9 Vectors and the Geometry of Space

9.1 Three-Dimensional Coordinate Systems

Suggested Time and Emphasis

$\frac{1}{2}$–$\frac{3}{4}$ class   Essential material

Points to Stress

1. The basics of points in three-dimensional space, including notation and the distance formula.
2. Equations of planes parallel to one of the coordinate planes.
3. The equation of a sphere.

Text Discussion

- Explain why the equation $y = x$ is the equation of a *plane* in three-dimensional space.
- How do we use the distance formula to get the equation of a sphere?

Materials for Lecture

- If possible, mark off one corner of the lecture room with electrical tape. Determine the coordinates of various students in the room, and/or find the equation of the plane of the chalkboard.
- Describe the unit cube $\{(x, y, z) \mid 0 \leq x, y, z \leq 1\}$ and its surfaces $\{(x, y, z) \mid x = 0, 0 \leq y \leq 1, 0 \leq z \leq 1\}$, $\{(x, y, z) \mid x = 1, 0 \leq y \leq 1, 0 \leq z \leq 1\}$, and so forth.
- Explain why the equation $z = \sqrt{R^2 - (x^2 + y^2)}$ represents the hemisphere $z \geq 0$, while the equation $x^2 + y^2 + z^2 = R^2$ represents a full sphere.
- Introduce the equation of a circular cylinder as an extension to $\mathbb{R}^3$ of the equation of the circle $x^2+y^2 = r^2$.
- Use inequalities to describe the quarter-unit sphere above the xy-plane and to the right of the yz-plane. Then use inequalities to describe the eighth of the unit sphere in the first octant.

Workshop/Discussion

- Describe some cylindrical surfaces such as $y = x^2$, $y = \sin x$, and $z = 1 - x^2$.
- Determine the curves given by the intersection of the sphere $x^2 + y^2 + z^2 = R^2$ with the various plane $x = k$, $y = k$, and $z = k$, indicating the necessary restrictions on k.
- Describe the set of points whose distance from each of $(0, 0, 1)$ and $(0, 0, -1)$ is 1. Then similarly discu the set of points whose distance from these points is 2, and then 0.5. Have the students explain why t set of all points equidistant in space from two given points is a plane.
- Describe the region in $\mathbb{R}^3$ determined by the equation $xy = 0$, then do the same for the inequality $xy >$

Group Work 1: Working with Surfaces in Three-Dimensional Space

Visualizing and sketching surfaces in three-dimensional space can be difficult for students. It is recommended that the answer key be distributed after the students have had time to work.

Group Work 2: Lines, Lines, Everywhere Lines

The idea is to help students visualize the triangular surface $x + y + z = 1, x, y, z \geq 0$. Note that at this stage, the students have not yet learned about the general equations of lines and planes. Question 3 is ambitious, and requires some formal thinking. Depending on the mathematical maturity of your class, you may want to either give a hint to the students, or perhaps do that part as a class. Questions 3 and 4 are printed on a separate page, so you can decide what to do based upon how the students are doing on the first two parts.

Group Work 3: Equidistance

Begin by having the students try to visualize and sketch in $\mathbb{R}^2$ the set of all points equidistant from a given point and a given line, starting with the set of all points equidistant from $(0, 2)$ and the x-axis.
Next, discuss how we can use algebraic equations to define an object described in words in this way. For example, the set of all points in the xy-plane equidistant from the point $(0, 1)$ and the line $y = -1$ must satisfy the equation $(y - 1)^2 + x^2 = (y + 1)^2$ or $y = \frac{1}{4}x^2$.
Now have the students compute an equation for the set of all points in space equidistant from the point $(0, 0, 2)$ and the xy-plane. They should graph their equation (a paraboloid) using technology to see if it matches their intuition. Then have them compute and graph the set of all points in space equidistant from the point $(0, 0, 2)$ and the x-axis (a parabolic cylinder). Note here that if P is the point (x, y, z), then the distance from P to the x-axis is $\sqrt{y^2 + z^2}$.

