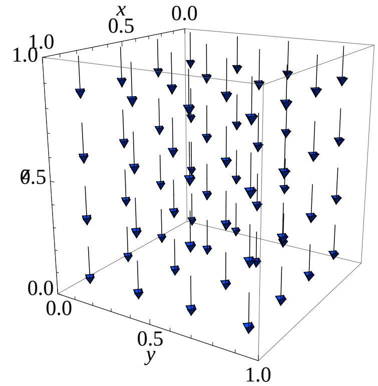


(IV)

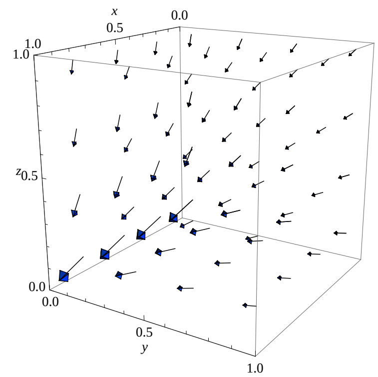
2. Match the following vector fields on $\mathbb{R}^3$ with their plots.



(I)



(II)



(III)

(a) $\vec{F}(x,y,z) = \langle 0, 0, -1 \rangle$.

Solution. This is a constant vector field; that is, at every point (x,y,z), we draw the same vector $\langle 0,0,-1\rangle$. This matches $\boxed{\text{(II)}}$.

(b) $\vec{F}(x,y,z) = \left\langle 0, -\dfrac{y}{y^2+z^2}, -\dfrac{z}{y^2+z^2} \right\rangle$.

Solution. If we draw the vector $\left\langle 0, -\dfrac{y}{y^2+z^2}, -\dfrac{z}{y^2+z^2} \right\rangle$ with its tail at the point (x,y,z), then it points toward the x-axis. This matches $\boxed{\text{(III)}}$.

(c) $\vec{F}(x,y,z) = \left\langle -\dfrac{x}{(x^2+y^2+z^2)^{3/2}}, -\dfrac{y}{(x^2+y^2+z^2)^{3/2}}, -\dfrac{z}{(x^2+y^2+z^2)^{3/2}} \right\rangle$.

Solution. If we draw the vector $\left\langle -\dfrac{x}{(x^2+y^2+z^2)^{3/2}}, -\dfrac{y}{(x^2+y^2+z^2)^{3/2}}, -\dfrac{z}{(x^2+y^2+z^2)^{3/2}} \right\rangle$ with its tail at the point (x,y,z), then it points toward the origin. This matches $\boxed{\text{(I)}}$.

3. Vector fields are used to model various things. For each of the following descriptions, decide which of the vector field plots in #2 (I, II, or III) gives the most appropriate model.

(a) *Force of gravity experienced by a fly in a room. More precisely, $\vec{F}(x,y,z)$ is the force due to gravity experienced by a fly located at point (x,y,z) in a room. (Remember that force is a vector.)*

Solution. No matter where the fly is in the room, it experiences the same force due to gravity — the magnitude of the force is just the fly's mass multiplied by acceleration due to gravity, or 9.8 m/s^2 (this is really the fly's weight), and the direction of the force is always straight toward the ground. Therefore, we want a constant vector field with all vectors pointing toward the ground, which means $\boxed{\text{(II)}}$ is the best model.

(b) *Force of Earth's gravity experienced by a space shuttle. More precisely, $\vec{F}(x,y,z)$ is the force that Earth's gravitational field exerts on a space shuttle located at the point (x,y,z). In the picture you've chosen, where is the Earth?*

Solution. No matter where the space shuttle is, the force exerted by Earth's gravitational field will be a vector pointing toward the Earth. The force should be stronger near the Earth (vectors of greater magnitude or length) and less strong away from the Earth (vectors of smaller magnitude). This matches $\boxed{\text{(I)}}$, with the Earth at the origin.

(c) ∇f, *where* $f(x, y, z)$ *is the temperature in a room in which there is a heater along one edge of the floor. In the picture you've chosen, where is the heater? (Hint: The gradient of a function* f *always points in the direction in which* f *is* ______________*?)*

Solution. We know that the gradient of a function f always points in the direction in which f is increasing the most (instantaneously). In this case, f represents temperature, so the gradient at any point points towards the warmest direction. This matches $\boxed{\text{(III)}}$, with the heater being positioned along the x-axis.

4. Let $\vec{F}$ be the vector field on $\mathbb{R}^2$ defined by $\vec{F}(x, y) = \langle 1, x\rangle$. (We saw this vector field already in #1.)

(a) *Let* C *be the bottom half of the unit circle* $x^2 + y^2 = 1$ *(in* $\mathbb{R}^2$*), traversed counter-clockwise. Evaluate* $\int_C \vec{F} \cdot d\vec{r}$.

Solution. First, we must parameterize the curve C. One possible parameterization is $\vec{r}(t) = \langle \cos t, \sin t\rangle$ with $\pi \le t \le 2\pi$.(1) Then, we can simply compute, using the definition of the line integral:

$$\begin{aligned}
\int_C \vec{F} \cdot d\vec{r} &= \int_\pi^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\ dt \\
&= \int_\pi^{2\pi} \vec{F}(\cos t, \sin t) \cdot \langle -\sin t, \cos t\rangle\ dt \\
&= \int_\pi^{2\pi} \langle 1, \cos t\rangle \cdot \langle -\sin t, \cos t\rangle\ dt \\
&= \int_\pi^{2\pi} (-\sin t + \cos^2 t)\ dt \\
&= \int_\pi^{2\pi} \left(-\sin t + \frac{1+\cos 2t}{2}\right)\ dt \\
&= \cos t + \frac{t}{2} + \frac{1}{4}\sin 2t \Big|_{t=\pi}^{t=2\pi} \\
&= \boxed{2 + \frac{\pi}{2}}
\end{aligned}$$

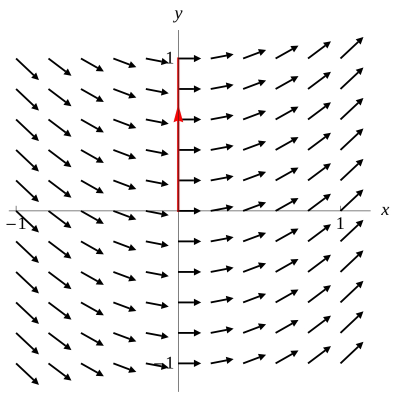
(b) *We write* $-C$ *to mean the same curve as* C *(in this case, the bottom half of the unit circle) but oriented in the opposite direction (so clockwise instead of counter-clockwise). What is* $\int_{-C} \vec{F} \cdot d\vec{r}$*?*

Solution. It is simply the negative of $\int_C \vec{F} \cdot d\vec{r}$, or $\boxed{-2 - \frac{\pi}{2}}$.

(1) One way to arrive at this parameterization is to think about a particle traveling along the curve. In polar coordinates, the unit circle is just $r = 1$. The particle starts at $\theta = \pi$ and moves toward $\theta = 2\pi$ (with θ increasing). So, in polar coordinates, we could parameterize the curve just by taking $r = 1$, $\theta = t$ with $\pi \le t \le 2\pi$. Translating this back to Cartesian coordinates, $x = \cos t$ and $y = \sin t$, so we have the parameterization $\vec{r}(t) = \langle \cos t, \sin t\rangle$.

(c) *Now, let C be the line segment from $(0,0)$ to $(0,1)$. Looking at the picture of $\vec{F}$ (in #1), do you think $\int_C \vec{F} \cdot d\vec{r}$ is positive, negative, or zero? Why?*

Solution. Here is a plot of the vector field, together with the curve (drawn in red):

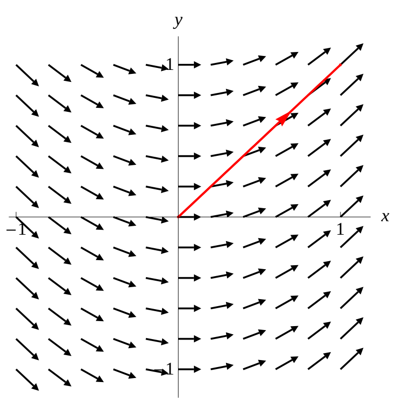


Along the path, $\vec{F}$ always points to the right, while the path goes up. In particular, $\vec{F}$ is always perpendicular to the path, so the line integral should be $\boxed{\text{zero}}$.

To be more precise about it, if we were to find a parameterization $\vec{r}(t)$ of the curve C, then $\vec{F}(\vec{r}(t))$ is always perpendicular to $\vec{r}'(t)$ (which is tangent to C), so the dot product $\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)$ is always 0. This is the thing we integrate to compute the line integral, so the line integral must be 0 as well.(2)

(d) *What if C is instead the line segment from $(0,0)$ to $(1,1)$? Is the line integral $\int_C \vec{F} \cdot d\vec{r}$ positive, negative, or zero?*

Solution. Here is a plot of the vector field, together with the curve (drawn in red):



Along this path, the vector field generally goes the same direction as the path; that is, the path makes an acute angle with the vectors in the vector field. So, the line integral is $\boxed{\text{positive}}$.

More precisely, if we were to find a parameterization $\vec{r}(t)$ of the curve C, then $\vec{F}(\vec{r}(t))$ (the vector

(2) To confirm, we could parameterize the curve and compute the line integral; one parameterization of C is $\vec{r}(t) = \langle 0, t \rangle$, $0 \leq t \leq 1$.

field $\vec{F}$ at a particular point on the path) always makes an acute angle with $\vec{r}'(t)$ (the direction the path is going), so the dot product $\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)$ is always positive.[3] Since we integrate this dot product to compute the line integral, the line integral will also be positive.[4]

5. Let $f(x,y) = e^x + xy$ and $\vec{F} = \nabla f$, a vector field on $\mathbb{R}^2$. Let C be the curve in $\mathbb{R}^2$ parameterized by $\vec{r}(t) = \langle t, t^2 \rangle$, $0 \le t \le 1$.

   (a) *Compute the line integral* $\int_C \vec{F} \cdot d\vec{r}$.

   Solution. We compute $\vec{F}(x,y) = \nabla f(x,y) = \langle e^x + y, x \rangle$. To find the line integral, we just use the definition of the line integral:

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= \int_0^1 \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\ dt \\ &= \int_0^1 \vec{F}(t,t^2) \cdot \langle 1, 2t \rangle\ dt \\ &= \int_0^1 \langle e^t + t^2, t \rangle \cdot \langle 1, 2t \rangle\ dt \\ &= \int_0^1 (e^t + 3t^2)\ dt \\ &= e^t + t^3 \Big|_{t=0}^{t=1} \\ &= \boxed{e}\end{aligned}$$

   (b) *What is $f(\vec{r}(t))$? Did you use this anywhere when you computed the line integral in (a)? Can you explain why this happened?*

   Solution. $f(\vec{r}(t)) = e^t + t^3$. We saw this in the second-to-last step of (a): it was what we got from integrating $\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)$ (so we ended up just plugging the starting and ending values of t into this).

   This can be explained by the Chain Rule, which says that $\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}'(t)$. In this case, $\nabla f = \vec{F}$, so $\frac{d}{dt} f(\vec{r}(t)) = \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)$. This was the thing we were integrating with respect to t, and we know by the Fundamental Theorem of Calculus that if we integrate $\frac{d}{dt}$(something) with respect to t, we'll just get that "something" back.

   (c) *Suppose we want to look at a new curve C, parameterized by $\vec{r}(t) = \left\langle (\sin t) e^{\cos t^2 + \sqrt{t}}, \sin t + \cos t \right\rangle$ with $0 \le t \le \pi$. Find* $\int_C \vec{F} \cdot d\vec{r}$.

[3] Remember that the dot product $\vec{u} \cdot \vec{v}$ can be written as $|\vec{u}||\vec{v}| \cos\theta$, where θ is the angle between the vectors $\vec{u}$ and $\vec{v}$. If θ is an acute angle, then $\cos\theta > 0$; if θ is an obtuse angle, then $\cos\theta < 0$. So, if the angle between $\vec{u}$ and $\vec{v}$ is acute, then $\vec{u} \cdot \vec{v} > 0$; if the angle between $\vec{u}$ and $\vec{v}$ is obtuse, then $\vec{u} \cdot \vec{v} < 0$.

[4] To confirm, we could parameterize the curve and compute the line integral; one parameterization of C is $\vec{r}(t) = \langle t, t \rangle$, $0 \le t \le 1$.

Solution. Using what we figured out (b), we can jump directly to the end of the computation:

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= f(\vec{r}(t))\Big|_{t=0}^{t=\pi} \\ &= f(\vec{r}(\pi)) - f(\vec{r}(0))\end{aligned}$$

From the given formula for $\vec{r}(t)$, $\vec{r}(\pi) = \langle 0, -1\rangle$ and $\vec{r}(0) = \langle 0, 1\rangle$, so the line integral is $f(0,-1) - f(0,1) = \boxed{0}$.

The Fundamental Theorem for Line Integrals; Gradient Vector Fields

1. Let $f(x, y) = \sin x + x^2 y$ and $\vec{F} = \nabla f$, a vector field on $\mathbb{R}^2$. Let C be the curve in $\mathbb{R}^2$ parameterized by $\vec{r}(t) = \langle t, t^2 \rangle$, $0 \le t \le \pi$.

   (a) Compute the line integral $\displaystyle\int_C \vec{F} \cdot d\vec{r}$.

   (b) What is $f(\vec{r}(t))$? Did you use this anywhere when you computed the line integral in (a)? Can you explain why this happened?

   (c) Suppose we want to look at a new curve C, parameterized by $\vec{r}(t) = \left\langle \ln t, \sin(\ln t) \cdot \sqrt{t^3 + 1} \right\rangle$, $1 \le t \le e^{2\pi}$. Find $\displaystyle\int_C \vec{F} \cdot d\vec{r}$.

Some terminology:

- A vector field $\vec{F}$ is called conservative (or a gradient vector field) if it is the gradient of a function f; that is, $\vec{F} = \nabla f$. In this case, f is called a potential function of $\vec{F}$.
- A vector field $\vec{F}$ is called independent of path if $\displaystyle\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$ for any two curves C_1 and C_2 that have the same starting point A and the same ending point B.
- A curve C is called closed (or a closed loop) if it starts and ends at the same point. A vector field $\vec{F}$ has the closed loop property if $\displaystyle\int_C \vec{F} \cdot d\vec{r} = 0$ whenever C is a closed loop.

2. In each part, $\vec{F}$ is a vector field on $\mathbb{R}^2$; what can you conclude from the given information about $\vec{F}$? Is $\vec{F}$ definitely conservative, definitely not conservative, or is there not enough information to tell?

   (a) $\displaystyle\int_C \vec{F} \cdot d\vec{r} = 1$, where C is the unit circle, traversed once counter-clockwise.

   (b) $\displaystyle\int_C \vec{F} \cdot d\vec{r} = 0$, where C is the unit circle, traversed once counter-clockwise.

Now, we will focus on vector fields on $\mathbb{R}^2$.

3. (a) $\vec{F}(x, y) = \langle y^2, x \rangle$ is not a conservative vector field. Why not? (Hint: If $\vec{F}$ was the gradient of a function f, what would f_x and f_y be? How about f_{xy} and f_{yx}?)

   (b) Let's generalize (a). Let $\vec{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$ be a vector field on $\mathbb{R}^2$. If $\vec{F}$ is a conservative vector field, then what must be true about P and Q?

   (c) Using your answer to (b), which of the following vector fields can you be sure are *not* conservative?

      i. $\vec{F}(x, y) = \langle y, -x \rangle$.

      ii. $\vec{F}(x, y) = \langle ye^x, e^x \rangle$.

      iii. $\vec{F}(x, y) = \langle x^2 y, xy^2 \rangle$.

4. The following vector fields $\vec{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$ have the property that $P_y = Q_x$. (You can check this easily.) In each part, is it valid to conclude from this information that $\vec{F}$ is conservative? If so, find a function f such that $\nabla f = \vec{F}$.

(a) $\vec{F}(x,y) = \langle y, x\rangle$.

(b) $\vec{F}(x,y) = \left\langle -\dfrac{y}{x^2+y^2} + x, \dfrac{x}{x^2+y^2} + y \right\rangle$.

(c) $\vec{F}(x,y) = \langle 1 + 2xy, x^2 + 3y^2\rangle$.

5. Let C be the top half of the unit circle, traversed counter-clockwise. For each of the vector fields $\vec{F}$ in #4, what is $\int_C \vec{F} \cdot d\vec{r}$?

Usually, we work with vector fields whose domains are open and connected. (§13.3, Theorem 4 in Stewart only works if the domain of the vector field is open and connected.) These terms are defined on pg. 926 of your book, but here's a basic idea of what they mean.

Intuitively, saying that a domain D is connected means that it consists of just one piece. For example:

- $\mathbb{R}^2$ is connected.

- The domain "all points (x, y) with $y \neq 0$" is not connected. It consists of two separate pieces: the piece with $y > 0$ and the piece with $y < 0$.

Saying that a domain is open basically means that the domain does not include any of its boundary points.[1] Here are some examples:

- The disk $x^2 + y^2 \leq 1$ is not open because its boundary is the circle $x^2 + y^2 = 1$, and this is part of the disk.

- On the other hand, the disk $x^2 + y^2 < 1$ *is* open because its boundary is the circle $x^2 + y^2 = 1$, but that's not included in the disk.

- $\mathbb{R}^2$ has no boundary, and it is an open set. (It doesn't contain any of its boundary points because it has no boundary points.)

[1] This is in contrast to closed, which meant that the domain included all of its boundary points. Although "open" and "closed" sound like opposites, they are not — there are domains (like $\mathbb{R}^2$) that are both open and closed, and there are also domains that are neither open nor closed (like "all points (x, y) with $x > 0$ and $y \geq 0$").

The Fundamental Theorem for Line Integrals; Gradient Vector Fields

1. Let $f(x,y) = \sin x + x^2 y$ and $\vec{F} = \nabla f$, a vector field on $\mathbb{R}^2$. Let C be the curve in $\mathbb{R}^2$ parameterized by $\vec{r}(t) = \langle t, t^2 \rangle$, $0 \le t \le \pi$.

   (a) *Compute the line integral* $\displaystyle\int_C \vec{F} \cdot d\vec{r}$*.*

   Solution. $\vec{F}(x,y) = \nabla f(x,y) = \langle \cos x + 2xy, x^2 \rangle$. To evaluate the line integral, we just use the definition of line integrals:

$$\begin{aligned}
\int_C \vec{F} \cdot d\vec{r} &= \int_0^\pi \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\ dt \\
&= \int_0^\pi \vec{F}(t,t^2) \cdot \langle 1, 2t \rangle\ dt \\
&= \int_0^\pi \langle \cos t + 2t^3, t^2 \rangle \cdot \langle 1, 2t \rangle\ dt \\
&= \int_0^\pi (\cos t + 4t^3)\ dt \\
&= \sin t + t^4 \Big|_{t=0}^{t=\pi} \\
&= \boxed{\pi^4}
\end{aligned}$$

   (b) *What is $f(\vec{r}(t))$? Did you use this anywhere when you computed the line integral in (a)? Can you explain why this happened?*

   Solution. $f(\vec{r}(t)) = \sin t + t^4$. We saw this in the second-to-last step of (a): it was what we got from integrating $\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)$ (so we ended up just plugging the starting and ending values of t into this).

   This can be explained by the Chain Rule, which says that $\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$. In this case, $\nabla f = \vec{F}$, so $\frac{d}{dt} f(\vec{r}(t)) = \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)$. This was exactly the thing we were integrating (with respect to t), and we know by the Fundamental Theorem of Calculus that if we integrate $\frac{d}{dt}$(something) with respect to t, we'll just get that "something" back.

   (c) *Suppose we want to look at a new curve C, parameterized by $\vec{r}(t) = \left\langle \ln t, \sin(\ln t) \cdot \sqrt{t^3+1} \right\rangle$, $1 \le t \le e^{2\pi}$. Find* $\displaystyle\int_C \vec{F} \cdot d\vec{r}$*.*

   Solution. Using what we figured out in (b), we can jump directly to the end of the computation:

$$\begin{aligned}
\int_C \vec{F} \cdot d\vec{r} &= f(\vec{r}(t)) \Big|_{t=1}^{t=e^{2\pi}} \\
&= f\left(\vec{r}\left(e^{2\pi}\right)\right) - f(\vec{r}(1))
\end{aligned}$$

   From the given formula for $\vec{r}(t)$, $\vec{r}\left(e^{2\pi}\right) = \langle 2\pi, 0 \rangle$ and $\vec{r}(1) = \langle 0, 0 \rangle$, so the line integral is $f(2\pi,0) - f(0,0) = \boxed{0}$.

2. In each part, $\vec{F}$ is a vector field on $\mathbb{R}^2$; what can you conclude from the given information about $\vec{F}$? Is $\vec{F}$ definitely conservative, definitely not conservative, or is there not enough information to tell?

(a) $\int_C \vec{F} \cdot d\vec{r} = 1$, *where C is the unit circle, traversed once counter-clockwise.*

Solution. We are given a closed loop C such that $\int_C \vec{F} \cdot d\vec{r} \neq 0$, so $\vec{F}$ does not satisfy the closed loop property. This means that $\vec{F}$ is $\boxed{\text{definitely not conservative}}$.

(b) $\int_C \vec{F} \cdot d\vec{r} = 0$, *where C is the unit circle, traversed once counter-clockwise.*

Solution. There is $\boxed{\text{not enough information}}$ to determine whether $\vec{F}$ is conservative. We see that, for *one* closed loop C, $\int_C \vec{F} \cdot d\vec{r} = 0$, but this doesn't tell us anything about the line integral of $\vec{F}$ along other closed loops.

3. (a) *$\vec{F}(x,y) = \langle y^2, x\rangle$ is not a conservative vector field. Why not? (Hint: If $\vec{F}$ was the gradient of a function f, what would f_x and f_y be? How about f_{xy} and f_{yx}?)*

Solution. Let's pretend that $\vec{F}$ *is* conservative; that means that $\vec{F} = \nabla f$ for some function $f(x,y)$. That is, $\vec{F} = \langle f_x, f_y\rangle$, so $f_x = y^2$ and $f_y = x$. If we differentiate $f_x = y^2$ with respect to y, we end up with $f_{xy} = 2y$. If we differentiate $f_y = x$ with respect to x, we end up with $f_{yx} = 1$. But this is bad because it says that $f_{xy} \neq f_{yx}$ (which we know is false!). So, $\vec{F}$ could not have been conservative.

(b) *Let's generalize (a). Let $\vec{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$ be a vector field on $\mathbb{R}^2$. If $\vec{F}$ is a conservative vector field, then what must be true about P and Q?*

Solution. $\boxed{P_y = Q_x.}$ After all, if $\vec{F}$ is the gradient of a function f, that means $\vec{F} = \nabla f = \langle f_x, f_y\rangle$, so $P = f_x$ and $Q = f_y$. Then, $P_y = f_{xy}$ and $Q_x = f_{yx}$, and these ought to be equal.

(c) Using your answer to (b), which of the following vector fields can you be sure are *not* conservative?

i. $\vec{F}(x,y) = \langle y, -x\rangle$.

Solution. Using the notation from (b), $P(x,y) = y$ and $Q(x,y) = -x$, so $P_y = 1$ and $Q_x = -1$. Since $P_y \neq Q_x$, $\boxed{\text{our answer to (b) tells us that } \vec{F} \text{ is not conservative}}$.

ii. $\vec{F}(x,y) = \langle ye^x, e^x\rangle$.

Solution. Using the notation from (b), $P(x,y) = ye^x$ and $Q(x,y) = e^x$, so $P_y = e^x$ and $Q_x = e^x$. Then, $P_y = Q_x$, and $\boxed{\text{our answer to (b) does not tell us anything}}$.

Note: Using what we learned later in class, we can actually conclude that $\vec{F}$ is conservative, since $P_y = Q_x$ and the domain of $\vec{F}$ is simply connected (the domain is all of $\mathbb{R}^2$).

iii. $\vec{F}(x,y) = \langle x^2y, xy^2\rangle$.

Solution. Using the notation from (b), $P(x,y) = x^2y$ and $Q(x,y) = xy^2$, so $P_y = x^2$ and $Q_x = y^2$. Since $P_y \neq Q_x$, and $\boxed{\text{our answer to (b) tells us that } \vec{F} \text{ is not conservative}}$.

4. The following vector fields $\vec{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$ have the property that $P_y = Q_x$. (You can

check this easily.) In each part, is it valid to conclude from this information that $\vec{F}$ is conservative? If so, find a function f such that $\nabla f = \vec{F}$.

(a) $\vec{F}(x, y) = \langle y, x \rangle$.

Solution. $\boxed{\text{Yes}}$, because the domain of $\vec{F}$ is all of $\mathbb{R}^2$ (that is, there are no places where $\vec{F}(x, y)$ is undefined), and $\mathbb{R}^2$ is simply connected (it has no holes).

We'd like to find a function f such that $\nabla f = \vec{F}$. So, the function f we are looking for has $f_x = y$ and $f_y = x$. If we integrate the equation $f_x = y$ with respect to x, we find that $f = xy + g(y)$ where $g(y)$ can be any function of just y (not involving x).(1) Plugging this into the equation $f_y = x$, we have $x + g'(y) = x$, so $g'(y) = 0$. This tells us that $g(y)$ is just a constant C, so $\boxed{f(x, y) = xy + C \text{ where } C \text{ is any constant}}$. (And indeed, it is easy to check that the gradient of this function is just $\langle y, x \rangle$.)

(b) $\vec{F}(x, y) = \left\langle -\frac{y}{x^2 + y^2} + x, \frac{x}{x^2 + y^2} + y \right\rangle$.

Solution. $\boxed{\text{No}}$, because the domain of $\vec{F}$ is all of $\mathbb{R}^2$ except the point $(0, 0)$ (that is, $\vec{F}$ is not defined at $(0, 0)$, although it is defined everywhere else), and this domain is not simply connected (it has a hole at $(0, 0)$).

(c) $\vec{F}(x, y) = \langle 1 + 2xy, x^2 + 3y^2 \rangle$.

Solution. $\boxed{\text{Yes}}$, because the domain of $\vec{F}$ is all of $\mathbb{R}^2$, which is simply connected.

We'd like to find a function f such that $\nabla f = \vec{F}$. So, we want a function f such that $f_x = 1 + 2xy$ and $f_y = x^2 + 3y^2$. If we integrate the equation $f_x = 1 + 2xy$ with respect to x, we find that $f = x + x^2y + g(y)$ where $g(y)$ can be any function involving just y (and no x). Plugging this into the equation $f_y = x^2 + 3y^2$, we have $x^2 + g'(y) = x^2 + 3y^2$, so $g'(y) = 3y^2$. This tells us that $g(y) = y^3 + C$, so $\boxed{f(x, y) = x + x^2y + y^3 + C \text{ where } C \text{ is any constant}}$. (Again, it is easy to check that the gradient of this function is indeed $\vec{F}$.)

5. *Let C be the top half of the unit circle, traversed counter-clockwise. For each of the vector fields $\vec{F}$ in #4, what is* $\int_C \vec{F} \cdot d\vec{r}$*?*

Solution. Let's look at each one.

(a) We found in #4 that $\vec{F}(x, y) = \langle y, x \rangle$ is the gradient of $f(x, y) = xy$,(2) so we can use the Fundamental Theorem for Line Integrals (essentially the reasoning in #1) to evaluate this line integral. Since the curve C starts at $(1, 0)$ and ends at $(-1, 0)$, $\int_C \vec{F} \cdot d\vec{r}$ is just equal to $f(-1, 0) - f(1, 0) = \boxed{0}$.

(b) We were not able to conclude in #4 that $\vec{F}$ was conservative, so we'd better just compute the line

(1) Why $+g(y)$ instead of $+C$? Basically, we can add on anything whose derivative with respect to x is 0, since that will not change f_x.

(2) Really, we found that it was the gradient of $xy + C$ for any constant C, so using $C = 0$ certainly works.

integral directly. Let's parameterize the curve by $\vec{r}(t) = \langle \cos t, \sin t \rangle$ with $0 \leq t \leq \pi$. Then,

$$\begin{aligned}
\int_C \vec{F} \cdot d\vec{r} &= \int_0^\pi \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\ dt \\
&= \int_0^\pi \vec{F}(\cos t, \sin t) \cdot \langle -\sin t, \cos t \rangle\ dt \\
&= \int_0^\pi \left\langle -\frac{\sin t}{\cos^2 t + \sin^2 t} + \cos t, \frac{\cos t}{\cos^2 t + \sin^2 t} + \sin t \right\rangle \cdot \langle -\sin t, \cos t \rangle\ dt \\
&= \int_0^\pi \langle -\sin t + \cos t, \cos t + \sin t \rangle \cdot \langle -\sin t, \cos t \rangle\ dt \\
&= \int_0^\pi (\sin^2 t - \cancel{\sin t \cos t} + \cos^2 t + \cancel{\sin t \cos t})\ dt \\
&= \int_0^\pi 1\ dt \\
&= \boxed{\pi}
\end{aligned}$$

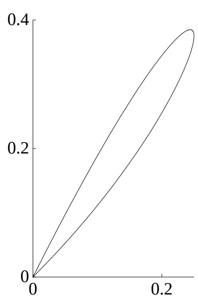
(c) We found in #4 that $\vec{F}(x, y) = \langle 1 + 2xy, x^2 + 3y^2 \rangle$ is the gradient of $f(x, y) = x + x^2 y + y^3$, so we can use the Fundamental Theorem for Line Integrals to evaluate this line integral. The curve C starts at $(1, 0)$ and ends at $(-1, 0)$, so $\int_C \vec{F} \cdot d\vec{r} = f(-1, 0) - f(1, 0) = \boxed{-2}$.

Green's Theorem

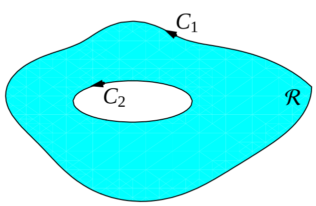
1. Let C be the boundary of the unit square $0 \le x \le 1, 0 \le y \le 1$, oriented counterclockwise, and let $\vec{F}$ be the vector field $\vec{F}(x,y) = \langle e^y + x, x^2 - y\rangle$. Find $\displaystyle\int_C \vec{F} \cdot d\vec{r}$.

2. Let C be the oriented curve consisting of line segments from $(0,0)$ to $(2,3)$ to $(2,0)$ back to $(0,0)$, and let $\vec{F}(x,y) = \langle y^2, x^2\rangle$. Find $\displaystyle\int_C \vec{F} \cdot d\vec{r}$.

3. Find the area of the region enclosed by the parameterized curve $\vec{r}(t) = \langle t - t^2, t - t^3\rangle$, $0 \le t \le 1$.



4. Let $\vec{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$ be any vector field defined on the region $\mathcal{R}$ (in $\mathbb{R}^2$) shown in the picture, and let C_1 and C_2 be the oriented curves shown in the picture. What does Green's Theorem say about $\displaystyle\int_{C_1} \vec{F} \cdot d\vec{r}$, $\displaystyle\int_{C_2} \vec{F} \cdot d\vec{r}$, and $\displaystyle\iint_{\mathcal{R}} (Q_x - P_y)\, dA$?



5. Let $\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle = \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right\rangle$. You can check that $P_y = Q_x$.

(a) What is wrong with the following reasoning? "$P_y = Q_x$, so $\vec{F}$ is conservative."

In the remainder of this problem, you will show that $\vec{F}$ is conservative by showing that $\vec{F}$ satisfies the closed loop property. (That is, if C is any closed curve, then $\int_C \vec{F} \cdot d\vec{r} = 0$.) We observed last time that this seemed like an impossible task; now that we know Green's Theorem, it's much more doable.

(b) Let C be any simple closed curve in $\mathbb{R}^2$ that does *not* enclose the origin, oriented counterclockwise. (A simple curve is a curve that does not cross itself.) Use Green's Theorem to explain why $\int_C \vec{F} \cdot d\vec{r} = 0$.

(c) Let a be a positive constant, and let C be the circle $x^2 + y^2 = a^2$, oriented counterclockwise. Parameterize C (check your parameterization by plugging it into the equation $x^2 + y^2 = a^2$), and use the definition of the line integral to show that $\int_C \vec{F} \cdot d\vec{r} = 0$. (Why doesn't the reasoning from (b) work in this case?)

(d) Let C be any simple closed curve in $\mathbb{R}^2$ that *does* enclose the origin, oriented counterclockwise. Explain why $\int_C \vec{F} \cdot d\vec{r} = 0$. (Hint: Use (c) and #4.)

(e) Is it valid to conclude from the above reasoning that, if $\vec{F}(x, y) = \langle P(x, y), Q(x, y)\rangle$ is a vector field defined everywhere except the origin and $P_y = Q_x$, then $\vec{F}$ is conservative?

6. In this problem, you'll prove Green's Theorem in the case where the region is a rectangle. Let $\vec{F}(x, y) = \langle P(x, y), Q(x, y)\rangle$ be a vector field on the rectangle $\mathcal{R} = [a, b] \times [c, d]$ in $\mathbb{R}^2$.

(a) Show that $\iint_{\mathcal{R}} [Q_x(x, y) - P_y(x, y)]\, dA = \int_c^d [Q(b, y) - Q(a, y)]\, dy - \int_a^b [P(x, d) - P(x, c)]\, dx.$

(b) Let C be the boundary of $\mathcal{R}$, traversed counterclockwise. Show that $\int_C \vec{F} \cdot d\vec{r}$ is also equal to $\int_c^d [Q(b, y) - Q(a, y)]\, dy - \int_a^b [P(x, d) - P(x, c)]\, dx.$

Green's Theorem

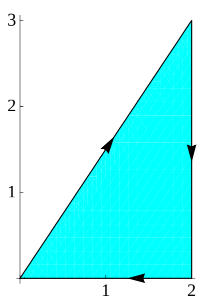
1. *Let C be the boundary of the unit square $0 \le x \le 1, 0 \le y \le 1$, oriented counterclockwise, and let $\vec{F}$ be the vector field $\vec{F}(x,y) = \langle e^y + x, x^2 - y \rangle$. Find $\int_C \vec{F} \cdot d\vec{r}$.*

Solution. Let's write $P(x,y) = e^y + x$ and $Q(x,y) = x^2 - y$, so that $\vec{F} = \langle P, Q \rangle$. Let $\mathcal{R}$ be the region $0 \le x \le 1$, $0 \le y \le 1$. The boundary of $\mathcal{R}$, oriented "correctly" (so that a penguin walking along it keeps $\mathcal{R}$ on his left), is the given curve C. So, Green's Theorem says that $\int_C \vec{F} \cdot d\vec{r} = \iint_{\mathcal{R}} (Q_x - P_y)\, dA = \iint_{\mathcal{R}} (2x - e^y)\, dA$. We compute this by converting it to an iterated integral:

$$\begin{aligned}
\int_C \vec{F} \cdot d\vec{r} &= \iint_{\mathcal{R}} (2x - e^y)\, dA \\
&= \int_0^1 \int_0^1 (2x - e^y)\, dx\, dy \\
&= \int_0^1 \left(x^2 - xe^y \Big|_{x=0}^{x=1} \right) dy \\
&= \int_0^1 (1 - e^y)\, dy \\
&= y - e^y \Big|_{y=0}^{y=1} \\
&= \boxed{2 - e}
\end{aligned}$$

2. *Let C be the oriented curve consisting of line segments from $(0,0)$ to $(2,3)$ to $(2,0)$ back to $(0,0)$, and let $\vec{F}(x,y) = \langle y^2, x^2 \rangle$. Find $\int_C \vec{F} \cdot d\vec{r}$.*

Solution. Here is a picture of the curve C, along with the interior of the triangle, which we'll call $\mathcal{R}$:

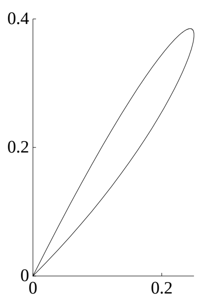


The boundary of $\mathcal{R}$, oriented "correctly" (so that a penguin walking along it keeps $\mathcal{R}$ on his left side), is $-C$ (that is, it's C with the opposite orientation). So, Green's Theorem says that $\int_{-C} \vec{F} \cdot d\vec{r} = \iint_{\mathcal{R}} (Q_x - P_y)\, dA$, where $F = \langle P, Q \rangle$. We are looking for $\int_C \vec{F} \cdot d\vec{r}$, which we know is the negative of

$\int_{-C} \vec{F} \cdot d\vec{r}$. Therefore,

$$\begin{aligned}
\int_C \vec{F} \cdot d\vec{r} &= -\iint_{\mathcal{R}} (Q_x - P_y)\ dA \\
&= -\iint_{\mathcal{R}} (2x - 2y)\ dA \\
&= -\int_0^2 \int_0^{3x/2} (2x - 2y)\ dy\ dx \\
&= -\int_0^2 \left(2xy - y^2 \Big|_{y=0}^{y=3x/2} \right)\ dx \\
&= -\int_0^2 \frac{3}{4}x^2\ dx \\
&= -\left(\frac{1}{4}x^3 \Big|_{x=0}^{x=2} \right) \\
&= \boxed{-2}
\end{aligned}$$

3. *Find the area of the region enclosed by the parameterized curve* $\vec{r}(t) = \langle t - t^2, t - t^3 \rangle$, $0 \le t \le 1$.

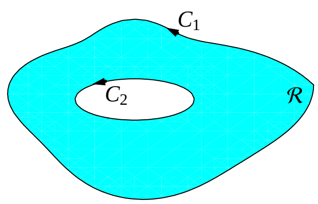


Solution. Let $\mathcal{R}$ be the region in question. We know from #2(a) on the worksheet "Double Integrals" that the area of $\mathcal{R}$ is $\iint_{\mathcal{R}} 1\ dA$. Normally, we would evaluate this by converting it to an iterated integral; in this case, that's quite challenging, so we'll instead use Green's Theorem to evaluate this integral. If we can come up with a vector field $\vec{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$ such that $Q_x - P_y = 1$, then Green's Theorem will say that $\iint_{\mathcal{R}} 1\ dA = \int_C \vec{F} \cdot d\vec{r}$, where C is the boundary of the region, traveled counterclockwise (so that a penguin walking along C keeps $\mathcal{R}$ on his left). One such vector field is $\vec{F}(x, y) = \langle 0, x \rangle$.

We are given a parameterization $\vec{r}(t)$ of the curve, and this parameterization does in fact travel the curve counterclockwise.[1] So,

$$\begin{aligned}
\iint_{\mathcal{R}} 1\ dA &= \int_C \vec{F}\cdot d\vec{r} \\
&= \int_0^1 \vec{F}(\vec{r}(t))\cdot \vec{r}'(t)\ dt \\
&= \int_0^1 \langle 0, t-t^2\rangle \cdot \langle 1-2t, 1-3t^2\rangle\ dt \\
&= \int_0^1 (t-t^2)(1-3t^2)\ dt \\
&= \int_0^1 (t-t^2-3t^3+3t^4)\ dt \\
&= \frac{1}{2}t^2 - \frac{1}{3}t^3 - \frac{3}{4}t^4 + \frac{3}{5}t^5\Big|_{t=0}^{t=1} \\
&= \boxed{\frac{1}{60}}
\end{aligned}$$

4. *Let $\vec{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$ be any vector field defined on the region $\mathcal{R}$ (in $\mathbb{R}^2$) shown in the picture, and let C_1 and C_2 be the oriented curves shown in the picture. What does Green's Theorem say about $\int_{C_1} \vec{F}\cdot d\vec{r}$, $\int_{C_2} \vec{F}\cdot d\vec{r}$, and $\iint_{\mathcal{R}} (Q_x - P_y)\ dA$?*



Solution. The boundary of $\mathcal{R}$ consists of two curves, C_1 and C_2. A penguin walking along C_1 in the indicated direction would indeed keep $\mathcal{R}$ on his left, but a penguin walking along C_2 in the indicated direction would have $\mathcal{R}$ on his right. So, the boundary of $\mathcal{R}$ is really C_1 together with $-C_2$, which means $\boxed{\iint_{\mathcal{R}} (Q_x - P_y)\ dA = \int_{C_1} \vec{F}\cdot d\vec{r} - \int_{C_2} \vec{F}\cdot d\vec{r}}$.