Discovery Project: Fun with Visualization

Ask the students to turn off their calculators and put down their pencils. Start by asking them to picture a plane in $\mathbb{R}^3$. Ask them how many regions in space are formed by a plane. (Answer: Two regions)
Then ask them how many regions are formed by *two* planes. Notice that there are two possible answers here. If the planes are parallel, they divide space into three regions, like a layer cake. If they cross, then four regions are formed. Have them get into groups to discuss how many regions can be made from three planes. For a shorter exercise, ask them to find only the maximum possible number of regions (8 regions). For a longer one, ask them to figure out all possible solutions (4, 6, 7, or 8 regions). If a group finishes ahead of the others, have them work on the case of four planes (maximum 15 regions.) Discuss the analogy between dividing space by planes and dividing a plane by lines.

Extended Lab Project: The Shapes of Things to Come

If the students have access to a CAS or calculator with three-dimensional graphing capabilities, have them look at surfaces of the form $Ax + By + Cz = 1$ and of the form $Ax^2 + By^2 + Cz^2 = 1$, varying the constants until they get a feel for the variety of shapes that each form can assume. Make sure that they try varying combinations of positive and negative constants. The goal is for the students to note that the first family is a collection of planes, and that the second takes on a variety of shapes.

Homework Problems

Core Exercises: 1, 5, 7, 9, 10, 14, 15, 18, 26

Sample Assignment: 1, 3, 4, 5, 7, 9, 10, 14, 15, 18, 21, 26, 28, 32, 33, 34

Note: Problem 1 of Focus on Problem Solving on page 703 would make a good project for motivated students.

Exercise	C	A	N	G	V
1					×
3	×				
4		×			×
5					×
7	×	×			
9		×			
10		×			

Exercise	C	A	N	G	V
14		×			
15		×			
18	×				×
19–28					×
32	×				
33	×			×	
34	×	×			

Group Work 1, Section 9.1
Working with Surfaces in Three-Dimensional Space

Provide a rough sketch and describe in words the surfaces described by the following five equations. The best way to do this exercise is to think first about the surfaces, before you calculate anything.

1. $x^2 + y^2 = 3^2$

2. $y^2 + z^2 = 1$

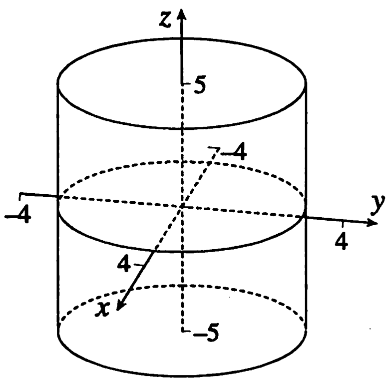
3. $z = y^2$

4. $xy = 1$

5. $x^2 + y^2 = z$

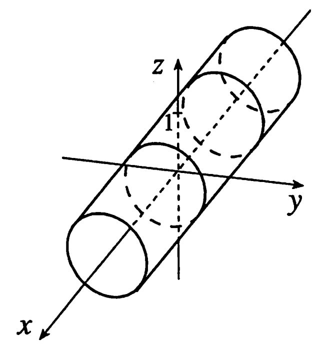
Group Work 1, Section 9.1
Working with Surfaces in Three-Dimensional Space (Solutions)

1. $x^2 + y^2 = 3^2$



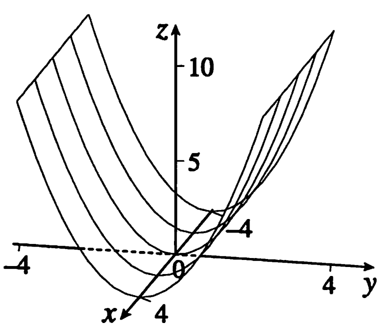
This is a right circular cylinder of radius 3 with axis the z-axis.