5. Let $\vec{F}(x,y) = \langle P(x,y), Q(x,y)\rangle = \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right\rangle$. You can check that $P_y = Q_x$.

   (a) *What is wrong with the following reasoning? "$P_y = Q_x$, so $\vec{F}$ is conservative."*

   Solution. $\vec{F}$ is not defined at the origin, so its domain is $\mathbb{R}^2$ except the point $(0,0)$. This domain is not simply connected, so we cannot conclude anything from the fact that $P_y = Q_x$.

   (b) *Let C be any simple closed curve in $\mathbb{R}^2$ that does not enclose the origin, oriented counterclockwise.*

[1] This is not completely obvious, but there's an easy way to tell at the end whether the parameterization went the right way — we are looking for an area, so our final answer must be positive.

(A simple curve is a curve that does not cross itself.) Use Green's Theorem to explain why $\int_C \vec{F} \cdot d\vec{r} = 0$.

Solution. Since C does not go around the origin, $\vec{F}$ is defined on the interior $\mathcal{R}$ of C. (The only point where $\vec{F}$ is not defined is the origin, but that's not in $\mathcal{R}$.) Therefore, we can use Green's Theorem, which says $\int_C \vec{F} \cdot d\vec{r} = \iint_{\mathcal{R}} (Q_x - P_y)\ dA$. Since $Q_x - P_y = 0$, this says that $\int_C \vec{F} \cdot d\vec{r} = 0$.

(c) *Let a be a positive constant, and let C be the circle $x^2 + y^2 = a^2$, oriented counterclockwise. Parameterize C (check your parameterization by plugging it into the equation $x^2 + y^2 = a^2$), and use the definition of the line integral to show that $\int_C \vec{F} \cdot d\vec{r} = 0$. (Why doesn't the reasoning from (b) work in this case?)*

Solution. One possible parameterization of C is $\vec{r}(t) = \langle a\cos t, a\sin t\rangle$, $0 \le t \le 2\pi$. Then,

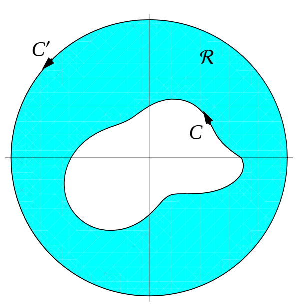
$$\begin{aligned}
\int_C \vec{F} \cdot d\vec{r} &= \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\ dt \\
&= \int_0^{2\pi} \left\langle \frac{a\cos t}{\sqrt{(a\cos t)^2 + (a\sin t)^2}}, \frac{a\sin t}{\sqrt{(a\cos t)^2 + (a\sin t)^2}} \right\rangle \cdot \langle -a\sin t, a\cos t\rangle\ dt \\
&= \int_0^{2\pi} 0\ dt \\
&= 0,
\end{aligned}$$

as we wanted.

We cannot use the reasoning from (b) since $\vec{F}$ is not defined in the whole interior of C (in particular, it's not defined at the origin, which is inside C).

(d) *Let C be any simple closed curve in $\mathbb{R}^2$ that does enclose the origin, oriented counterclockwise. Explain why $\int_C \vec{F} \cdot d\vec{r} = 0$. (Hint: Use (c) and #4.)*

Solution. No matter what C looks like, we can draw a giant circle $x^2 + y^2 = a^2$ around the origin that encloses all of C. Let's orient this giant circle counterclockwise and call it C', and let's have $\mathcal{R}$ be the region between C and C' $\mathcal{R}$:



Notice that $\vec{F}$ is defined on all of $\mathcal{R}$ (because it is defined everywhere except the origin, and $\mathcal{R}$

doesn't include the origin). So, #4 tells us that

$$\iint_{\mathcal{R}} (Q_x - P_y)\ dA = \int_{C'} \vec{F} \cdot d\vec{r} - \int_C \vec{F} \cdot d\vec{r}.$$

We showed in (c) that $\int_{C'} \vec{F} \cdot d\vec{r} = 0$, so this simplifies to

$$\iint_{\mathcal{R}} (Q_x - P_y)\ dA = -\int_C \vec{F} \cdot d\vec{r}.$$

Since $Q_x = P_y$ inside of $\mathcal{R}$, the double integral is really a double integral of 0, so it's equal to 0. Therefore, we conclude that $\int_C \vec{F} \cdot d\vec{r} = 0$ as well.

(e) *Is it valid to conclude from the above reasoning that, if $\vec{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$ is a vector field defined everywhere except the origin and $P_y = Q_x$, then $\vec{F}$ is conservative?*

Solution. No! The calculation in (c) only applied to this particular vector field $\vec{F}(x,y) = \langle P(x,y), Q(x,y)\rangle = \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right\rangle$.

There are vector fields that are defined everywhere except the origin and satisfy $P_y = Q_x$ but are still not conservative; the vector field in #4(b) of the worksheet "The Fundamental Theorem for Line Integrals; Gradient Vector Fields" is an example.

6. In this problem, you'll prove Green's Theorem in the case where the region is a rectangle. Let $\vec{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$ be a vector field on the rectangle $\mathcal{R} = [a,b] \times [c,d]$ in $\mathbb{R}^2$.

(a) *Show that* $\iint_{\mathcal{R}} [Q_x(x,y) - P_y(x,y)]\ dA = \int_c^d [Q(b,y) - Q(a,y)]\ dy - \int_a^b [P(x,d) - P(x,c)]\ dx.$

Solution. Let's first break the given double integral into a difference of two double integrals:

$$\iint_{\mathcal{R}} [Q_x(x,y) - P_y(x,y)]\ dA = \iint_{\mathcal{R}} Q_x(x,y)\ dA - \iint_{\mathcal{R}} P_y(x,y)\ dA.$$

Now, we'll convert the double integrals on the right side to iterated integrals. This is easy, since the region $\mathcal{R}$ is just a rectangle. However, we're going to do the two iterated integrals in different orders: it makes sense to first integrate Q_x with respect to x (since it's a derivative with respect to x) and to first integrate P_y with respect to y:

$$\iint_{\mathcal{R}} [Q_x(x,y) - P_y(x,y)]\ dA = \int_c^d \int_a^b Q_x(x,y)\ dx\ dy - \int_a^b \int_c^d P_y(x,y)\ dy\ dx.$$

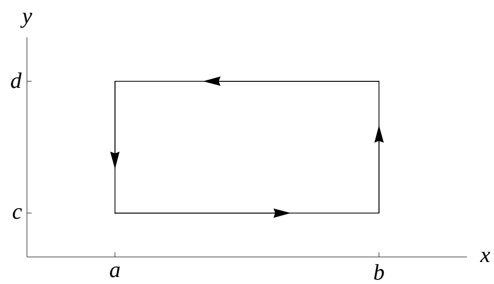
When we integrate Q_x with respect to x, we just get Q; similarly, when we integrate P_y with respect to y, we just get P:

$$\begin{aligned}\iint_{\mathcal{R}} [Q_x(x,y) - P_y(x,y)]\ dA &= \int_c^d \left(Q(x,y)\Big|_{x=a}^{x=b} \right) dy - \int_a^b \left(P(x,y)\Big|_{y=c}^{y=d} \right) dx \\ &= \int_c^d [Q(b,y) - Q(a,y)]\ dy - \int_a^b [P(x,d) - P(x,c)]\ dx,\end{aligned}$$

which is exactly what we were asked to show.

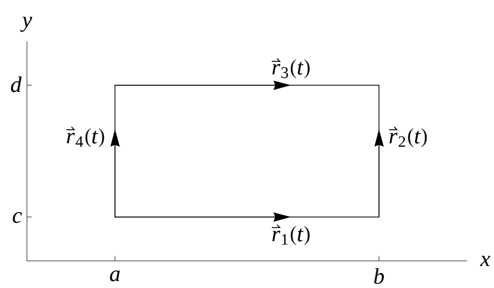
(b) *Let C be the boundary of $\mathcal{R}$, traversed counterclockwise. Show that* $\int_C \vec{F} \cdot d\vec{r}$ *is also equal to* $\int_c^d [Q(b,y) - Q(a,y)]\ dy - \int_a^b [P(x,d) - P(x,c)]\ dx.$

Solution. Here is a picture of C:



As we can see, it's composed of 4 pieces, and we'll parameterize each separately. The bottom piece has $y = c$, so only x varies, and we can parameterize it using $\vec{r}_1(t) = \langle t, c\rangle$ with $a \le t \le b$. The right piece has $x = b$, so only y varies, and we can parameterize it using $\vec{r}_2(t) = \langle b, t\rangle$, $c \le t \le d$.

The top piece has $y = d$, so only x varies, and we'd like to parameterize it using $\vec{r}_3(t) = \langle t, d\rangle$. The slight problem with this is that it goes the wrong direction: as t increases, $\langle t, d\rangle$ goes to the right. This is actually not a problem, as long as we account for it later. So, we'll go ahead and use $\vec{r}_3(t) = \langle t, d\rangle$ with $a \le t \le b$. Similarly, for the left piece, we'll use $\vec{r}_4(t) = \langle a, t\rangle$, $c \le t \le d$. Here's a diagram showing the various things we've parameterized:



As we can see from the two diagrams,

$$\int_C \vec{F} \cdot d\vec{r} = \int_{\vec{r}_1(t)} \vec{F} \cdot d\vec{r} + \int_{\vec{r}_2(t)} \vec{F} \cdot d\vec{r} - \int_{\vec{r}_3(t)} \vec{F} \cdot d\vec{r} - \int_{\vec{r}_4(t)} \vec{F} \cdot d\vec{r}.$$

Plugging the four parameterizations into this, $\int_C \vec{F} \cdot d\vec{r}$ is equal to

$$\int_a^b \vec{F}(t,c) \cdot \langle 1, 0\rangle\ dt + \int_c^d \vec{F}(b,t) \cdot \langle 0, 1\rangle\ dt - \int_a^b \vec{F}(t,d) \cdot \langle 1, 0\rangle\ dt - \int_c^d \vec{F}(a,t) \cdot \langle 0, 1\rangle\ dt.$$

Writing $\vec{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$, we can simplify this to

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b P(t,c)\ dt + \int_c^d Q(b,t)\ dt - \int_a^b P(t,d)\ dt - \int_c^d Q(a,t)\ dt.$$

This is exactly what we were supposed to show, which is more obvious if we rename t to be x in the first and third integrals, rename t to be y in the second and fourth integrals, and rearrange the terms:

$$\begin{aligned}
\int_C \vec{F} \cdot d\vec{r} &= \int_a^b P(x,c)\ dx + \int_c^d Q(b,y)\ dy - \int_a^b P(x,d)\ dx - \int_c^d Q(a,y)\ dy \\
&= \int_c^d Q(b,y)\ dy - \int_c^d Q(a,y)\ dy - \int_a^b P(x,d)\ dx + \int_a^b P(x,c)\ dx \\
&= \int_c^d [Q(b,y) - Q(a,y)]\ dy - \int_a^b [P(x,d) - P(x,c)]\ dx
\end{aligned}$$

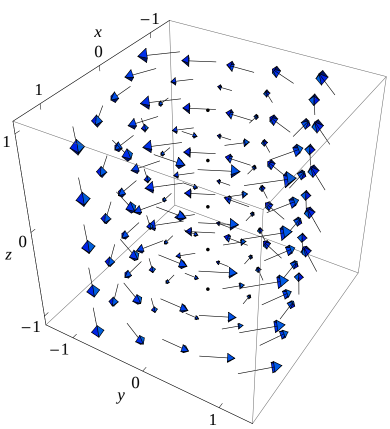
Curl and Divergence

1. (a) $\vec{F}(x,y,z) = \langle y+z, x+2y, x+x^2 \rangle$ is not a conservative vector field. Why not?

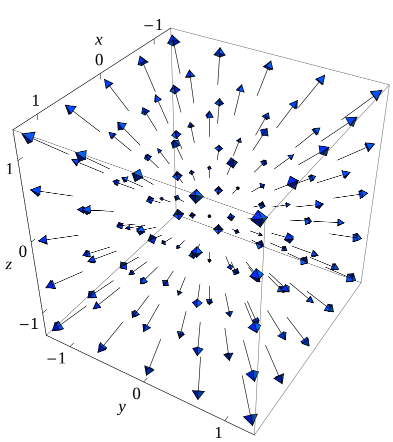
   (b) Let's generalize (a). Let $\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$ be a vector field on $\mathbb{R}^3$. If $\vec{F}$ is a conservative vector field, then what must be true about P, Q, and R?

2. Find the curl and divergence of each vector field.

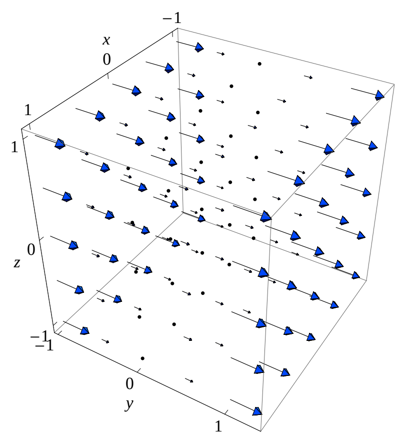
   (a) $\vec{F}(x,y,z) = \langle -y, x, 0 \rangle$



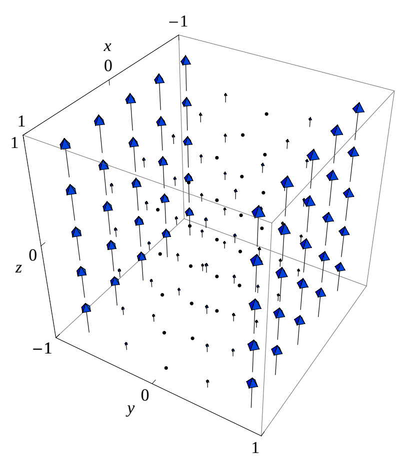
   (b) $\vec{F}(x,y,z) = \langle x, y, z \rangle$



   (c) $\vec{F}(x,y,z) = \langle 0, y^2, 0 \rangle$



   (d) $\vec{F}(x,y,z) = \langle 0, 0, y^2 \rangle$



3. Fill in each blank with either "scalar-valued function of 3 variables" (also sometimes called a "scalar field on $\mathbb{R}^3$") or "vector field on $\mathbb{R}^3$".

   (a) The gradient of a ________________________________ is a ________________________________.

   (b) The curl of a ________________________________ is a ________________________________.

   (c) The divergence of a ________________________________ is a ________________________________.

4. If $\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$ is a vector field on $\mathbb{R}^3$, what is $\operatorname{div}(\operatorname{curl}\vec{F})$ in terms of P, Q, and R? How about $\operatorname{curl}(\operatorname{div}\vec{F})$?

5. Suppose the surface of a very small planet is described by the equation $x^2+y^2+z^2=1$ (where the axes are marked in miles). The population density of green aliens at (x,y,z) is $f(x,y,z) = 10+x+y+z$ green aliens per square mile. How many green aliens live on the planet? (You may leave your answer as an iterated integral.)

Curl and Divergence

1. (a) *$\vec{F}(x,y,z) = \langle y+z, x+2y, x+x^2\rangle$ is not a conservative vector field. Why not?*

   Solution. Let's pretend that $\vec{F}$ *is* conservative. Remember that the definition of conservative is simply that a conservative vector field is one that is the gradient of some function. So, if our vector field $\vec{F}$ is conservative, then that means that $\vec{F} = \nabla f$ some function $f(x,y,z)$. That is, $\vec{F} = \langle f_x, f_y, f_z\rangle$, so $f_x = y+z$, $f_y = x+2y$, and $f_z = x+x^2$. Differentiating the first equation with respect to z, $f_{xz} = 1$. Differentiating the third equation with respect to x, $f_{zx} = 1+2x$. But this is bad because it says that $f_{xz} \neq f_{zx}$ (which we know is false!). So, $\vec{F}$ could not have been conservative.

   (b) *Let's generalize (a). Let $\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$ be a vector field on $\mathbb{R}^3$. If $\vec{F}$ is a conservative vector field, then what must be true about P, Q, and R?*

   Solution. $P_y = Q_x$, $P_z = R_x$, and $Q_z = R_y$. In other words, $\operatorname{curl}\vec{F} = \langle 0,0,0\rangle$.

2. Find the curl and divergence of each vector field.

   (a) $\vec{F}(x,y,z) = \langle -y, x, 0\rangle$

   Solution. $\operatorname{curl}(\vec{F})(x,y,z) = \langle 0,0,2\rangle$ and $\operatorname{div}(\vec{F})(x,y,z) = 0$.

   (b) $\vec{F}(x,y,z) = \langle x, y, z\rangle$

   Solution. $\operatorname{curl}(\vec{F})(x,y,z) = \langle 0,0,0\rangle$ and $\operatorname{div}(\vec{F})(x,y,z) = 3$.

   (c) $\vec{F}(x,y,z) = \langle 0, y^2, 0\rangle$

   Solution. $\operatorname{curl}(\vec{F})(x,y,z) = \langle 0,0,0\rangle$ and $\operatorname{div}(\vec{F})(x,y,z) = 2y$.

   (d) $\vec{F}(x,y,z) = \langle 0, 0, y^2\rangle$

   Solution. $\operatorname{curl}(\vec{F})(x,y,z) = \langle 2y,0,0\rangle$ and $\operatorname{div}(\vec{F})(x,y,z) = 0$.

3. Fill in each blank with either "scalar-valued function of 3 variables" (also sometimes called a "scalar field on $\mathbb{R}^3$") or "vector field on $\mathbb{R}^3$".

   (a) The gradient of a **scalar-valued function of 3 variables** is a **vector field on $\mathbb{R}^3$**.

   (b) The curl of a **vector field on $\mathbb{R}^3$** is a **vector field on $\mathbb{R}^3$**.

   (c) The divergence of a **vector field on $\mathbb{R}^3$** is a **scalar-valued function of 3 variables**.

4. *If $\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$ is a vector field on $\mathbb{R}^3$, what is $\operatorname{div}(\operatorname{curl}\vec{F})$ in terms of P, Q, and R? How about $\operatorname{curl}(\operatorname{div}\vec{F})$?*

Solution. We know that $\operatorname{curl}\vec{F} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y\rangle$, so

$$\begin{aligned}
\operatorname{div}(\operatorname{curl}\vec{F}) &= (R_y - Q_z)_x + (P_z - R_x)_y + (Q_x - P_y)_z \\
&= R_{yx} - Q_{zx} + P_{zy} - R_{xy} + Q_{xz} - P_{yz} \\
&= (R_{yx} - R_{xy}) + (Q_{xz} - Q_{zx}) + (P_{zy} - P_{yz}) \\
&= 0
\end{aligned}$$

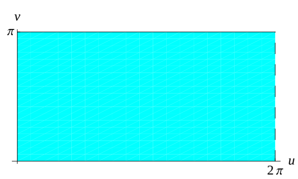
by Clairaut's Theorem.

On the other hand, $\operatorname{curl}(\operatorname{div}\vec{F})$ does not make any sense: by #3, $\operatorname{div}\vec{F}$ is a scalar-valued function of 3 variables, but we can only take the curl of a vector field.

5. *Suppose the surface of a very small planet is described by the equation* $x^2 + y^2 + z^2 = 1$ *(where the axes are marked in miles). The population density of green aliens at* (x, y, z) *is* $f(x, y, z) = 10 + x + y + z$ *green aliens per square mile. How many green aliens live on the planet? (You may leave your answer as an iterated integral.)*

   Solution. We are being asked to find the scalar surface integral $\iint_S f(x, y, z)\ dS$, where S is the surface $x^2 + y^2 + z^2 = 1$. To do this, we must first parameterize S.