2. $y^2 + z^2 = 1$



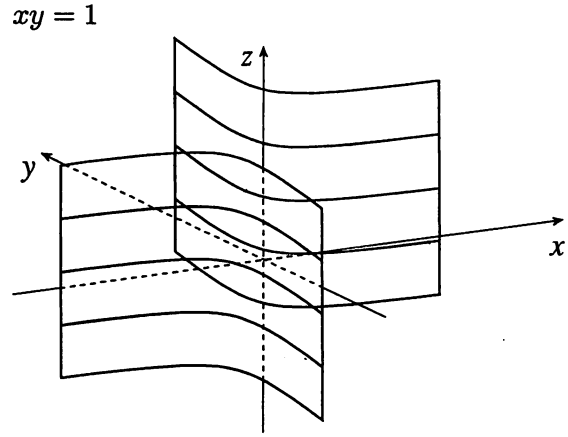
This is a right circular cylinder of radius 1 with axis the x-axis.

3. $z = y^2$



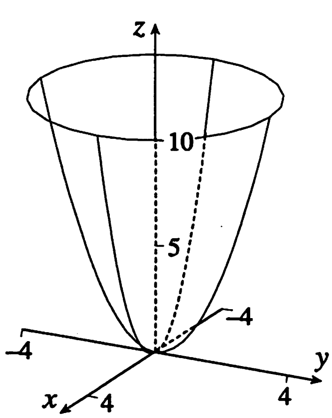
This is a parabolic cylinder parallel to the x-axis.

4. $xy = 1$



This is a hyperbolic cylinder parallel to the z-axis.

5. $x^2 + y^2 = z$



This is a paraboloid opening upward.

Group Work 2, Section 9.1

Lines, Lines, Everywhere Lines

Consider the surface S: $x + y + z = 1$, $x \geq 0$, $y \geq 0$, $z \geq 0$.

1. Show that every point on the line segment in the xy-plane, $x + y = 1$ ($x \geq 0$, $y \geq 0$) is also a point on this surface.

2. Show that every point on the line segments $x + z = 1$ ($x \geq 0$, $z \geq 0$) and $y + z = 1$ ($y \geq 0$, $z \geq 0$) in the xz- and yz-planes is also a point on this surface.

3. Given that two points $P_1\,(x_1, y_1, z_1)$ and $P_2\,(x_2, y_2, z_2)$ are on the surface S, show that their midpoint $P\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$ must also be a point on S.

4. Draw a picture of S.

9.2 Vectors

Suggested Time and Emphasis

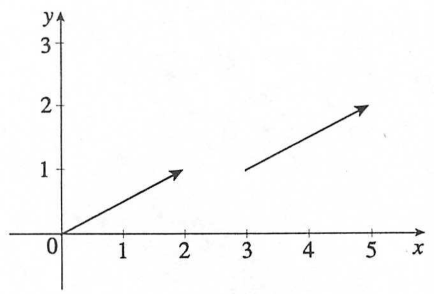
$\frac{1}{2}$–$\frac{3}{4}$ class   Essential material

Points to Stress

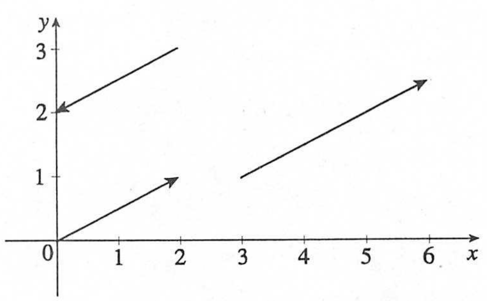
1. The basics of vectors, including the definition, length of vectors, vector addition and scalar multiplication.
2. The relationship between the vector representation $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, the point $P(a_1, a_2, a_3)$, and the position representation $\overrightarrow{OP}$.
3. The standard basis vectors **i**, **j**, and **k**, and general unit vectors.

Text Discussion

- The following two vectors have the same magnitude and direction. Are they the same vector, or two different vectors?



- Are the following three vectors parallel? (Some students can be confused by the definition of parallel vectors.)