   We did this already in #4(c) on the worksheet "Applications of Double Integrals: Center of Mass and Surface Area" (when we were finding the surface area of this sphere). There, we came up with the parameterization $\vec{r}(u, v) = \langle \sin v \cos u, \sin v \sin u, \cos v\rangle$ with the possible (u, v) values being described by $0 \le u < 2\pi$, $0 \le v \le \pi$. This described a rectangle $\mathcal{R}$ in the uv-plane:



   Now that we have parameterized the surface, we use the definition

$$\iint_S f(x, y, z)\ dS = \iint_{\mathcal{R}} f(\vec{r}(u, v))|\vec{r}_u \times \vec{r}_v|\ dA$$

   to compute the surface integral. Let's first figure out the integrand. Plugging $\vec{r}(u, v)$ into f gives $f(\vec{r}(u, v)) = f(\sin v \cos u, \sin v \sin u, \cos v) = 10 + \sin v \cos u + \sin v \sin u + \cos v$. We found in #4(c) on the worksheet "Applications of Double Integrals: Center of Mass and Surface Area" that $|\vec{r}_u \times \vec{r}_v| = \sin v$ for this parameterization. So,

$$\iint_S f(x, y, z)\ dS = \iint_{\mathcal{R}} (10 + \sin v \cos u + \sin v \sin u + \cos v) \sin v\ dA.$$

   Remember that the right side is a double integral over the region $\mathcal{R}$. We are asked to convert this to an iterated integral. Since $\mathcal{R}$ is a rectangle, this is straightforward:

$$\iint_S f(x, y, z)\ dS = \boxed{\int_0^{\pi} \int_0^{2\pi} (10 + \sin v \cos u + \sin v \sin u + \cos v) \sin v\ du\ dv}.$$

Flux Integrals

The pictures for problems #1 - #4 are on the last page.

1. Let's orient each of the three pictured surfaces so that the light side is considered to be the "positive" side. Decide whether each of the following flux integrals is positive, negative, or zero. ($\vec{F}$ and $\vec{G}$ are the pictured vector fields.)

   (a) $\iint_{\mathcal{S}_1} \vec{F} \cdot d\vec{S}$.

   (b) $\iint_{\mathcal{S}_2} \vec{F} \cdot d\vec{S}$.

   (c) $\iint_{\mathcal{S}_3} \vec{F} \cdot d\vec{S}$.

   (d) $\iint_{\mathcal{S}_1} \vec{G} \cdot d\vec{S}$.

   (e) $\iint_{\mathcal{S}_2} \vec{G} \cdot d\vec{S}$.

   (f) $\iint_{\mathcal{S}_3} \vec{G} \cdot d\vec{S}$.

2. In each part, you are given an orientation of one of the pictured surfaces. Decide whether this orientation means that the light side or dark side of the surface is the "positive" side, or if the description just doesn't make sense.

   (a) $\mathcal{S}_1$, oriented with normals pointing upward.

   (b) $\mathcal{S}_2$, oriented with normals pointing upward.

   (c) $\mathcal{S}_2$, oriented with normals pointing toward the y-axis.

   (d) $\mathcal{S}_3$, oriented with normals pointing outward.

   (e) $\mathcal{S}_3$, oriented with normals pointing toward the origin.

3. In each part, you are given a parameterization of one of the three pictured surfaces. Decide whether the orientation induced by the parameterization has the light side or dark side of the surface as the "positive" side.

   (a) For $\mathcal{S}_1$, $\vec{r}(u, v) = \langle u, v, \sqrt{u^2 + v^2} \rangle$ with $u^2 + v^2 < 1$.

(b) For $\mathcal{S}_1$, $\vec{r}(u,v) = \langle u\cos v, u\sin v, u\rangle$ with $0 \le u < 1$ and $0 \le v < 2\pi$.

(c) For $\mathcal{S}_2$, $\vec{r}(u,v) = \langle \cos v, u, \sin v\rangle$ with $-1 < u < 1$ and $0 \le v < 2\pi$.

(d) For $\mathcal{S}_3$, $\vec{r}(u,v) = \langle \sin v\cos u, \sin v\sin u, \cos v\rangle$ with $0 \le u < 2\pi$ and $0 \le v \le \pi$.

4. Compute the following flux integrals (remember that parameterizations of the surfaces are given in #3). Do the signs of your answers agree with your answers to #1?

(a) $\displaystyle\iint_{\mathcal{S}_1} \vec{F} \cdot d\vec{S}$, where $\mathcal{S}_1$ is oriented with normals pointing upward. ($\vec{F}(x,y,z) = \langle 0,0,-z\rangle$, as before.)

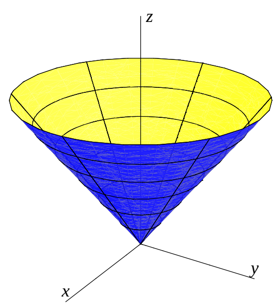
(b) $\displaystyle\iint_{\mathcal{S}_2} \vec{G} \cdot d\vec{S}$, where $\mathcal{S}_2$ is oriented with normals pointing toward the y-axis. ($\vec{G}(x,y,z) = \langle 0,y,0\rangle$, as before.)

(c) $\iint_{\mathcal{S}_3} \vec{F} \cdot d\vec{S}$, where $\mathcal{S}_3$ is oriented with normals pointing outward. ($\vec{F}(x, y, z) = \langle 0, 0, -z \rangle$, as before.)

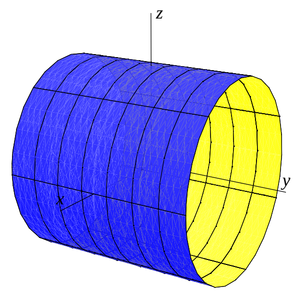
5. Let $\mathcal{S}$ be the portion of the surface $3x - 3y + z = 12$ lying inside the cylinder $x^2 + y^2 = 1$, oriented with normals pointing upward. Let $\vec{F}(x, y, z) = \langle -x^2, 0, -3y^2 \rangle$. Evaluate the flux integral $\iint_{\mathcal{S}} \vec{F} \cdot d\vec{S}$.

These are the surfaces for problems #1 - #4. Each is colored so that one side of the surface is light and the other side is dark.

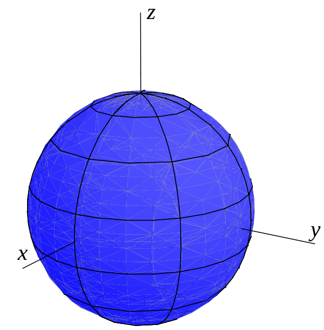
$\mathcal{S}_1$ is the portion of the cone $z = \sqrt{x^2 + y^2}$ under the plane $z = 1$.



$\mathcal{S}_2$ is the portion of the cylinder $x^2 + z^2 = 1$ between the planes $y = -1$ and $y = 1$.

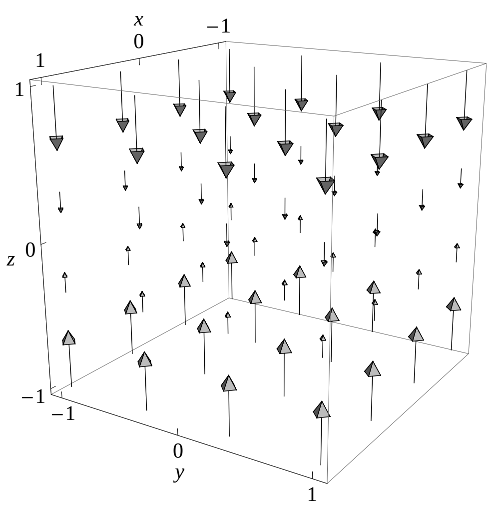


$\mathcal{S}_3$ is the unit sphere $x^2 + y^2 + z^2 = 1$.

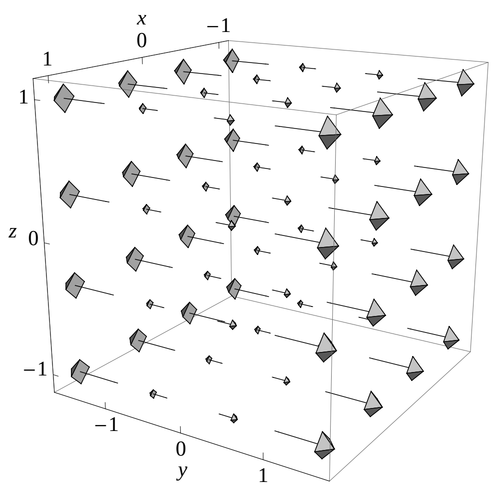


These are the vector fields $\vec{F}$ and $\vec{G}$ for problems #1 - #4. (Note that the origin is located in the middle of each box.)

$\vec{F}(x, y, z) = \langle 0, 0, -z \rangle$



$\vec{G}(x, y, z) = \langle 0, y, 0 \rangle$



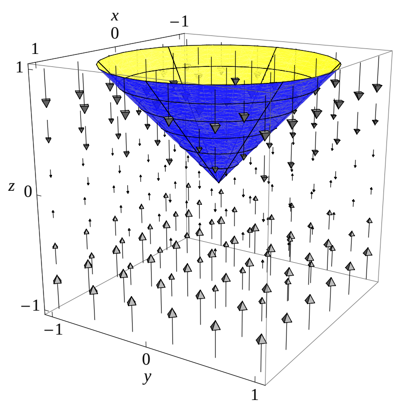
Flux Integrals

The pictures for problems #1 - #4 are on the last page.

1. Let's orient each of the three pictured surfaces so that the light side is considered to be the "positive" side. Decide whether each of the following flux integrals is positive, negative, or zero. ($\vec{F}$ and $\vec{G}$ are the pictured vector fields.)

   (a) $\iint_{\mathcal{S}_1} \vec{F} \cdot d\vec{S}$.

   Solution. We want to visualize the surface together with the vector field. Here's a picture of exactly that:

   As we can see, vectors in the vector field $\vec{F}$ that go through the surface $\mathcal{S}_1$ all go from the yellow side to the blue side. (We only care about the vectors that actually go through the surface; so, for instance, we can completely ignore the vectors in the bottom half of the picture since they don't go through the surface.) Since the surface is oriented so that the yellow side is considered to be the "positive" side, this means all of the vectors are going from the "positive" side to the "negative" side, so the flux is $\boxed{\text{negative}}$.

   (b) $\iint_{\mathcal{S}_2} \vec{F} \cdot d\vec{S}$.

   Solution. Vectors in the vector field $\vec{F}$ that go through the surface $\mathcal{S}_2$ go from the blue side to the yellow side. Since the surface is oriented so that the yellow side is considered to be the "positive" side, this means all of the vectors are going from the "negative" side to the "positive" side, so the flux is $\boxed{\text{positive}}$.

   (c) $\iint_{\mathcal{S}_3} \vec{F} \cdot d\vec{S}$.

   Solution. Vectors in the vector field $\vec{F}$ that go through the surface $\mathcal{S}_3$ go from the blue side to the yellow side. Since the surface is oriented so that the yellow side is considered to be the "positive" side, this means all of the vectors are going from the "negative" side to the "positive" side, so the flux is $\boxed{\text{positive}}$.

(d) $\iint_{\mathcal{S}_1} \vec{G} \cdot d\vec{S}$.

Solution. Vectors in the vector field $\vec{G}$ that go through the surface $\mathcal{S}_1$ go from the yellow side to the blue side. Since the surface is oriented so that the yellow side is considered to be the "positive" side, this means all of the vectors are going from the "positive" side to the "negative" side, so the flux is $\boxed{\text{negative}}$.

(e) $\iint_{\mathcal{S}_2} \vec{G} \cdot d\vec{S}$.

Solution. None of the vectors in the vector field $\vec{G}$ actually go *through* the surface $\mathcal{S}_2$, so the flux is $\boxed{\text{zero}}$.

(f) $\iint_{\mathcal{S}_3} \vec{G} \cdot d\vec{S}$.

Solution. Vectors in the vector field $\vec{G}$ that go through the surface $\mathcal{S}_3$ go from the yellow side to the blue side. Since the surface is oriented so that the yellow side is considered to be the "positive" side, this means all of the vectors are going from the "positive" side to the "negative" side, so the flux is $\boxed{\text{negative}}$.

2. In each part, you are given an orientation of one of the pictured surfaces. Decide whether this orientation means that the light side or dark side of the surface is the "positive" side, or if the description just doesn't make sense.

(a) $\mathcal{S}_1$, *oriented with normals pointing upward.*

Solution. In order for the normals to be pointing upward, they must be sticking out of the yellow (light) side of the surface. (If they stuck out of the blue side, they would point downward.) So, the $\boxed{\text{yellow (light) side is "positive"}}$.

(b) $\mathcal{S}_2$, *oriented with normals pointing upward.*

Solution. This $\boxed{\text{doesn't make sense}}$. If the normals at the top of the cylinder point up, then the normals at the bottom must point down. There's no way for *all* of the normal vectors to point upward.

(c) $\mathcal{S}_2$, *oriented with normals pointing toward the y-axis.*

Solution. This means that the $\boxed{\text{yellow (light) side is "positive"}}$.

(d) $\mathcal{S}_3$, *oriented with normals pointing outward.*

Solution. This means that the $\boxed{\text{blue (dark) side is "positive"}}$.

(e) $\mathcal{S}_3$, *oriented with normals pointing toward the origin.*

Solution. This means that the $\boxed{\text{yellow (light) side is "positive"}}$.

3. In each part, you are given a parameterization of one of the three pictured surfaces. Decide whether the orientation induced by the parameterization has the light side or dark side of the surface as the

"positive" side.

(a) *For* $\mathcal{S}_1$, $\vec{r}(u,v) = \langle u, v, \sqrt{u^2+v^2}\rangle$ *with* $u^2+v^2 < 1$.

Solution. To figure out what the normals look like, we'll compute $\vec{r}_u \times \vec{r}_v$:

$$\begin{aligned} \vec{r}_u &= \left\langle 1, 0, \frac{u}{\sqrt{u^2+v^2}} \right\rangle \\ \vec{r}_v &= \left\langle 0, 1, \frac{v}{\sqrt{u^2+v^2}} \right\rangle \\ \vec{r}_u \times \vec{r}_v &= \left\langle -\frac{u}{\sqrt{u^2+v^2}}, -\frac{v}{\sqrt{u^2+v^2}}, 1 \right\rangle \end{aligned}$$

Once we've computed $\vec{r}_u \times \vec{r}_v$, it's not always easy to tell what the resulting orientation looks like. In this particular case, notice that the last component of $\vec{r}_u \times \vec{r}_v$ is just 1, which is of course always positive. This means that all of the normal vectors $\vec{r}_u \times \vec{r}_v$ point upwards. Looking at the picture of the surface $\mathcal{S}_1$, this means that the yellow (light) side is "positive". (Alternatively, we could look back at #2(a), where we looked at this surface with upward normals.)

(b) *For* $\mathcal{S}_1$, $\vec{r}(u,v) = \langle u\cos v, u\sin v, u\rangle$ *with* $0 \le u < 1$ *and* $0 \le v < 2\pi$.

Solution. We first compute $\vec{r}_u \times \vec{r}_v$:

$$\begin{aligned} \vec{r}_u &= \langle \cos v, \sin v, 1\rangle \\ \vec{r}_v &= \langle -u\sin v, u\cos v, 0\rangle \\ \vec{r}_u \times \vec{r}_v &= \langle -u\cos v, -u\sin v, u\rangle \end{aligned}$$

Notice that the last component of $\vec{r}_u \times \vec{r}_v$ is just u, and our parameterization has $0 \le u < 1$. So, the normal vectors $\vec{r}_u \times \vec{r}_v$ point upwards. Looking at the picture of the surface $\mathcal{S}_1$, this means that the yellow (light) side is "positive".

(c) *For* $\mathcal{S}_2$, $\vec{r}(u,v) = \langle \cos v, u, \sin v\rangle$ *with* $-1 < u < 1$ *and* $0 \le v < 2\pi$.

Solution. We first compute $\vec{r}_u \times \vec{r}_v$:

$$\begin{aligned} \vec{r}_u &= \langle 0, 1, 0\rangle \\ \vec{r}_v &= \langle -\sin v, 0, \cos v\rangle \\ \vec{r}_u \times \vec{r}_v &= \langle \cos v, 0, \sin v\rangle \end{aligned}$$

Here, it's not totally obvious what's going on with the normal vectors. At the point $\vec{r}(u,v) = (\cos v, u, \sin v)$, the normal vector points in the direction $\langle \cos v, 0, \sin v\rangle$. One way to figure out what's going on is just to find the normal vector at a particular point that's easy to visualize. For instance, let's try to find the normal vector at the point $(0,0,1)$. To do this, we need to first find the values of u and v corresponding to this point. That is, we have to solve $\vec{r}(u,v) = \langle 0,0,1\rangle$ for u and v; in this case, $\vec{r}(u,v) = \langle 0,0,1\rangle$ is the same as saying $\cos v = 0$, $u = 0$, and $\sin v = 1$. Therefore, $u = 0$ and $v = \frac{\pi}{2}$. When $u = 0$ and $v = \frac{\pi}{2}$, the normal vector $\vec{r}_u \times \vec{r}_v$ is $\langle 0,0,1\rangle$. This tells us that the normal vector at the point $(0,0,1)$ on top of the cylinder is just $\langle 0,0,1\rangle$, which points upward. Looking at the picture of $\mathcal{S}_2$, this means that the blue (dark) side is "positive".

(d) *For* $\mathcal{S}_3$, $\vec{r}(u,v) = \langle \sin v\cos u, \sin v\sin u, \cos v\rangle$ *with* $0 \le u < 2\pi$ *and* $0 \le v \le \pi$.

Solution. We first compute $\vec{r}_u \times \vec{r}_v$:

$$\begin{aligned}\vec{r}_u &= \langle -\sin v \sin u, \sin v \cos u, 0\rangle \\ \vec{r}_v &= \langle \cos v \cos u, \cos v \sin u, -\sin v\rangle \\ \vec{r}_u \times \vec{r}_v &= \langle -\sin^2 v \cos u, -\sin^2 v \sin u, -\sin v \cos v\rangle\end{aligned}$$

Since this looks a bit complicated, let's try to just figure out the normal vector at a point we can understand easily, like the point $(1, 0, 0)$.(1) To figure out the u and v values corresponding to this point, we solve $\vec{r}(u, v) = \langle 1, 0, 0\rangle$ for u and v. This gives three equations:

$$\begin{aligned}\sin v \cos u &= 1 \\ \sin v \sin u &= 0 \\ \cos v &= 0\end{aligned}$$

By the third equation (and the fact that $0 \le v \le \pi$), we know that $v = \frac{\pi}{2}$. Then, $\sin v = 1$, so the first two equations say that $\cos u = 1$ and $\sin u = 0$, which means $u = 0$. So, the point $(1, 0, 0)$ corresponds to $u = 0$, $v = \frac{\pi}{2}$. With these two values, $\vec{r}_u \times \vec{r}_v = \langle -1, 0, 0\rangle$. That is, the normal vector at $(1, 0, 0)$ points toward the origin. From our picture, this means that the **yellow (light) side is "positive"**.

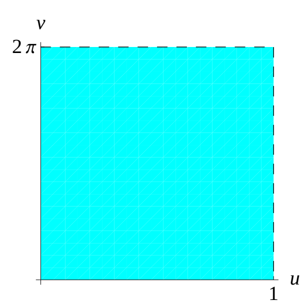
4. Compute the following flux integrals (remember that parameterizations of the surfaces are given in #3). Do the signs of your answers agree with your answers to #1?

(a) $\iint_{\mathcal{S}_1} \vec{F} \cdot d\vec{S}$, *where* $\mathcal{S}_1$ *is oriented with normals pointing upward.* *(*$\vec{F}(x, y, z) = \langle 0, 0, -z\rangle$*, as before.)*

Solution. I like to organize my thinking into a few steps.

- **Step 1 - Parameterize the surface, and figure out the region in the uv-plane describing the possible parameter values.**

  We were given this information in #3; actually, #3(a) and #3(b) gave us two different parameterizations. Let's use the parameterization from #3(b), $\vec{r}(u, v) = \langle u \cos v, u \sin v, u\rangle$ with $0 \le u < 1$, $0 \le v < 2\pi$.(2) The region $\mathcal{R}$ in the uv-plane described by $0 \le u < 1$, $0 \le v < 2\pi$ is a rectangle:



(1)If you try the point $(0, 0, 1)$ in this example, you'll find that the normal vector is $\langle 0, 0, 0\rangle$. This happens occasionally, and it really doesn't give you any useful information. Just pick another point and try again.

(2)Why did I choose this one instead of the parameterization from #3(a)? Well, in the parameterization from #3(a), the region in the uv-plane describing the possible parameters is a disk; in the parameterization from #3(b), the region in the uv-plane describing the possible parameters is a rectangle, which seems like an easier region of integration to deal with.