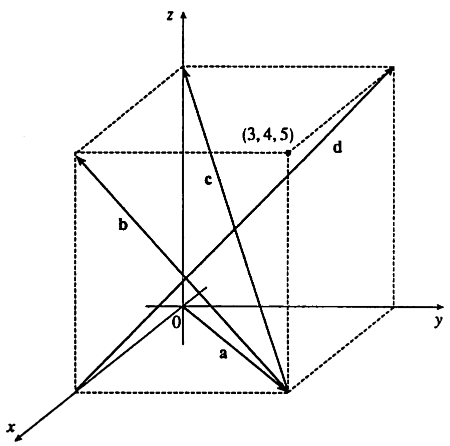


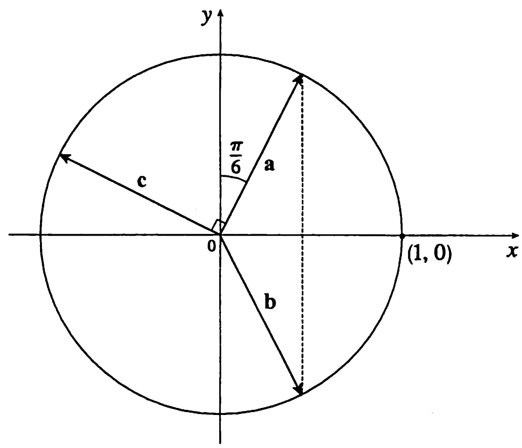
- If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ has $a_2 > 0$ and $a_3 < 0$, then the z-component of $-3\mathbf{a}$ has what sign?
- If $\overrightarrow{AB}$ is a representation of $\mathbf{a} = \langle a_1, a_2 \rangle$ and the initial point is $A\,(x, y)$, what are the coordinates of B?

Materials for Lecture

- Emphasize that the position representation of $\mathbf{a} = \langle a_1, a_2 \rangle$ is $\overrightarrow{OP}$, where P is the point (a_1, a_2) and that the position representation $\overrightarrow{OP}$ for $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ has endpoint $P\,(a_1, a_2, a_3)$. Indicate why $\mathbf{i} + \mathbf{j}$ is not a unit vector, even though **i** and **j** are unit vectors.
- Present geometric examples of the parallelogram law (such as Figure 4 on page 653) and scalar multiplication (such as Figure 7 on page 654).

- For a vector **a**, discuss the vector line $l = \{c\mathbf{a} \mid c \in \mathbb{R}\}$. Then introduce the idea that a line is determined by a point and a vector.
- Foreshadow the process of resolving vectors into components by working the following two problems. In the first, start with a rectangular solid with one corner at $(0,0,0)$ and the other at $(3,4,5)$, and find the component representation $\langle x,y,z\rangle$ of each of the vectors labeled **a**, **b**, **c**, and **d** in the diagram. In the second, start with a unit circle, and an angle of $\frac{\pi}{6}$with respect to the y-axis, and attempt to find the component representations $\langle x,y\rangle$ of the vectors labeled **a**, **b**, and **c**.



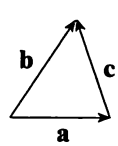


- Expand on Example 7 (page 658) by showing, in detail, how to break a two-dimensional vector into its horizontal and vertical components. One possible example is to calculate the work required to push a 200 lb object up a 30° incline (foreshadowing Section 9.3).

Workshop/Discussion

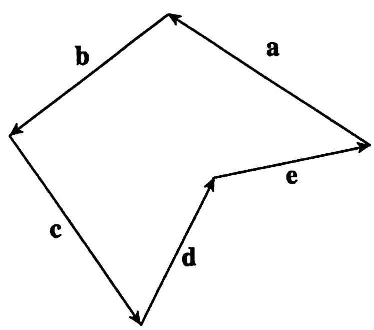
- Expand on the notion of a vector as a quantity with both magnitude and direction. If the students have a background in physics, make a list of quantities such as the ones below and have the students choose "vector" or "scalar" by a show of hands.

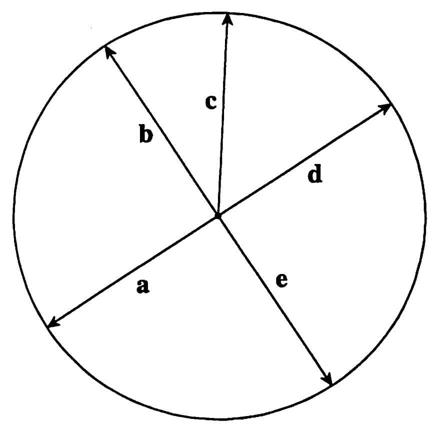
  Examples: speed, velocity, force, work, momentum, energy, friction
- Make sure to reinforce the fact that the location of the initial point of a vector can be chosen at will, and that the position vector uses the origin 0.
- Ask students to represent the vectors $\mathbf{i}+\mathbf{j}$, $\mathbf{i}+\mathbf{k}$, $\mathbf{j}+\mathbf{k}$, and $\mathbf{i}+\mathbf{j}+\mathbf{k}$ geometrically.
- Find unit vectors in the directions of $\langle 8,0,0\rangle$, $\langle 5,5,0\rangle$ and $\langle 1,2,3\rangle$, and explain what is happening geometrically. Then find a unit vector in the direction opposite that of $\langle -1,1,1\rangle$.
- Ask students to describe **c** in terms of **a** and **b**.



- Write **a** as $s\mathbf{b} + t\mathbf{c}$ where $\mathbf{b} = \langle 1,1\rangle$ and $\mathbf{c} = \langle -1,2\rangle$ and **a** is chosen by the students. Repeat for a different vector **a**. Do not stress the arithmetic for this introduction; instead emphasize the fact that it can be done for any choice of **a**. Follow up by letting $\mathbf{b} = \langle 2,4\rangle$ and $\mathbf{c} = \langle 1,2\rangle$, and ask the students to try to express $\langle 1, 1\rangle$ as a linear combination of $\mathbf{b}$ and $\mathbf{c}$. Conclude that some pairs of vectors in $\mathbb{R}^2$ have this "combining" property while others do not, and mention that we will get back to this concept.

- Draw the following two diagrams on the board, and compute $\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d} + \mathbf{e}$ for each of them. (Answers: $\mathbf{0}$, $\mathbf{c}$)





Group Work 1: The Position Vector

Group Work 2: Where Do They Point?

This group work extends the idea of adding several vectors without coordinates. For Problem 1, the students can be given the hint to first try computing $\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d} + \mathbf{e}$. The answers are $-2\mathbf{e}$ for Problem 1, $\left\langle -\frac{1}{2}r, -\frac{\sqrt{3}}{2}r \right\rangle$ for Problem 2(a), and $-(\mathbf{a} + \mathbf{d}) = \langle -r, 0\rangle$ for Problem 2(b), since $\mathbf{a} + \mathbf{b} = \langle r, 0\rangle$.

Group Work 3: The Return of Geometry

This is a challenging group work for stronger students. Question 1 is based on Exercise 37. It is best to have some groups do Question 1 first, and some start with Question 2. A group that finishes early can start on the other problem. Before starting the exercise, it may be helpful to remind the students of the definitions of "midpoint" and "bisect". An advanced class may not need the handout sheets with the diagrams; they may just need to have the problem statement written on the board. Other classes might need hints as they go along, given that proofs of this type will be new to most of the students.

After they have started Question 2, you may want to announce that $r\mathbf{a} + s\mathbf{b} = \mathbf{0}$ if and only if $r = s = 0$ (since $\mathbf{a}$ and $\mathbf{b}$ are not parallel).

Lab Project: Turning a Vector

Start by having the students use paper and pencil to rotate a vector $\langle a, b\rangle$ by 180°, 90°, and 45°. Then, have them use a CAS to explore general rotations and scaling of vectors (for example, replacing $\langle a, b\rangle$ by $\langle a + 1, b + 1\rangle$ and noticing that the resultant vector is not in general a scaling of $\langle a, b\rangle$).