- **Step 2 - Decide whether the orientation given by the parameterization matches the desired orientation.**

  In #3(b), we had already figured out that this parameterization had the yellow side as the "positive" side.

  The problem asks us to orient $\mathcal{S}_1$ with its normals pointing upward, which also means that the yellow side should be the "positive" side.

  So, the orientation described by the parameterization does match the desired orientation.

- **Step 3 - Compute!**

  Since the orientation described by our parameterization matches the orientation we want, we know the flux integral is

$$\iint_{\mathcal{S}_1} \vec{F} \cdot d\vec{S} = \iint_{\mathcal{R}} \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\ dA.$$

  To find $\vec{F}(\vec{r}(u,v))$, we just plug our parameterization into $\vec{F}(x,y,z) = \langle 0, 0, -z\rangle$, which gives $\vec{F}(\vec{r}(u,v)) = \langle 0, 0, -u\rangle$. In #3(b), we found $\vec{r}_u \times \vec{r}_v = \langle -u\cos v, -u\sin v, u\rangle$. So, we have

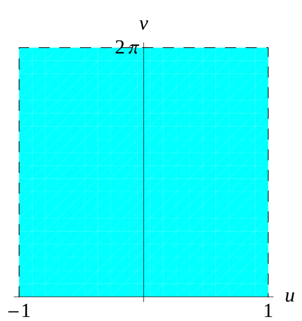
$$\begin{aligned}\iint_{\mathcal{S}_1} \vec{F} \cdot d\vec{S} &= \iint_{\mathcal{R}} \langle 0, 0, -u\rangle \cdot \langle -u\cos v, -u\sin v, u\rangle\ dA \\ &= \iint_{\mathcal{R}} -u^2\ dA \\ &= \int_0^{2\pi} \int_0^1 -u^2\ du\ dv \\ &= \boxed{-\frac{2\pi}{3}}\end{aligned}$$

Note that the sign matches our guess from #1(a); we had said that, if the yellow side was considered the "positive" side (which it is, according to Step 1), then the flux should be negative.

(b) $\iint_{\mathcal{S}_2} \vec{G} \cdot d\vec{S}$, *where $\mathcal{S}_2$ is oriented with normals pointing toward the y-axis. ($\vec{G}(x,y,z) = \langle 0, y, 0\rangle$, as before.)*

Solution. Let's follow the same three steps as in (a):

1. From #3(c), we have the parameterization $\vec{r}(u,v) = \langle \cos v, u, \sin v\rangle$ with $-1 < u < 1$ and $0 \le v < 2\pi$. The region $\mathcal{R}$ in the uv-plane described by these inequalities is a rectangle:



2. In #3(c), we decided that this parameterization had the blue side as the "positive" side, which does *not* match the described orientation (the described orientation has the yellow side as the "positive" side).

3. Since the orientation described by our parameterization does not match the orientation we want, we know the flux integral is

$$\iint_{\mathcal{S}_2} \vec{G} \cdot d\vec{S} = -\iint_{\mathcal{R}} \vec{G}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\ dA.$$

We already found in #3(c) that $\vec{r}_u \times \vec{r}_v = \langle \cos v, 0, \sin v \rangle$, so

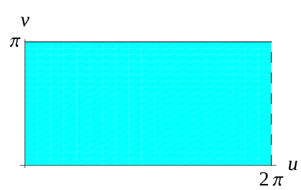
$$\begin{aligned} \iint_{\mathcal{S}_2} \vec{G} \cdot d\vec{S} &= -\iint_{\mathcal{R}} \langle 0, u, 0 \rangle \cdot \langle \cos v, 0, \sin v \rangle\ dA \\ &= -\iint_{\mathcal{R}} 0\ dA \\ &= \boxed{0} \end{aligned}$$

Note that this matches our guess from #1(e), where we guessed that the flux would be zero.

(c) $\iint_{\mathcal{S}_3} \vec{F} \cdot d\vec{S}$, *where* $\mathcal{S}_3$ *is oriented with normals pointing outward.* ($\vec{F}(x,y,z) = \langle 0, 0, -z \rangle$, *as before.)*

Solution. Let's follow the same three steps as in (a):

1. From #3(d), we have the parameterization $\vec{r}(u,v) = \langle \sin v \cos u, \sin v \sin u, \cos v \rangle$ with $0 \le u < 2\pi$ and $0 \le v \le \pi$. The region $\mathcal{R}$ in the uv-plane described by these inequalities is a rectangle:



2. In #3(d), we decided that this parameterization had the yellow side as the "positive" side, which does *not* match the described orientation (the described orientation has the blue side as the "positive" side).

3. Since the orientation described by our parameterization does not match the orientation we want, we know the flux integral is

$$\iint_{\mathcal{S}_3} \vec{F} \cdot d\vec{S} = -\iint_{\mathcal{R}} \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\ dA.$$

We already found in #3(d) that $\vec{r}_u \times \vec{r}_v = \langle -\sin^2 v \cos u, -\sin^2 v \sin u, -\sin v \cos v\rangle$, so

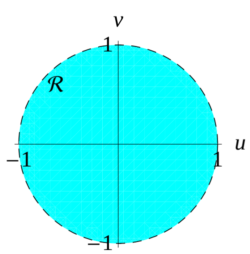
$$\begin{aligned}
\iint_{\mathcal{S}_3} \vec{F} \cdot d\vec{S} &= -\iint_{\mathcal{R}} \langle 0, 0, -\cos v\rangle \cdot \langle -\sin^2 v \cos u, -\sin^2 v \sin u, -\sin v \cos v\rangle \, dA \\
&= -\iint_{\mathcal{R}} \sin v \cos^2 v \, dA \\
&= -\int_0^{\pi} \int_0^{2\pi} \sin v \cos^2 v \, du \, dv \\
&= -\int_0^{\pi} 2\pi \sin v \cos^2 v \, dv \\
&= \frac{2\pi}{3} \cos^3 v \Big|_{v=0}^{v=\pi} \\
&= \boxed{-\frac{4\pi}{3}}
\end{aligned}$$

Note that the sign matches our answer from #1(c): we had decided that, if the yellow side of the sphere was the "positive" side, then the flux would be positive. This means that, if the blue side of the sphere is "positive" (as it is in this problem), then the flux should be negative.

5. *Let $\mathcal{S}$ be the portion of the surface $3x - 3y + z = 12$ lying inside the cylinder $x^2 + y^2 = 1$, oriented with normals pointing upward. Let $\vec{F}(x, y, z) = \langle -x^2, 0, -3y^2\rangle$. Evaluate the flux integral $\iint_{\mathcal{S}} \vec{F} \cdot d\vec{S}$.*

Solution. Let's follow the same three steps as in #4(a):

1. First, we need to parameterize the surface. If we rewrite the equation $3x - 3y + z = 12$ as $z = 12 - 3x + 3y$, then we see that we can parameterize it by $\vec{r}(u, v) = \langle u, v, 12 - 3u + 3v\rangle$. We want only the portion with $x^2 + y^2 < 1$. In terms of our parameters, this says $u^2 + v^2 < 1$, so the region $\mathcal{R}$ in the uv-plane describing the possible parameter values is the disk $u^2 + v^2 < 1$:



2. Next, we need to see what orientation this parameterization describes, as we did in #3. First, we compute $\vec{r}_u \times \vec{r}_v$:

$$\begin{aligned}
\vec{r}_u &= \langle 1, 0, -3\rangle \\
\vec{r}_v &= \langle 0, 1, 3\rangle \\
\vec{r}_u \times \vec{r}_v &= \langle 3, -3, 1\rangle
\end{aligned}$$

This always points upward, so this matches the orientation we want (after all, we are told that the normals should point upward).

3. Now, we just compute. Since the orientation given by $\vec{r}(u, v)$ matches the orientation we want,

we know the flux integral is

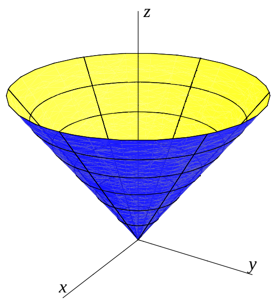
$$\begin{aligned}\iint_{\mathcal{S}} \vec{F} \cdot d\vec{S} &= \iint_{\mathcal{R}} \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\ dA \\ &= \iint_{\mathcal{R}} \langle -u^2, 0, -3v^2 \rangle \cdot \langle 3, -3, 1 \rangle\ dA \\ &= \iint_{\mathcal{R}} (-3u^2 - 3v^2)\ dA\end{aligned}$$

Since $\mathcal{R}$ is a disk, this integral will be easier to do in polar coordinates. In polar coordinates (thinking of u and v as x and y), the region is $0 \le r < 1$, $0 \le \theta < 2\pi$, so the integral is

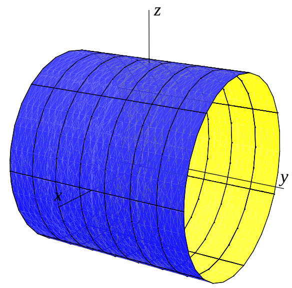
$$\begin{aligned}\int_0^{2\pi} \int_0^1 -3r^2 \cdot r\ dr\ d\theta &= \int_0^{2\pi} \left(-\frac{3}{4} r^4 \Big|_{r=0}^{r=1} \right) d\theta \\ &= \int_0^{2\pi} -\frac{3}{4}\ d\theta \\ &= \boxed{-\frac{3}{2}\pi}\end{aligned}$$

These are the surfaces for problems #1 - #4. Each is colored so that one side of the surface is light and the other side is dark.

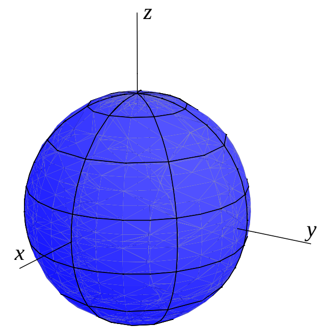
$\mathcal{S}_1$ is the portion of the cone $z = \sqrt{x^2 + y^2}$ under the plane $z = 1$.



$\mathcal{S}_2$ is the portion of the cylinder $x^2 + z^2 = 1$ between the planes $y = -1$ and $y = 1$.

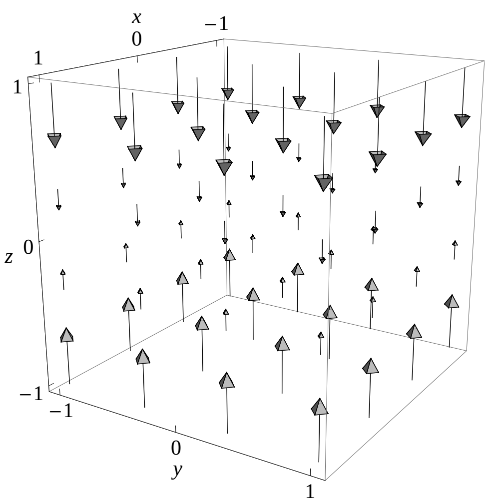


$\mathcal{S}_3$ is the unit sphere $x^2 + y^2 + z^2 = 1$.

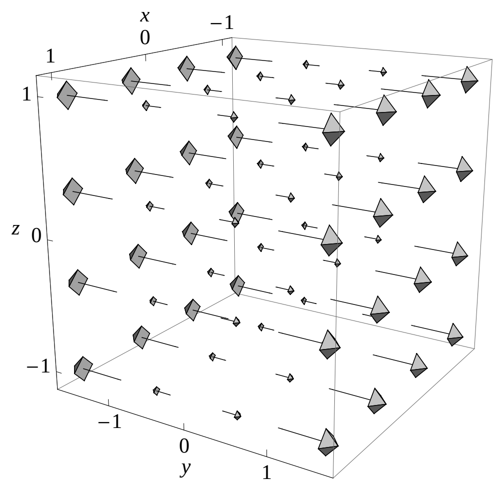


These are the vector fields $\vec{F}$ and $\vec{G}$ for problems #1 - #4. (Note that the origin is located in the middle of each box.)

$\vec{F}(x, y, z) = \langle 0, 0, -z \rangle$



$\vec{G}(x, y, z) = \langle 0, y, 0 \rangle$



Stokes' Theorem

1. Let $\vec{F}(x, y, z) = \langle -y, x, xyz \rangle$ and $\vec{G} = \operatorname{curl} \vec{F}$. Let $\mathcal{S}$ be the part of the sphere $x^2 + y^2 + z^2 = 25$ that lies below the plane $z = 4$, oriented so that the unit normal vector at $(0, 0, -5)$ is $\langle 0, 0, -1 \rangle$. Use Stokes' Theorem to find $\iint_{\mathcal{S}} \vec{G} \cdot d\vec{S}$.

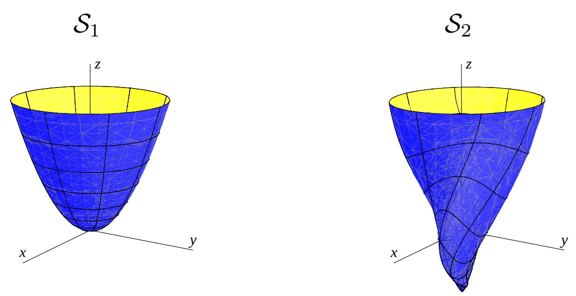
2. Let $\vec{F}(x, y, z) = \langle -y, x, z \rangle$. Let $\mathcal{S}$ be the part of the paraboloid $z = 7 - x^2 - 4y^2$ that lies above the plane $z = 3$, oriented with upward pointing normals. Use Stokes' Theorem to find $\iint_{\mathcal{S}} \operatorname{curl} \vec{F} \cdot d\vec{S}$.

3. The plane $z = x + 4$ and the cylinder $x^2 + y^2 = 4$ intersect in a curve C. Suppose C is oriented counterclockwise when viewed from above. Let $\vec{F}(x, y, z) = \langle x^3 + 2y, \sin y + z, x + \sin z^2 \rangle$. Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$.

4. Let C be the oriented curve parameterized by $\vec{r}(t) = \langle \cos t, \sin t, 8 - \cos^2 t - \sin t \rangle$, $0 \le t < 2\pi$, and let $\vec{F}$ be the vector field $\vec{F}(x, y, z) = \left\langle z^2 - y^2, -2xy^2, e^{\sqrt{z}} \cos z \right\rangle$. Evaluate $\displaystyle\int_C \vec{F} \cdot d\vec{r}$.

5. Let C be the curve of intersection of $2x^2 + 2y^2 + z^2 = 9$ with $z = \frac{1}{2}\sqrt{x^2 + y^2}$, oriented counterclockwise when viewed from above, and let $\vec{F}(x, y, z) = \langle 3y, 2yz, xz^3 + \sin z^2 \rangle$. Evaluate $\displaystyle\int_C \vec{F} \cdot d\vec{r}$.

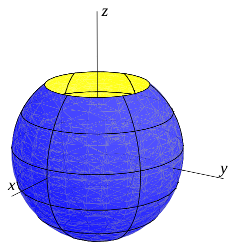
6. The two surfaces shown have the same boundary. Suppose they are both oriented so that the light side is the "positive" side. Is the following reasoning correct? "Since $\mathcal{S}_1$ and $\mathcal{S}_2$ have the same (oriented) boundary, the flux integrals $\displaystyle\iint_{\mathcal{S}_1} \vec{G} \cdot d\vec{S}$ and $\displaystyle\iint_{\mathcal{S}_2} \vec{G} \cdot d\vec{S}$ must be equal for any vector field $\vec{G}$. Therefore, you can compute any flux integral using the simpler surface."



Stokes' Theorem

1. *Let* $\vec{F}(x,y,z) = \langle -y, x, xyz\rangle$ *and* $\vec{G} = \operatorname{curl}\vec{F}$. *Let* $\mathcal{S}$ *be the part of the sphere* $x^2+y^2+z^2 = 25$ *that lies below the plane* $z = 4$, *oriented so that the unit normal vector at* $(0,0,-5)$ *is* $\langle 0,0,-1\rangle$. *Use Stokes' Theorem to find* $\iint_{\mathcal{S}} \vec{G}\cdot d\vec{S}$.

Solution. Here's a picture of the surface $\mathcal{S}$.



To use Stokes' Theorem, we need to first find the boundary C of $\mathcal{S}$ and figure out how it should be oriented. The boundary is where $x^2+y^2+z^2 = 25$ and $z = 4$. Substituting $z = 4$ into the first equation, we can also describe the boundary as where $x^2+y^2 = 9$ and $z = 4$.

To figure out how C should be oriented, we first need to understand the orientation of $\mathcal{S}$. We are told that $\mathcal{S}$ is oriented so that the unit normal vector at $(0,0,-5)$ (which is the lowest point of the sphere) is $\langle 0,0,-1\rangle$ (which points down). This tells us that the blue side must be the "positive" side.

We want to orient the boundary so that, if a penguin walks near the boundary of $\mathcal{S}$ on the "positive" side (which we've already decided is the blue side), he keeps the surface on his left. If we imagine looking down on the surface from a really high point like $(0,0,100)$, then the penguin should walk clockwise (from our vantage point).

So, using Stokes' Theorem, we have changed the original problem into a new one:

> Evaluate the line integral $\int_C \vec{F}\cdot d\vec{r}$, where C is the curve described by $x^2+y^2 = 9$ and $z = 4$, oriented clockwise when viewed from above.

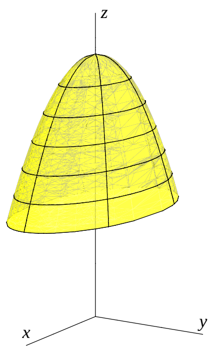
Now, we just need to evaluate the line integral, using the definition of the line integral. (This is like #4(a) on the worksheet "Vector Fields and Line Integrals".) We start by parameterizing C. One possible parameterization is $\vec{r}(t) = \langle 3\cos t, 3\sin t, 4\rangle$, $0 \le t < 2\pi$.(1) If we look at this from above, it is oriented counterclockwise, which is the wrong orientation. Therefore,

$$\begin{aligned}
\int_C \vec{F}\cdot d\vec{r} &= -\int_0^{2\pi} \vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t)\;dt \\
&= -\int_0^{2\pi} \langle -3\sin t, 3\cos t, 36\cos t\sin t\rangle \cdot \langle -3\sin t, 3\cos t, 0\rangle\;dt \\
&= -\int_0^{2\pi} 9\;dt \\
&= \boxed{-18\pi}
\end{aligned}$$

(1)To come up with this, remember that we can parameterize a circle $x^2+y^2 = 1$ in $\mathbb{R}^2$ by $x = \cos t$, $y = \sin t$ (and, as t increases, this goes around the circle counterclockwise). Here, we're looking at $x^2+y^2 = 9$; if we rewrite this as $\left(\frac{x}{3}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$, then we can write $\frac{x}{3} = \cos t$, $\frac{y}{3} = \sin t$.

2. *Let $\vec{F}(x,y,z) = \langle -y, x, z\rangle$. Let $\mathcal{S}$ be the part of the paraboloid $z = 7 - x^2 - 4y^2$ that lies above the plane $z = 3$, oriented with upward pointing normals. Use Stokes' Theorem to find $\iint_{\mathcal{S}} \operatorname{curl} \vec{F} \cdot d\vec{S}$.*

Solution. Here is a picture of the surface $\mathcal{S}$.



The strategy is exactly the same as in #1. The boundary is where $z = 7 - x^2 - 4y^2$ and $z = 3$, which is the same as $x^2 + 4y^2 = 4$ and $z = 3$.

Since $\mathcal{S}$ is oriented with normals pointing upward, the top side of the paraboloid (the yellow side in the picture) is the "positive" side. If we imagine looking down on the surface from above, then a penguin walking around on the "positive" (yellow, in this case) side keeps the surface on his left by walking counterclockwise.

Therefore, by Stokes' Theorem, the original problem can be rewritten as

> Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$, where C is the curve described by $x^2 + 4y^2 = 4$ and $z = 3$, oriented counterclockwise when viewed from above.