Homework Problems

Core Exercises: 4(a), 4(d), 5, 6(a), 6(d), 6(f), 9, 12, 18, 19, 23, 24

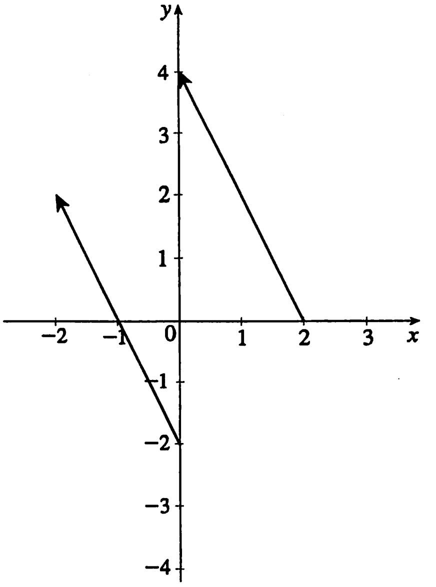
Sample Assignment: 4, 5, 6, 9, 12, 15, 18, 19, 23, 24, 33

Exercise	C	A	N	G	V
4					×
5					×
6					×
9		×			×
12		×			×
15		×			

Exercise	C	A	N	G	V
18		×			
19		×			
23		×		×	
24	×	×			
33	×				×

Group Work 1, Section 9.2
The Position Vector

Recall that a vector, say $\langle -2, 4\rangle$, is always the same vector no matter where it is placed in space (or on the plane).



Also recall that we call a particular representation of a vector (namely the one starting at the origin) the *position vector.* Let **a** be the two-dimensional vector $\langle -2, 4\rangle$.

1. Draw the position vector of **a** and find the angle that it makes with the positive x-axis.

2. Show that the tip of the position vector of $\frac{\mathbf{a}}{|\mathbf{a}|}$ is on the unit circle $x^2 + y^2 = 1$, and illustrate this fact on the diagram.

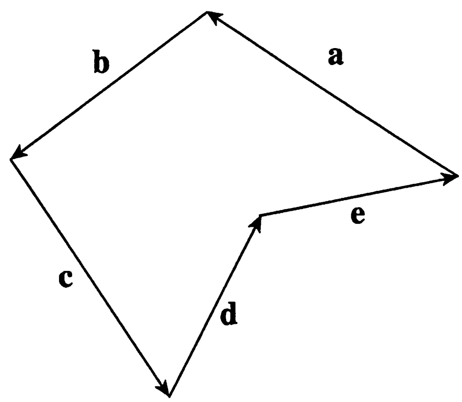
3. Now consider a general vector $\mathbf{b} \neq \mathbf{0}$ whose position vector makes an angle θ with the positive x-axis. Show that the unit vector in the direction of **b** is $\frac{\mathbf{b}}{|\mathbf{b}|} = \langle \cos\theta, \sin\theta\rangle$.

4. Explain why, no matter what b is ($b \neq 0$), the tip of the position vector of $\frac{b}{|b|}$ is on the unit circle.

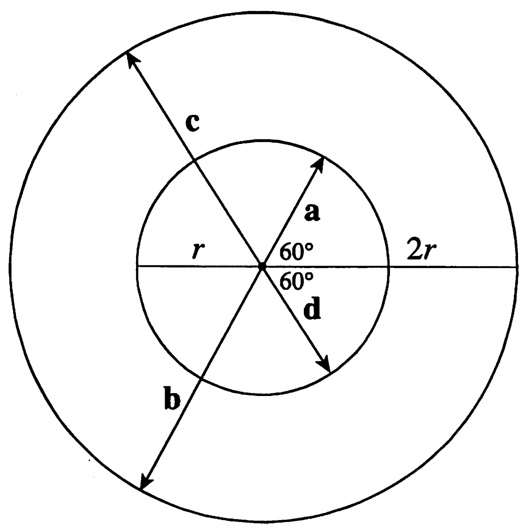
5. Consider the three-dimensional position vector $b = \langle 1, 1, 1 \rangle$. Can you make a statement, similar to what was done in Problem 2, about the tip of the position vector of $\frac{b}{|b|}$? Which vector has greater magnitude, b or $\frac{b}{|b|}$? Answer the same questions for $c = \langle \frac{1}{2}, 0, \frac{1}{3} \rangle$.

Group Work 2, Section 9.2
Where Do They Point?

1. Compute a + b + c + d − e for the following diagram.



2. (a) Compute the position representation $\overrightarrow{OP}$ for a + b in the following diagram. Give the coordinates of the point P.

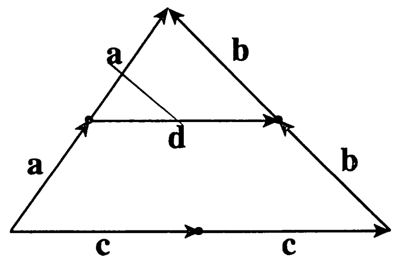


(b) Compute a + b + c + d.

Group Work 3, Section 9.2
The Return of Geometry

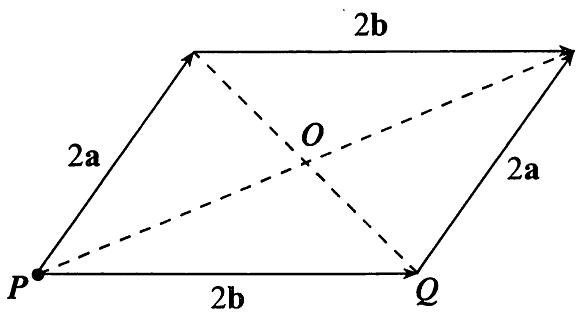
Vectors sometimes make it easier to prove geometric properties. In this exercise, you will get the chance to prove some useful geometric formulas using your knowledge of vectors.

1. Use vectors to show that the a line segment connecting the midpoints of two sides of a triangle is half as long as the opposite side, and also parallel to the opposite side.



Hint: You need to show that $\mathbf{d} = \mathbf{c}$. One way to do it is to express $\mathbf{c}$ in terms of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{d}$ in two different ways.

2. Use vectors to show that the diagonals of a parallelogram bisect each other



Hint: Let $\mathbf{c}$ be the vector from point P to point O, and let $\mathbf{d}$ be the vector from point Q to point O. Then there exist numbers k and l such that $\mathbf{c} = k(\mathbf{a} + \mathbf{b})$ and $\mathbf{d} = l(\mathbf{a} - \mathbf{b})$. Show that $k = l = 1$.

9.3 The Dot Product

Suggested Time and Emphasis

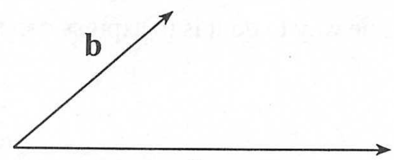
1 class   Essential Material

Points to Stress

1. The algebraic and geometric formulations of the dot product.
2. The interpretation of the sign of $\mathbf{a} \cdot \mathbf{b}$ in terms of the angle between $\mathbf{a}$ and $\mathbf{b}$.
3. Orthogonal vectors.
4. Vector and scalar projections.

Text Discussion

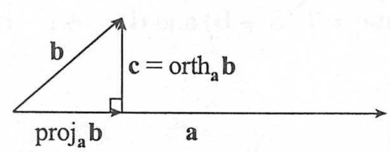
- Consider these two vectors:



Draw the vector projection of $\mathbf{b}$ onto $\mathbf{a}$, and the vector projection of $\mathbf{a}$ onto $\mathbf{b}$.