A parameterization of this curve is $\vec{r}(t) = \langle 2\cos t, \sin t, 3\rangle$.(2) This goes counterclockwise when viewed from above (as we want), so

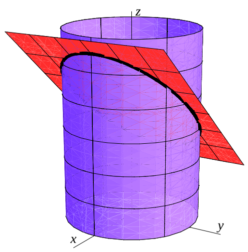
$$\begin{aligned}
\int_C \vec{F} \cdot d\vec{r} &= \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\ dt \\
&= \int_0^{2\pi} \langle -\sin t, 2\cos t, 3\rangle \cdot \langle -2\sin t, \cos t, 0\rangle\ dt \\
&= \int_0^{2\pi} (2\sin^2 t + 2\cos^2 t)\ dt \\
&= \int_0^{2\pi} 2\ dt \\
&= \boxed{4\pi}
\end{aligned}$$

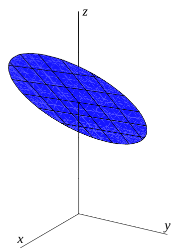
(2) To come up with this parameterization, rewrite $x^2 + 4y^2 = 4$ as $\left(\frac{x}{2}\right)^2 + y^2 = 1$ and then use $\frac{x}{2} = \cos t$, $y = \sin t$. It's easy to check that it's reasonable: if we plug in $x = 2\cos t$, $y = \sin t$, and $z = 3$, then the equations $x^2 + 4y^2 = 4$ and $z = 3$ are indeed satisfied.

3. *The plane $z = x + 4$ and the cylinder $x^2 + y^2 = 4$ intersect in a curve C. Suppose C is oriented counterclockwise when viewed from above. Let $\vec{F}(x, y, z) = \langle x^3 + 2y, \sin y + z, x + \sin z^2\rangle$. Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$.*

Solution. We'll use Stokes' Theorem. To do this, we need to think of an oriented surface $\mathcal{S}$ whose (oriented) boundary is C (that is, we need to think of a surface $\mathcal{S}$ and orient it so that the given orientation of C matches). Then, Stokes' Theorem says that $\int_C \vec{F} \cdot d\vec{r} = \iint_{\mathcal{S}} \operatorname{curl} \vec{F} \cdot d\vec{S}$. Let's compute $\operatorname{curl} \vec{F}$ first. (It's worthwhile to do this first because, if we find out it's $\vec{0}$, then we know the integral will be 0 without any more work.) In this case, $\operatorname{curl} \vec{F} = \langle -1, -1, -2\rangle$.

Now, let's think of a surface whose boundary is the given curve C. We are told that C is the intersection of a plane and a cylinder (left picture), so one surface we could use is the part of the plane inside the cylinder (right picture):





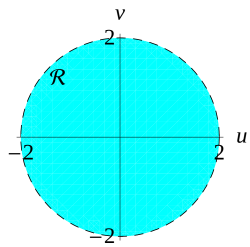
Let's call this $\mathcal{S}$ and figure out how it should be oriented. We want to orient $\mathcal{S}$ so that, if a penguin walks along the given curve C (going counterclockwise when viewed from above) on the "positive" side of $\mathcal{S}$, he keeps the surface on his left. This means that we want the top side of $\mathcal{S}$ to be the "positive" side, so we should orient $\mathcal{S}$ with normals pointing upward.

So, using Stokes' Theorem, we have changed the original problem into a new one:

> Evaluate the flux integral $\iint_{\mathcal{S}} \vec{G} \cdot d\vec{S}$, where $\mathcal{S}$ is the part of the plane $z = x + 4$ inside the cylinder $x^2 + y^2 = 4$, oriented with normals pointing upward, and $\vec{G}$ is the vector field $\vec{G}(x, y, z) = \langle -1, -1, -2\rangle$.

To do this new problem, let's follow the same three steps we used in #4(a) on the worksheet "Flux Integrals".

First, we parameterize $\mathcal{S}$. Since the plane has equation $z = x+4$, we can use x and y as our parameters. If we let $x = u$ and $y = v$, then $z = u + 4$. This gives the parameterization $\vec{r}(u, v) = \langle u, v, u + 4\rangle$. Since we are only interested in the part of the plane inside the cylinder $x^2 + y^2 = 4$, we want $x^2 + y^2 < 4$. In terms of u and v, this says $u^2 + v^2 < 4$, so the region $\mathcal{R}$ in the uv-plane describing the possible parameter values is a disk:



Next, we need to see what orientation this parameterization describes. To do this, we compute $\vec{r}_u \times \vec{r}_v$:

$$\begin{aligned} \vec{r}_u &= \langle 1, 0, 1\rangle \\ \vec{r}_v &= \langle 0, 1, 0\rangle \\ \vec{r}_u \times \vec{r}_v &= \langle -1, 0, 1\rangle \end{aligned}$$

This always points upward, which matches the orientation we want. So, the flux integral is

$$\begin{aligned} \iint_{\mathcal{S}} \vec{G} \cdot d\vec{S} &= \iint_{\mathcal{R}} \vec{G}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\ dA \\ &= \iint_{\mathcal{R}} \langle -1, -1, -2\rangle \cdot \langle -1, 0, 1\rangle\ dA \\ &= \iint_{\mathcal{R}} -1\ dA \end{aligned}$$

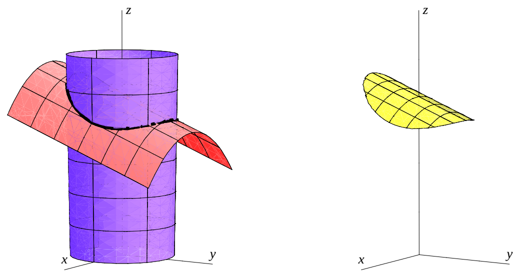
Although we could evaluate this double integral by converting it to an iterated integral, there is an easier way — remember that $\iint_{\mathcal{R}} 1\ dA$ gives the area of $\mathcal{R}$ (see #2(a) on the worksheet "Double Integrals"). Therefore,

$$\begin{aligned} \iint_{\mathcal{S}} \vec{G} \cdot d\vec{S} &= -\iint_{\mathcal{R}} 1\ dA \\ &= -(\text{area of } \mathcal{R}) \\ &= \boxed{-4\pi} \end{aligned}$$

4. *Let C be the oriented curve parameterized by $\vec{r}(t) = \langle \cos t, \sin t, 8 - \cos^2 t - \sin t\rangle$, $0 \le t < 2\pi$, and let $\vec{F}$ be the vector field $\vec{F}(x,y,z) = \left\langle z^2 - y^2, -2xy^2, e^{\sqrt{z}} \cos z \right\rangle$. Evaluate $\int_C \vec{F} \cdot d\vec{r}$.*

Solution. The line integral is very difficult to compute directly, so we'll use Stokes' Theorem. The curl of the given vector field $\vec{F}$ is $\operatorname{curl} \vec{F} = \langle 0, 2z, 2y - 2y^2\rangle$.

To use Stokes' Theorem, we need to think of a surface whose boundary is the given curve C. First, let's try to understand C a little better. We are given a parameterization $\vec{r}(t)$ of C. In this parameterization, $x = \cos t$, $y = \sin t$, and $z = 8 - \cos^2 t - \sin t$. So, we can see that $x^2 + y^2 = 1$ and $z = 8 - x^2 - y$. In other words, C must be the intersection of the surface $x^2 + y^2 = 1$ (which is a cylinder) and the surface $z = 8 - x^2 - y$ (which we don't need to visualize particularly well, beyond noticing that it's the graph of a function $f(x, y) = 8 - x^2 - y$). So, one surface we could use is the part of the surface $z = 8 - x^2 - y$ inside the cylinder $x^2 + y^2 = 1$ (right picture).



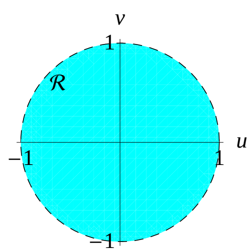
Let's call this surface $\mathcal{S}$ and figure out how it should be oriented. The original curve was parameterized using $x = \cos t$, $y = \sin t$, so when viewed from above, it was oriented counterclockwise. Therefore, we want to orient $\mathcal{S}$ so that is top is the "positive" side (a penguin walking on the top along the boundary, going counterclockwise when viewed from above, keeps the surface on his left). So, we should orient $\mathcal{S}$ with normals pointing upward.

So, using Stokes' Theorem, we have changed the original problem into a new one:

> Evaluate the flux integral $\iint_{\mathcal{S}} \vec{G} \cdot d\vec{S}$, where $\mathcal{S}$ is the part of the surface $z = 8 - x^2 - y$ inside the cylinder $x^2 + y^2 = 1$, oriented with normals pointing upward, and $\vec{G}$ is the vector field $\vec{G}(x, y, z) = \langle 0, 2z, 2y - 2y^2 \rangle$.

To do this new problem, let's follow the same three steps we used in #4(a) on the worksheet "Flux Integrals".

First, we parameterize $\mathcal{S}$. Since the surface has equation $z = 8 - x^2 - y$, we can parameterize it as $\vec{r}(u, v) = \langle u, v, 8 - u^2 - v \rangle$. Since we are only interested in the part of the surface inside the cylinder $x^2 + y^2 = 1$, we want $x^2 + y^2 < 1$; in terms of u and v, this says $u^2 + v^2 < 1$, so the region $\mathcal{R}$ in the uv-plane describing the possible parameter values is a disk:



Next, we need to see what orientation this parameterization describes. To do this, we compute $\vec{r}_u \times \vec{r}_v$:

$$\begin{aligned} \vec{r}_u &= \langle 1, 0, -2u \rangle \\ \vec{r}_v &= \langle 0, 1, -1 \rangle \\ \vec{r}_u \times \vec{r}_v &= \langle 2u, 1, 1 \rangle \end{aligned}$$

This always points upward, which matches the orientation we want. So, the flux integral is

$$\begin{aligned} \iint_{\mathcal{S}} \vec{G} \cdot d\vec{S} &= \iint_{\mathcal{R}} \vec{G}(\vec{r}(u, v)) \cdot (\vec{r}_u \times \vec{r}_v)\ dA \\ &= \iint_{\mathcal{R}} \langle 0, 2(8 - u^2 - v), 2v - 2v^2 \rangle \cdot \langle 2u, 1, 1 \rangle\ dA \\ &= \iint_{\mathcal{R}} (16 - 2u^2 - 2v^2)\ dA \end{aligned}$$

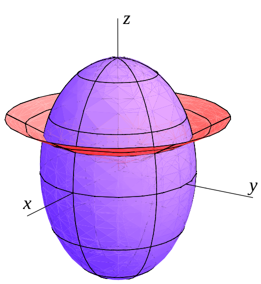
Since our region $\mathcal{R}$ is a disk, let's do this integral in polar coordinates. The disk $\mathcal{R}$ can be described as $0 \le r < 1$, $0 \le \theta < 2\pi$, so

$$\begin{aligned}
\iint_{\mathcal{S}} \vec{G} \cdot d\vec{S} &= \int_0^{2\pi} \int_0^1 (16 - 2r^2) \cdot r \, dr \, d\theta \\
&= \int_0^{2\pi} \int_0^1 (16r - 2r^3) \, dr \, d\theta \\
&= \int_0^{2\pi} \left(8r^2 - \frac{1}{2} r^4 \Big|_{r=0}^{r=1} \right) d\theta \\
&= \int_0^{2\pi} \frac{15}{2} \, d\theta \\
&= \boxed{15\pi}
\end{aligned}$$

5. *Let C be the curve of intersection of $2x^2 + 2y^2 + z^2 = 9$ with $z = \frac{1}{2}\sqrt{x^2 + y^2}$, oriented counterclockwise when viewed from above, and let $\vec{F}(x, y, z) = \langle 3y, 2yz, xz^3 + \sin z^2 \rangle$. Evaluate $\int_C \vec{F} \cdot d\vec{r}$.*

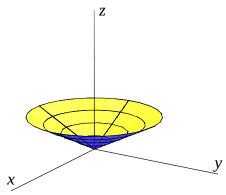
Solution. Again, we'll use Stokes' Theorem. The curl of the given vector field is curl $\vec{F} = \big\langle -2y, -z^3, -3 \big\rangle$.

We'll start by thinking of an oriented surface $\mathcal{S}$ whose (oriented) boundary is the given curve C. The curve is the intersection of an ellipsoid with a cone:

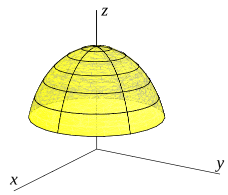


It appears from the picture that the curve lies in a plane parallel to the xy-plane. To verify this, let's look at the equations $2x^2 + 2y^2 + z^2 = 9$ and $z = \frac{1}{2}\sqrt{x^2 + y^2}$ defining the curve. The second equation can be rewritten as $x^2 + y^2 = 4z^2$. Plugging this into the first equation, $9z^2 = 9$, so $z = \pm 1$. Since the cone is only defined for $z \ge 0$, we know the intersection is where $z = 1$, in which case $x^2 + y^2 = 4$.

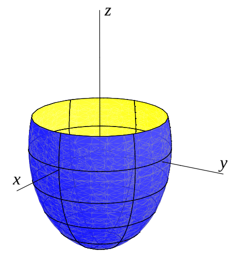
There are many surfaces whose boundary is the given curve C. From what we know so far, there are several surfaces that might come to mind:



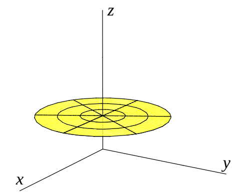
The part of the cone inside the ellipsoid.



The part of the ellipsoid lying above the curve.



The part of the ellipsoid lying below the curve.

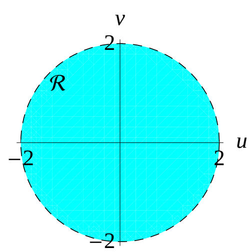


The part of the plane $z = 1$ lying inside $x^2 + y^2 = 4$.

Each of these should be oriented so that the yellow side is the "positive" side. The last is probably the simplest, so let's use that. To have the yellow side be the "positive" side, we want the normals to point upward. Thus, we have rewritten the original problem as:

> Evaluate the flux integral $\iint_{\mathcal{S}} \vec{G} \cdot d\vec{S}$, where $\mathcal{S}$ is the part of the plane $z = 1$ lying inside $x^2+y^2 = 4$, oriented with normals pointing upward, and $\vec{G}$ is the vector field $\vec{G}(x, y, z) = \left\langle -2y, -z^3, -3 \right\rangle$.

To do this, we first parameterize the surface. Since it is part of the plane $z = 1$, we can parameterize it by $\vec{r}(u, v) = \langle u, v, 1\rangle$. Since we want only the part inside $x^2 + y^2 = 4$, the region $\mathcal{R}$ in the uv-plane describing the possible parameter values is the disk $u^2 + v^2 < 4$:



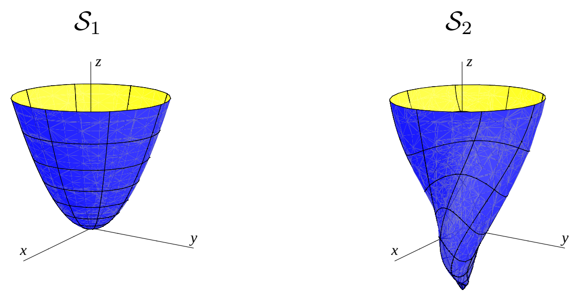
Next, we see what orientation this parameterization describes. To do this, we compute $\vec{r}_u \times \vec{r}_v$:

$$\begin{aligned} \vec{r}_u &= \langle 1, 0, 0\rangle \\ \vec{r}_v &= \langle 0, 1, 0\rangle \\ \vec{r}_u \times \vec{r}_v &= \langle 0, 0, 1\rangle \end{aligned}$$

This always points upward, so our parameterization matches the orientation we want. Therefore, the flux integral is

$$\begin{aligned} \iint_{\mathcal{S}} \vec{G} \cdot d\vec{S} &= \iint_{\mathcal{R}} \vec{G}(\vec{r}(u, v)) \cdot (\vec{r}_u \times \vec{r}_v)\ dA \\ &= \iint_{\mathcal{R}} \langle -2v, -1, -3\rangle \cdot \langle 0, 0, 1\rangle\ dA \\ &= \iint_{\mathcal{R}} -3\ dA \\ &= -3 \iint_{\mathcal{R}} 1\ dA \\ &= -3(\text{area of } \mathcal{R}) \text{ by the worksheet "Double Integrals", \#2(a)} \\ &= \boxed{-12\pi} \end{aligned}$$

6. *The two surfaces shown have the same boundary. Suppose they are both oriented so that the light side is the "positive" side. Is the following reasoning correct? "Since $\mathcal{S}_1$ and $\mathcal{S}_2$ have the same (oriented) boundary, the flux integrals $\iint_{\mathcal{S}_1} \vec{G} \cdot d\vec{S}$ and $\iint_{\mathcal{S}_2} \vec{G} \cdot d\vec{S}$ must be equal for any vector field $\vec{G}$. Therefore, you can compute any flux integral using the simpler surface."*



Solution. False. The statement *is* true if the vector field $\vec{G}$ is the curl of some other vector field, say $\vec{G} = \text{curl}\,\vec{F}$. In that case, if C is the (properly oriented) boundary, Stokes' Theorem says that $\iint_{\mathcal{S}_1} \vec{G} \cdot d\vec{S}$ and $\iint_{\mathcal{S}_1} \vec{G} \cdot d\vec{S}$ are both equal to $\int_C \vec{F} \cdot d\vec{r}$. But there's no reason that $\vec{G}$ has to be the curl of some other vector field, so the statement is false in general.

The Divergence Theorem

1. Describe the boundary of each of the following solids. (Your description should be thorough enough that somebody reading it would have enough information to find the surface area of the boundary).

   (a) The solid $x^2 + 4y^2 + 9z^2 \leq 36$.

   (b) The solid $x^2 + y^2 \leq z \leq 9$.

   (c) The solid consisting of all points (x, y, z) inside both the sphere $x^2 + y^2 + z^2 = 4$ and the cylinder $x^2 + y^2 = 3$.

2. Let $\vec{F}(x, y, z) = \langle x^2, 2y, e^z \rangle$. Let $\mathcal{S}$ be the surface of the cube whose vertices are $(\pm 1, \pm 1, \pm 1)$, oriented with outward normals. Evaluate the flux integral $\iint_{\mathcal{S}} \vec{F} \cdot d\vec{S}$.

3. Let $\vec{F}(x, y, z) = \langle x^3, z^2, 3y^2 z \rangle$. Let $\mathcal{S}$ be the surface $z = x^2 + y^2$, $z \leq 4$ together with the surface $z = 8 - (x^2 + y^2)$, $z \geq 4$. Evaluate the flux integral $\iint_{\mathcal{S}} \vec{F} \cdot d\vec{S}$ if $\mathcal{S}$ is oriented with outward normals.

4. True or false: If $\vec{F}$ is a vector field whose divergence is 0 and $\mathcal{S}$ is any surface, then the Divergence Theorem implies that the flux integral $\iint_{\mathcal{S}} \vec{F} \cdot d\vec{S}$ is equal to 0.

5. Let $\mathcal{S}_1$ be the surface consisting of the top and the four sides (but not the bottom) of the cube whose vertices are $(\pm 1, \pm 1, \pm 1)$, oriented the same way as in #2. Let $\vec{F}(x, y, z) = \langle x^2, 2y, e^z \rangle$, as in #2. Evaluate the flux integral $\iint_{\mathcal{S}_1} \vec{F} \cdot d\vec{S}$. (Hint: Use #2.)

6. Let $\vec{F}$ be the vector field $\vec{F}(x, y, z) = \left\langle z^3 \sin e^y, z^3 e^{x^2 \sin z}, y^2 + z \right\rangle$, and let $\mathcal{S}$ be the bottom half of the sphere $x^2 + y^2 + z^2 = 4$, oriented with normals pointing upward. Find $\iint_{\mathcal{S}} \vec{F} \cdot d\vec{S}$.

The Divergence Theorem

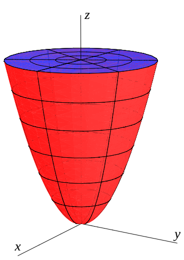
1. Describe the boundary of each of the following solids. (Your description should be thorough enough that somebody reading it would have enough information to find the surface area of the boundary).

   (a) *The solid* $x^2 + 4y^2 + 9z^2 \leq 36$.

   Solution. The boundary is just the ellipsoid $x^2 + 4y^2 + 9z^2 = 36$.

   (b) *The solid* $x^2 + y^2 \leq z \leq 9$.

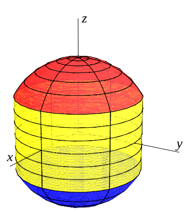
   Solution. The boundary has two pieces:



   One (the red one) is the part of the paraboloid $z = x^2 + y^2$ lying below the plane $z = 9$, and the other is the part of the plane $z = 9$ satisfying $x^2 + y^2 \leq 9$.

   (c) *The solid consisting of all points* (x, y, z) *inside both the sphere* $x^2 + y^2 + z^2 = 4$ *and the cylinder* $x^2 + y^2 = 3$.

   Solution. The boundary of this solid has 3 pieces, as we can see from the picture:



   The cylinder and the sphere intersect when $z^2 = 1$, or $z = \pm 1$. Therefore, the top (red) piece is the part of the sphere $x^2 + y^2 + z^2 = 4$ above the plane $z = 1$. The bottom (blue) piece is the part of the sphere $x^2 + y^2 + z^2 = 4$ below the plane $z = -1$. The middle (yellow) piece is the part of the cylinder $x^2 + y^2 = 3$ between the planes $z = -1$ and $z = 1$.

2. *Let* $\vec{F}(x, y, z) = \langle x^2, 2y, e^z \rangle$. *Let* $\mathcal{S}$ *be the surface of the cube whose vertices are* $(\pm 1, \pm 1, \pm 1)$, *oriented with outward normals. Evaluate the flux integral* $\iint_{\mathcal{S}} \vec{F} \cdot d\vec{S}$.

   Solution. Although it is possible to compute the flux integral directly, we would have to parameterize all 6 sides of the cube, compute a flux integral through each, and add up the answers. That doesn't seem like much fun.

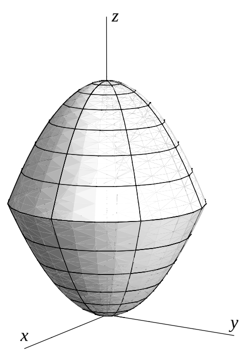
Instead, we'll use the Divergence Theorem. Since $\mathcal{S}$ is oriented with outward normals, the Divergence Theorem says that $\iint_{\mathcal{S}} \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F}\, dV$, where E is the solid interior of $\mathcal{S}$. We just calculate $\operatorname{div} \vec{F}$: it is $\operatorname{div} \vec{F} = 2x + 2 + e^z$.

So, our goal now is: "Compute the triple integral $\iiint_E (2x + 2 + e^z)\, dV$ where E is the interior of the cube with vertices $(\pm 1, \pm 1, \pm 1)$." Like any other triple integral, we do this by converting it to an iterated integral:

$$\begin{aligned}\iiint_E (2x + 2 + e^z)\, dV &= \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} (2x + 2 + e^z)\, dx\, dy\, dz \\ &= \boxed{16 - \frac{4}{e} + 4e}\end{aligned}$$

3. *Let $\vec{F}(x, y, z) = \langle x^3, z^2, 3y^2 z\rangle$. Let $\mathcal{S}$ be the surface $z = x^2 + y^2$, $z \leq 4$ together with the surface $z = 8 - (x^2 + y^2)$, $z \geq 4$. Evaluate the flux integral $\iint_{\mathcal{S}} \vec{F} \cdot d\vec{S}$ if $\mathcal{S}$ is oriented with outward normals.*

Solution. Here is a picture of the surface.



Since $\mathcal{S}$ is oriented with outward normals, the Divergence Theorem tells us that $\iint_{\mathcal{S}} \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F}\, dV$, where E is the solid interior of $\mathcal{S}$. In this case, $\operatorname{div} \vec{F} = 3x^2 + 3y^2$.

So, our goal now is: "Compute the triple integral $\iiint_E (3x^2 + 3y^2)\, dV$ where E is the solid enclosed by $z = x^2 + y^2$ and $z = 8 - (x^2 - y^2)$."

It's probably easiest to do this as an iterated integral in cylindrical coordinates. In fact, we've already done a triple integral with this solid as the region of integration before, in #1 on the worksheet "Triple Integrals in Cylindrical or Spherical Coordinates". Using the work we did there, we can rewrite the triple integral as an iterated integral

$$\begin{aligned}
\iiint_E 3(x^2+y^2)\ dV &= \int_0^{2\pi}\int_0^2\int_{r^2}^{8-r^2} 3r^2\cdot r\ dz\ dr\ d\theta \\
&= \int_0^{2\pi}\int_0^2 \left(3r^3 z\Big|_{z=r^2}^{z=8-r^2}\right)\ dr\ d\theta \\
&= \int_0^{2\pi}\int_0^2 (24r^3-6r^5)\ dr\ d\theta \\
&= \int_0^{2\pi}\left(6r^4-r^6\Big|_{r=0}^{r=2}\right)\ d\theta \\
&= \int_0^{2\pi} 32\ d\theta \\
&= \boxed{64\pi}
\end{aligned}$$

4. *True or false: If $\vec{F}$ is a vector field whose divergence is 0 and $\mathcal{S}$ is any surface, then the Divergence Theorem implies that the flux integral $\iint_{\mathcal{S}} \vec{F}\cdot d\vec{S}$ is equal to 0.*

   Solution. $\boxed{\text{False.}}$ The statement is only true if $\mathcal{S}$ is the boundary of some solid. Many surfaces do not fall into this category. For instance, the disk $x^2+y^2\le 1$ in the plane $z=0$ is not the boundary of any solid.

5. *Let $\mathcal{S}_1$ be the surface consisting of the top and the four sides (but not the bottom) of the cube whose vertices are $(\pm1,\pm1,\pm1)$, oriented the same way as in #2. Let $\vec{F}(x,y,z)=\langle x^2, 2y, e^z\rangle$, as in #2. Evaluate the flux integral $\iint_{\mathcal{S}_1}\vec{F}\cdot d\vec{S}$. (Hint: Use #2.)*

   Solution. Let $\mathcal{S}_2$ be the bottom of the cube, oriented with normals pointing downward (so that the bottom side of $\mathcal{S}_2$ is the "positive" side). Then, $\mathcal{S}_1$ and $\mathcal{S}_2$ together form the surface $\mathcal{S}$ in #2. So, by #2, we know that

$$\iint_{\mathcal{S}_1}\vec{F}\cdot d\vec{S}+\iint_{\mathcal{S}_2}\vec{F}\cdot d\vec{S}=16-\frac{4}{e}+4e \tag{1}$$

   If we can evaluate $\iint_{\mathcal{S}_2}\vec{F}\cdot d\vec{S}$, then this equation will tell us what $\iint_{\mathcal{S}_1}\vec{F}\cdot d\vec{S}$ is as well. Evaluating $\iint_{\mathcal{S}_2}\vec{F}\cdot d\vec{S}$ should be a lot easier than evaluating $\iint_{\mathcal{S}_1}\vec{F}\cdot d\vec{S}$, since $\mathcal{S}_2$ consists of one face of the cube, while $\mathcal{S}_1$ consists of five faces.

   To evaluate $\iint_{\mathcal{S}_2}\vec{F}\cdot d\vec{S}$, we'll use the definition of the flux integral. That is, we start by parameterizing the surface $\mathcal{S}_2$. In this case, $\mathcal{S}_2$ is the portion of the plane $z=-1$ with $-1\le x\le 1$ and $-1\le y\le 1$. So, we can parameterize it by $x=u$, $y=v$, $z=-1$, or $\vec{r}(u,v)=\langle u,v,-1\rangle$, with $-1\le u\le 1$ and $-1\le v\le 1$. (The region $\mathcal{R}$ in the uv-plane described by the inequalities $-1\le u\le 1$, $-1\le v\le 1$ is a square.) We should check whether this gives the right orientation:

$$\begin{aligned}
\vec{r}_u &= \langle 1,0,0\rangle \\
\vec{r}_v &= \langle 0,1,0\rangle \\
\vec{r}_u\times\vec{r}_v &= \langle 0,0,1\rangle
\end{aligned}$$

This points upward, but we wanted to orient $\mathcal{S}_2$ with normals pointing downward, so we have the wrong orientation. Therefore,

$$\begin{aligned}
\iint_{\mathcal{S}_2} \vec{F} \cdot d\vec{S} &= -\iint_{\mathcal{R}} \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\ dA \\
&= -\iint_{\mathcal{R}} \left\langle u^2, 2v, \frac{1}{e} \right\rangle \cdot \langle 0,0,1 \rangle\ dA \\
&= -\iint_{\mathcal{R}} \frac{1}{e}\ dA \\
&= -\frac{1}{e} \iint_{\mathcal{R}} 1\ dA \\
&= -\frac{1}{e}(\text{area of } \mathcal{R}) \text{ by the worksheet “Double Integrals”, \#2(a)} \\
&= -\frac{4}{e}
\end{aligned}$$

Plugging this into (1), we find that $\iint_{\mathcal{S}} \vec{F} \cdot d\vec{S} = 16 + 4e$.

6. *Let $\vec{F}$ be the vector field $\vec{F}(x,y,z) = \left\langle z^3 \sin e^y, z^3 e^{x^2 \sin z}, y^2 + z \right\rangle$, and let $\mathcal{S}$ be the bottom half of the sphere $x^2 + y^2 + z^2 = 4$, oriented with normals pointing upward. Find $\iint_{\mathcal{S}} \vec{F} \cdot d\vec{S}$.*

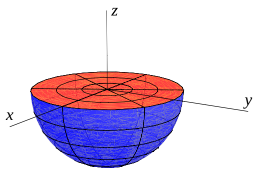
   Solution. Since the vector field $\vec{F}$ has quite a complicated definition, we can guess that doing the flux integral directly would be horrible. Since this problem is on a Divergence Theorem worksheet, we can guess that we should use the Divergence Theorem.(1)

   However, we can't directly apply the Divergence Theorem because the given surface $\mathcal{S}$ is not the boundary of a solid. So, we'll have to be a bit more clever and use the strategy of #5. That is, we want to think of another surface $\mathcal{S}'$ which, together with $\mathcal{S}$, encloses a solid E. Then, the Divergence Theorem will relate $\iint_{\mathcal{S}} \vec{F} \cdot d\vec{S}$, $\iint_{\mathcal{S}'} \vec{F} \cdot d\vec{S}$, and $\iiint_E \operatorname{div} \vec{F}\ dV$. So, if we can compute $\iint_{\mathcal{S}'} \vec{F} \cdot d\vec{S}$ and $\iiint_E \operatorname{div} \vec{F}\ dV$ directly, we'll be able to find $\iint_{\mathcal{S}} \vec{F} \cdot d\vec{S}$.

   There are lots of surfaces $\mathcal{S}'$ which, together with $\mathcal{S}$, enclose a solid E. Let's take $\mathcal{S}'$ to be the disk $x^2 + y^2 \le 4$ in the plane $z = 0$, oriented with its normals pointing upward (so that the top side of $\mathcal{S}'$ is the “positive” side).(2) Then, $\mathcal{S}$ and $\mathcal{S}'$ together enclose the bottom half of the ball $x^2 + y^2 + z^2 \le 4$, and we'll call this solid E.

(1) A better way to tell that the Divergence Theorem is a good approach is to think through our options. We've already decided we don't want to compute the flux integral directly, so that means we need to use an integral theorem. The ones we know that tell us something about flux integrals are Stokes' Theorem and the Divergence Theorem. Stokes' Theorem only applies to flux integrals in which the vector field being integrated is the curl of something. In this case, $\vec{F}$ is *not* the curl of anything because $\operatorname{div} \vec{F} \ne 0$, and #4 on the worksheet “Curl and Divergence” says that the divergence of a vector field that is the curl of something must be 0.

(2) Here, we just make an arbitrary choice of which way to orient. It doesn't make any difference, as long as we are consistent throughout the rest of the problem.



The boundary of E, oriented outwards, is $\mathcal{S}$ oriented with downward normals (opposite of what we're asked for) and $\mathcal{S}'$ oriented with upward normals (the way we decided to do it). So, the Divergence Theorem says

$$\iint_{\mathcal{S}'} \vec{F} \cdot d\vec{S} - \iint_{\mathcal{S}} \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F}\, dV \tag{2}$$

(In this equation, we're thinking of $\mathcal{S}$ as oriented with upward normals and $\mathcal{S}'$ as oriented with upward normals.) Let's first calculate the right side. $\operatorname{div} \vec{F} = 1$, so $\iiint_E \operatorname{div} \vec{F}\, dV = \iiint_E 1\, dV$. We know that this is equal to the volume of E (see #1(a) on the worksheet "Triple Integrals"), which is $\frac{16}{3}\pi$.

Next, let's calculate $\iint_{\mathcal{S}'} \vec{F} \cdot d\vec{S}$. To do this, we need to parameterize $\mathcal{S}'$. Since $\mathcal{S}'$ is the part of the plane $z = 0$ satisfying $x^2 + y^2 \leq 4$, we can just parameterize it as $\vec{r}(u, v) = \langle u, v, 0\rangle$ (and the restriction $x^2 + y^2 \leq 4$ translates to $u^2 + v^2 \leq 4$). Next, we check whether we have the right orientation:

$$\begin{aligned} \vec{r}_u &= \langle 1, 0, 0\rangle \\ \vec{r}_v &= \langle 0, 1, 0\rangle \\ \vec{r}_u \times \vec{r}_v &= \langle 0, 0, 1\rangle \end{aligned}$$

This always points upward, which is what we want. So, if we let $\mathcal{R}$ be the disk $u^2 + v^2 \leq 4$ in the uv-plane,

$$\begin{aligned} \iint_{\mathcal{S}'} \vec{F} \cdot d\vec{S} &= \iint_{\mathcal{R}} \vec{F}(\vec{r}(u, v)) \cdot (\vec{r}_u \times \vec{r}_v) \\ &= \iint_{\mathcal{R}} \langle 0, 0, v^2\rangle \cdot \langle 0, 0, 1\rangle\, dA \\ &= \iint_{\mathcal{R}} v^2\, dA \end{aligned} \tag{3}$$

Since $\mathcal{R}$ is a disk, this double integral is probably a bit easier in polar coordinates: it is

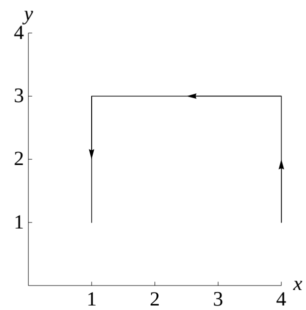
$$\begin{aligned} \iint_{\mathcal{S}'} \vec{F} \cdot d\vec{S} &= \int_0^{2\pi} \int_0^2 (r \sin\theta)^2 \cdot r\, dr\, d\theta \\ &= 4\pi \end{aligned}$$

Plugging this all into (2), $4\pi - \iint_{\mathcal{S}} \vec{F} \cdot d\vec{S} = \dfrac{16}{3}\pi$, so $\iint_{\mathcal{S}} \vec{F} \cdot d\vec{S} = \boxed{-\dfrac{4}{3}\pi}$.

Notice that, when we calculated the flux integral $\iint_{\mathcal{S}'} \vec{F} \cdot d\vec{S}$, $\vec{F}(\vec{r}(u, v))$ became very simple in (3): it was just $\langle 0, 0, v^2\rangle$. In fact, we could tell as soon as we chose the surface $\mathcal{S}'$ that this term would become very simple: $\vec{r}(u, v)$ parameterizes $\mathcal{S}'$, and the surface $\mathcal{S}'$ that we chose had $z = 0$. When $z = 0$, the vector field $\vec{F}$ becomes very simple. This is one reason the choice of $\mathcal{S}'$ that we made was a good one.

The Integral Theorems

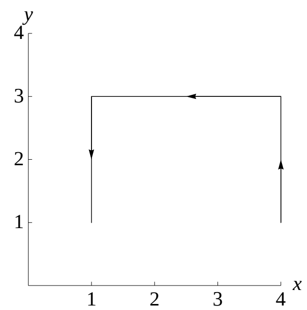
1. Let C be the curve in $\mathbb{R}^2$ consisting of line segments from $(4,1)$ to $(4,3)$ to $(1,3)$ to $(1,1)$. Let $\vec{F}(x,y) = \langle x+y, (y-1)^3 e^{\sin y} \rangle$. Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$.



2. Let C be the (oriented) curve parameterized by $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$, $0 \le t \le 2\pi$. Let $\vec{F}(x,y,z) = \langle e^{x^2}, (\sin y + 3)^y, z^2 \rangle$. Evaluate $\int_C \vec{F} \cdot d\vec{r}$.

The Integral Theorems

1. *Let C be the curve in $\mathbb{R}^2$ consisting of line segments from $(4,1)$ to $(4,3)$ to $(1,3)$ to $(1,1)$. Let $\vec{F}(x,y) = \left\langle x+y, (y-1)^3 e^{\sin y} \right\rangle$. Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$.*



Solution. Let's consider our options. This problem is asking us to evaluate a line integral in $\mathbb{R}^2$, so we seem to have three choices:

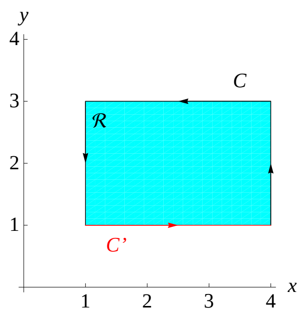
- Evaluate using the definition of line integral. (Parameterize the curve, check the orientation, etc., like what we did in #4(a) on the worksheet "Vector Fields and Line Integrals" or what you did on homework in §13.2, #24(a).)
- Use the Fundamental Theorem for Line Integrals, which tells us how to integrate $\int_C \vec{F} \cdot d\vec{r}$ if $\vec{F}$ is conservative (a gradient vector field).
- Use Green's Theorem.

The first option is not great, partly because C consists of three separate pieces, and partly because if you try to parameterize the two vertical pieces, you'll end up with integrals that are impossible to do by hand.

The second option seems nice, but we can only use it if $\vec{F}$ is conservative. In this case, we can check fairly easily that $\vec{F}$ is not conservative: if we write $\vec{F} = \langle P, Q \rangle$, then $Q_x = 0$ and $P_y = 1$. Since $Q_x \neq P_y$, $\vec{F}$ is not conservative.(1) So, the Fundamental Theorem for Line Integrals really isn't an option in this case.

The third possibility is to use Green's Theorem. However, we cannot use Green's Theorem directly because Green's Theorem is talking about a region and the boundary of that region, and our curve C is not the boundary of any region. However, we can use a trick similar to the one we used in #5 and #6 on the worksheet "The Divergence Theorem". Let C' be the line segment from $(1,1)$ to $(4,1)$. Then, C and C' together enclose a region, which we'll call $\mathcal{R}$:

(1) We're using (#3(b) on the worksheet "The Fundamental Theorem for Line Integrals; Gradient Vector Fields" here.



Then, since C and C' together form the boundary of $\mathcal{R}$, oriented so that a penguin walking along it keeps $\mathcal{R}$ on his left, Green's Theorem tells us that

$$\int_C \vec{F} \cdot d\vec{r} + \int_{C'} \vec{F} \cdot d\vec{r} = \iint_{\mathcal{R}} (Q_x - P_y)\ dA, \tag{1}$$

where we write $\vec{F} = \langle P, Q \rangle$.

Let's evaluate the right side first: $\iint_{\mathcal{R}} (Q_x - P_y)\ dA = \iint_{\mathcal{R}} -1\ dA = -\iint_{\mathcal{R}} 1\ dA$. We could change this to an iterated integral to evaluate, but it's easier to just remember that $\iint_{\mathcal{R}} 1\ dA$ represents the area of $\mathcal{R}$,(2) which we can see is 6. So, $\iint_{\mathcal{R}} (Q_x - P_y)\ dA = -6$.

Next, let's evaluate $\int_{C'} \vec{F} \cdot d\vec{r}$. To do this, we'll use the definition of the line integral. C' is part of the line $y = 1$, so we can parameterize it as $\vec{r}(t) = \langle t, 1 \rangle$, with $1 \le t \le 4$. This gives the correct orientation of C', so

$$\begin{aligned}
\int_{C'} \vec{F} \cdot d\vec{r} &= \int_1^4 \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\ dt \\
&= \int_1^4 \langle t+1, 0 \rangle \cdot \langle 1, 0 \rangle\ dt \\
&= \int_1^4 (t+1)\ dt \\
&= \frac{21}{2}
\end{aligned}$$

Plugging this into (1), we get that $\int_C \vec{F} \cdot d\vec{r} = \boxed{-\frac{33}{2}}$.

2. *Let C be the (oriented) curve parameterized by $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$, $0 \le t \le 2\pi$. Let $\vec{F}(x, y, z) = \langle e^{x^2}, (\sin y + 3)^y, z^2 \rangle$. Evaluate $\int_C \vec{F} \cdot d\vec{r}$.*

Solution. We are asked to evaluate a line integral in $\mathbb{R}^3$, so we have three options.

- Use the definition of the line integral.
- Use the Fundamental Theorem for Line Integrals.

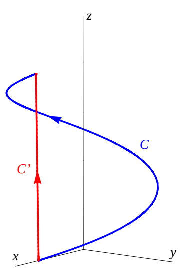
(2) See #2(a) on the worksheet "Double Integrals"

- Use Stokes' Theorem.

In this case, the definition of the line integral results in an integral that is impossible to do by hand (try and see).

To see if we can use the second option, we need to see if $\vec{F}$ is conservative. Since $\vec{F}$ is defined on all of $\mathbb{R}^3$ (so its domain is $\mathbb{R}^3$), we can check this simply by seeing if $\operatorname{curl} \vec{F} = \vec{0}$, and it is. So, $\vec{F}$ is conservative, and we can use the Fundamental Theorem for Line Integrals. However, it's not so easy to find a function f such that $\nabla f = \vec{F}$ (again, try and see).

However, we can instead use another fact we know about conservative vector fields: they are independent of path. That is, if C' is any path that starts where C starts ($\vec{r}(0) = \langle 1, 0, 0\rangle$) and ends where C ends ($\vec{r}(2\pi) = \langle 1, 0, 2\pi\rangle$), then $\int_{C'} \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot d\vec{r}$.(3) So, perhaps we can find a simple path C' from $(1, 0, 0)$ to $(1, 0, 2\pi)$ and evaluate the line integral along that path. The most obvious choice is to use the straight line path from $(1, 0, 0)$ to $(1, 0, 2\pi)$. Let's call this C'.



C' can be parameterized by $\vec{r}(t) = \langle 1, 0, t\rangle$, with $0 \le t \le 2\pi$. This gives the correct orientation, so

$$\begin{aligned} \int_{C'} \vec{F} \cdot d\vec{r} &= \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\ dt \\ &= \int_0^{2\pi} \left\langle e, 1, t^2 \right\rangle \cdot \langle 0, 0, 1\rangle\ dt \\ &= \int_0^{2\pi} t^2\ dt \\ &= \boxed{\frac{8\pi^3}{3}} \end{aligned}$$

(3) This follows directly from the Fundamental Theorem for Line Integrals, since, if $\vec{F} = \nabla f$, both integrals are equal to $f(1, 0, 2\pi) - f(1, 0, 0)$.

Practice Problems for Sections 13.6 - 13.9

1. Use the definition of the flux integral to evaluate $\iint_{\mathcal{S}} \vec{F} \cdot d\vec{S}$, where $\vec{F}(x, y, z) = \langle 9y, -x, z \rangle$ and $\mathcal{S}$ is the half of the ellipsoid $x^2 + 9y^2 + 4z^2 = 16$ with $y > 0$, oriented with normals pointing in the positive y-direction.

2. Let C be the edges of the parallelogram with vertices $(0, 0, 0)$, $(1, 2, 3)$, $(2, 5, 3)$, and $(1, 3, 0)$, traveled counterclockwise when viewed from above. Let $\vec{F}(x, y, z) = \langle y, z^2, x \rangle$. Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$.

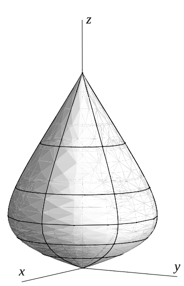
3. True or false: If C is any curve in $\mathbb{R}^3$ and $\vec{F}$ is a vector field on $\mathbb{R}^3$ whose curl is $\vec{0}$, then Stokes' Theorem implies that $\int_C \vec{F} \cdot d\vec{r} = 0$.

4. Let $\mathcal{S}$ be the part of the plane $y = z$ contained inside the ellipsoid $4x^2 + y^2 + z^2 = 8$, oriented with normals pointing upward. Evaluate $\iint_{\mathcal{S}} \operatorname{curl} \vec{F} \cdot d\vec{S}$, where $\vec{F}(x, y, z) = \langle 2x, z, 2x^2 + y^2 \rangle$.

5. Evaluate $\iint_{\mathcal{S}} \operatorname{curl} \vec{F} \cdot d\vec{S}$, where $\vec{F}(x, y, z) = \langle y^2 + \sin yz, xz^3, x(y^2 + 1)^4 \rangle$ and $\mathcal{S}$ is the ellipsoid $x^2 + 4y^2 + z^2 = 16$.

6. Evaluate $\iiint_E \operatorname{div} \vec{F} \, dV$, where $\vec{F}(x, y, z) = \langle x^2 y, -xy^2, -z \rangle$ and E is the solid ball $x^2 + y^2 + z^2 \leq 1$.

7. Find the volume of the solid enclosed by the parameterized surface $\vec{r}(u, v) = \langle (u - u^2) \cos v, (u - u^2) \sin v, u + u^4 \rangle$, $0 \leq u \leq 1$, $0 \leq v \leq 2\pi$.



8. Let $\mathcal{S}$ be the part of the sphere $x^2 + y^2 + (z - 1)^2 = 2$ above the plane $z = 0$, oriented so that the unit normal vector at $(0, 0, 1 + \sqrt{2})$ is $\langle 0, 0, 1 \rangle$. Let $\vec{F}(x, y, z) = \langle 2xyz \cos z^2, -xy \sin xz, -\cos xz - y \sin z^2 \rangle$. Evaluate $\iint_{\mathcal{S}} \vec{F} \cdot d\vec{S}$.

Extra Problems on Double and Triple Integrals

1. A flat plate is in the shape of the region $\mathcal{R}$ described by $x^2 + y^2 \leq 4$. Its density at the point (x, y) is $f(x, y) = x^2 - 2x + 2y^2 + 2$, measured in grams per cubic cm.

   (a) Which point of the plate has the highest density? What is the density at that point?

   (b) Which point of the plate has the lowest density? What is the density at that point?

   (c) Find the mass of the plate.

   (d) Is your answer to (c) consistent with your answers to (a) and (b)? Explain.

2. A plot of land is shaped like the triangular region $\mathcal{R}$ with vertices $(0, 0)$, $(2, 0)$, and $(2, 3)$.

   (a) Vegetation density at the point (x, y) of the plot is given by the function $f(x, y) = y$. Use a double integral to find the amount of vegetation in the plot.

   (b) If you took Math 1b last semester, then you should be able to use the method of slicing to write a single integral that computes the amount of vegetation in the plot. How does that single integral relate to the double integral you found in (a)?

   (c) Suppose the vegetation density at the point (x, y) is instead given by the function $f(x, y) = ye^{x^3}$. Compute the amount of vegetation in the plot.

   (d) If you took Math 1b, can you do (c) using just what you learned in 1b? Why or why not?

3. Find the surface area of the portion of the plane $4x - y - 4z = 1$ which lies inside the paraboloid $y = x^2 + 4z^2$.

4. Suppose we'd like to know the surface area of the piece of the sphere $\rho = 10$ (in spherical coordinates) with $\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} + 0.2$ and $\frac{\pi}{6} \leq \phi \leq \frac{\pi}{6} + 0.1$. Which of the following is the best estimate? (Hint: You don't need to do any integrating here; instead, think about how we came up with the surface area formula.)

   0.1   1   5   50

5. Let $\mathcal{U}$ be the solid enclosed by the surfaces $z = 0$, $z = x^2y + 1$, $y = x^2$, and $y = 2 - x$. In this problem, you'll find the volume of $\mathcal{U}$ in two different ways.

   (a) Write a double integral that gives the volume of $\mathcal{U}$. Convert this to an iterated integral in the order $dy\ dx$.

   (b) Write a triple integral that gives the volume of $\mathcal{U}$. Convert this to an iterated integral in the order $dz\ dy\ dx$. How does this integral relate to the one you wrote down in (a)?

6. Let $\mathcal{U}$ be the solid bounded by $y = x^2 + 4z^2$ and $y = 4$. Write an iterated integral which gives the volume of $\mathcal{U}$.(1)

(1) You need not evaluate; it's a bit of a pain, no matter what order you use. However, if you know the formula for the area of an ellipse, you should be able to show that the volume is 4π.

7. Find the volume of the solid that lies within the sphere $x^2 + y^2 + z^2 = 4$, above the xy-plane, and below the cone $z = \sqrt{x^2 + y^2}$.

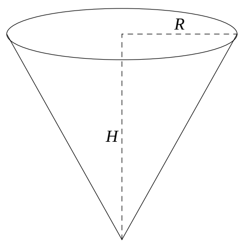
8. Let $\mathcal{U}$ be the solid $\sqrt{x^2 + y^2} \leq z \leq 4 - \sqrt{x^2 + y^2}$. The density of $\mathcal{U}$ at the point (x, y, z) is given by the function $f(x, y, z) = z$.

   (a) Let $\mathcal{S}$ be the surface of $\mathcal{U}$. Find the surface area of $\mathcal{S}$.

   (b) Find the volume of $\mathcal{U}$.

   (c) Find the mass of $\mathcal{U}$.

9. In this problem, you'll look at a right circular cone of height H and base radius R.



   (a) Show that the surface area of the cone is $\pi R\sqrt{R^2 + H^2} + \pi R^2$.

   (b) Show that the volume of the cone is $\frac{1}{3}\pi R^2 H$.

Practice Problems for Week 7 (and Earlier)

1. True or false:

   (a) The level sets of the function $f(x, y, z) = x^2 + y^2$ are circles.

   (b) The graph of the function $f(x, y, z) = x^2 + y^2 + z^2$ is a sphere.

   (c) The level sets of the function $f(x, y, z) = x + y^2 + z^2$ are elliptic paraboloids.

2. *One of the most confusing topics in Math 21a is using gradients to find tangent planes of graphs of functions $f(x, y)$. This problem looks at a simpler, analogous case — using a gradient to find the tangent line of the graph of a function $f(x)$.*

   Consider the parabola $y = x^2$ (in $\mathbb{R}^2$). Use the fact that the gradient of a function is perpendicular to level sets of the function to find the line tangent to the parabola at $(3, 9)$. Be thorough and precise in your explanation: what function are you using the gradient of? How many variables is it a function of?

   Use single-variable calculus to check that you have the right equation for the tangent line.

3. We know two ways of finding the tangent plane for a surface. (Can you explain both of them?) Let S be a surface described as $z = f(x, y)$; that is, S is the graph of $f(x, y)$. Find the plane tangent to S at $(a, b, f(a, b))$ using both methods. Make sure your answers agree with each other.

4. In each part, $f(x, y)$ is a function of two variables and P is a point in $\mathbb{R}^2$. Based on the given information, decide which conclusion you can draw from the given information. Here are the choices:

   (A) P is a local minimum of f.

   (B) P is a local maximum of f.

   (C) P is a saddle point of f.

   (D) P is a critical point of f, but there is not enough information to determine what kind of critical point.

   (E) P is not a critical point of f.

   (F) There is not enough information to determine whether P is a critical point of f.

   Questions:

   (a) $\nabla f(P) = \vec{0}$, $f_{xx}(P) = 1$, $f_{yy}(P) = 2$, and $f_{xy}(P) = 3$.

   (b) $\nabla f(P) = \vec{0}$, $f_{xx}(P) = 1$, $f_{yy}(P) = 2$, and $f_{xy}(P) = 1$.

   (c) $\nabla f(P) = \vec{0}$, $f_{xx}(P) = 2$, $f_{yy}(P) = -1$.

   (d) $\nabla f(P) = \vec{0}$, $f_{xx}(P) = -2$, $f_{yy}(P) = -1$.

   (e) $f_x = 0$ and $D_{\vec{u}} f(P) = 1$, where $\vec{u} = \left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle$.

   (f) $D_{\vec{u}} f(P) = 0$ and $D_{\vec{v}} f(P) = 0$, where $\vec{u} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$ and $\vec{v} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$.

(g) $D_{\vec{u}}f(P) = 0$ and $D_{\vec{v}}f(P) = 0$, where $\vec{u} = \langle \frac{3}{5}, \frac{4}{5} \rangle$ and $\vec{v} = \langle -\frac{3}{5}, -\frac{4}{5} \rangle$.

(h) $D_{\vec{u}}f(P) = 0$, $D_{\vec{v}}f(P) = 0$, $[D_{\vec{u}}(D_{\vec{u}}f)](P) = 1$, and $[D_{\vec{v}}(D_{\vec{v}}f)](P) = 1$, where $\vec{u} = \left\langle \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \right\rangle$ and $\vec{v} = \left\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle$.

(i) $D_{\vec{u}}f(P) = 0$, $D_{\vec{v}}f(P) = 0$, $[D_{\vec{u}}(D_{\vec{u}}f)](P) = 1$, and $[D_{\vec{v}}(D_{\vec{v}}f)](P) = -1$, where $\vec{u} = \left\langle \frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}} \right\rangle$ and $\vec{v} = \left\langle -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle$.

(j) $f_{xx}(P) = 5$ and $[D_{\vec{u}}(D_{\vec{u}}f)](P) = -1$, where $\vec{u} = \left\langle \frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}} \right\rangle$.

5. (a) Find the maximum and minimum values of $f(x, y, z) = x^2 + y^2 + 2z^2$ subject to the constraint that $x^2 + xy + y^2 + z^2 = 1$. (You may assume that the maximum and minimum values are in fact attained.)

   (b) Find the maximum and minimum values of $f(x, y, z) = x^2 + y^2 + 2z^2$ subject to the constraint that $x^2 + y^2 + z^2 = 1$. (The maximum and minimum values are in fact attained: why?)

6. For which of the regions D described below is it true that every continuous function $f(x, y)$ must attain an absolute maximum value and absolute minimum value on D? (There may be more than one.)

   (a) D is the set of points (x, y) such that $|x| \leq 4$ and $|y| < 2$.

   (b) D is the set of points (x, y) such that $|x + y| \leq 1$.

   (c) D is the set of points (x, y) such that $x^2 + 4y^2 \leq 1$.

   (d) D is the set of points (x, y) such that $x^2 + 4y \leq 1$.

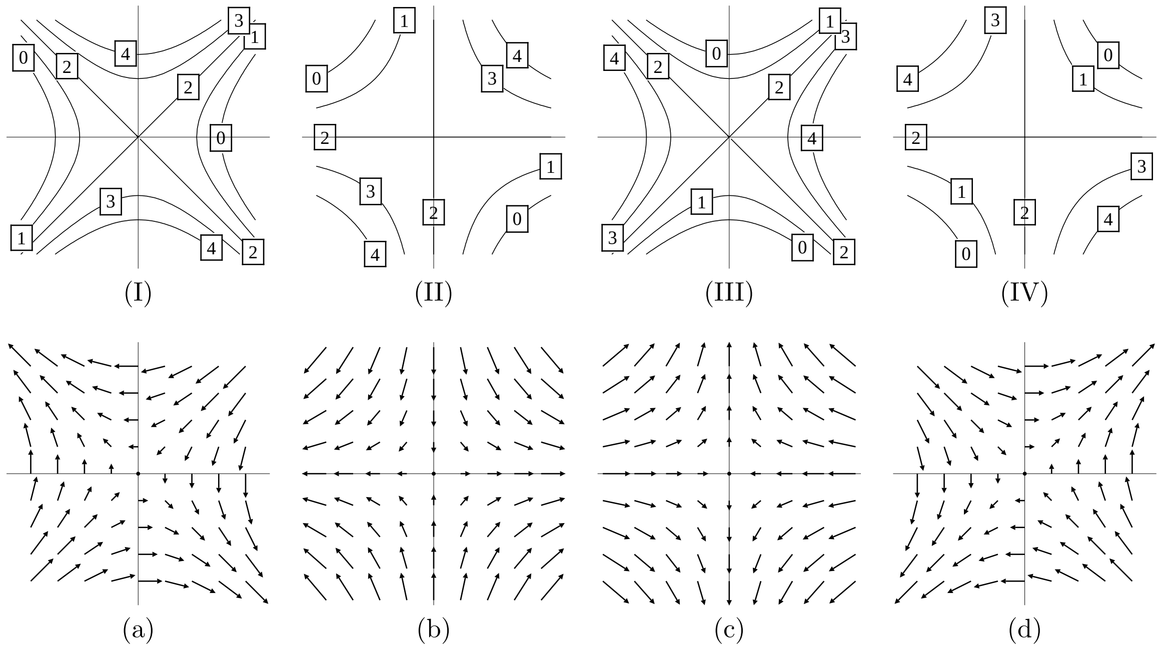
   (e) D is the set of points (x, y) such that $-x \leq y \leq x$.

   For each region D that you picked, find the absolute mininimum and absolute maximum value of $f(x, y) = x^2 - 4x + y^2$ on the region.

7. The temperature in a room is described by the function $T(x, y, z) = x^2y + z$. A bug is walking on a surface in the room, which can be described parametrically by $\vec{r}(u, v) = \langle u, e^v, u + v \rangle$, $0 \leq u \leq 1$, $0 \leq v \leq 1$. What is the warmest point the bug can reach? What is the coolest?

Extra Problems on Sections 13.1 - 13.4

1. The first row shows level set diagrams (contour maps) of four functions f. The second row shows the vector fields ∇f for the same four functions. Match each level set diagram with the corresponding vector field. Explain your reasoning.



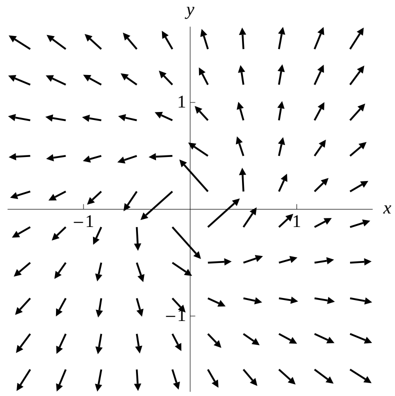
2. Let $\vec{F}$ be the vector field on $\mathbb{R}^2$ defined by $\vec{F}(x,y) = \langle 2xy, x^2 - \sin y\rangle$.

   (a) Let (a,b) be any point in $\mathbb{R}^2$, and let C be the straight-line path from $(0,0)$ to (a,b). Parameterize C, and use your parameterization to compute $\int_C \vec{F} \cdot d\vec{r}$.

   (b) Show that $\vec{F}$ is a gradient vector field.

   (c) Find a function f such that $\nabla f = \vec{F}$. (How does your answer relate to your answer from (a)?)

3. Consider the vector field $\vec{F}(x,y) = \langle P(x,y), Q(x,y)\rangle = \left\langle -\frac{y}{x^2+y^2} + x, \frac{x}{x^2+y^2} + y \right\rangle$ from #4(b) on the worksheet "The Fundamental Theorem for Line Integrals; Gradient Vector Fields". In class, we did not decide conclusively whether this vector field was conservative. Now, show that it is in fact **not** conservative.



4. In each part, evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$, using any method you like.

(a) $\vec{F}(x,y) = \langle y, -x \rangle$ and C is the ellipse $4x^2 + 9y^2 = 16$, oriented counterclockwise.

(b) $\vec{F}(x,y) = \left\langle \tan(x^3), e^{y^2} \right\rangle$ and C is the circle $(x-2)^2 + y^2 = 1$, oriented counterclockwise.

(c) $\vec{F}(x,y) = \langle y, x \rangle$ and C is the path parameterized by $\langle \sin 5t + \cos 3t, \sin 4t + \sin^2 2t + \cos 3t + \cos^2 t \rangle$, $0 \le t \le \pi$.

(d) $\vec{F}(x,y) = \langle e^{x^3}, y^2 \rangle$ and C is the right half of the ellipse $9x^2 + y^2 = 1$, oriented counterclockwise.

(e) $\vec{F}(x,y) = \langle 3x^2 y \sin(x^3) + y + 2, -\cos(x^3) \rangle$ and C is the top half of the circle $x^2 + y^2 = 1$, oriented counterclockwise.