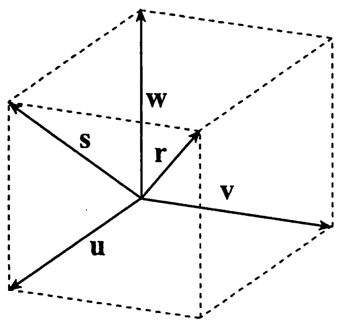
Materials for Lecture

- Another approach to the dot product is to introduce it algebraically using the coordinates of vectors ($\langle a_1, a_2, a_3 \rangle \cdot \langle b_1, b_2, b_3 \rangle = a_1 b_1 + a_2 b_2 + a_3 b_3$). Assert that the result is the same even if the vectors are translated (this is clear) or rotated (this is less so, but special cases would be discussed below to justify the assertions.) Then show that if two vectors in $\mathbb{R}^2$ are rotated so that one is lying on the x-axis, we obtain $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos \theta$.
- Suppose $\mathbf{a} \neq \mathbf{0}$ and $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$. Ask the class if it is reasonable to assume that $\mathbf{b} = \mathbf{c}$. Then show that this is not the case; that we only know that $\mathbf{b}$ and $\mathbf{c}$ have the same projection onto $\mathbf{a}$. Perhaps use the example where $\mathbf{a} = \langle 1, 2 \rangle, \mathbf{b} = \langle 2, 1 \rangle, \mathbf{c} = \langle 4, 0 \rangle$. Then convince the students that $\mathbf{a} \perp (\mathbf{b} - \mathbf{c})$ and give a geometric interpretation.
- Elaborate on Exercises 25 and 26 as follows: Referring to the figure below, conclude that if $\mathbf{a} \neq \mathbf{0}$ and $\mathbf{b} \neq \mathbf{0}$, we can write $\mathbf{b} = \mathbf{c} + \text{proj}_{\mathbf{a}} \mathbf{b}$, with $\mathbf{c} \perp \mathbf{a}$ and $\text{proj}_{\mathbf{a}} \mathbf{b} \parallel \mathbf{a}$. $\mathbf{c}$ is called the orthogonal projection of $\mathbf{b}$ onto $\mathbf{a}$.



- The definition $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos \theta$ implies that the dot product is invariant under rotations. Illustrate this fact by choosing unit vectors $\mathbf{a}_1 = \left\langle \cos \frac{\pi}{6}, \sin \frac{\pi}{6} \right\rangle$, $\mathbf{b}_1 = \left\langle \cos \frac{3\pi}{2}, \sin \frac{3\pi}{2} \right\rangle$, $\mathbf{a}_2 = \left\langle \cos\left(\frac{\pi}{6} + \alpha\right), \sin\left(\frac{\pi}{6} + \alpha\right) \right\rangle$, and $\mathbf{b}_2 = \left\langle \cos\left(\frac{3\pi}{2} + \alpha\right), \sin\left(\frac{3\pi}{2} + \alpha\right) \right\rangle$, and showing that $\mathbf{a}_1 \cdot \mathbf{b}_1 = \mathbf{a}_2 \cdot \mathbf{b}_2$ regardless of α.

- Discuss Exercise 37. If possible, bring in a model tetrahedron.
- Compute **r**, **s**, and **r** · **s** if **u**, **v**, and **w** are mutually orthogonal unit vectors.



Workshop/Discussion

- Go over Exercise 1, perhaps allowing students to answer each question by a show of hands before going through the solutions.
- Compute the angle between the vectors $\langle -1, 1\rangle$ and $\langle \sqrt{3}, 1\rangle$.
- Compute values for a such that $\mathbf{a} = \langle 1, a, -1\rangle$ and $\mathbf{b} = \langle -1, 1, 1\rangle$ (a) make an angle of 45°, and (b) satisfy $\mathbf{a} \perp \mathbf{b}$.
- Go over two methods of solving Exercise 33, first using vectors, and then using calculus to minimize the distance from (x_1, y_1) to $\left(x_0, \dfrac{-ax_0 - c}{b}\right)$.
- Show that if $\mathbf{a} \perp \mathbf{b}$ and $\mathbf{a} \perp \mathbf{c}$ then $\mathbf{a} \perp (r\mathbf{b} + s\mathbf{c})$ for any real numbers r and s.
- Bring in a model square pyramid. Compute the angle between the faces, and then compute the angle between an edge and a face.

Group Work 1: The Right Stuff

Give each group of students a different set of three points, and have them use vectors to determine if they form a right triangle. They can do this either using dot products, or by calculating side lengths and using the Pythagorean Theorem. Perhaps have the students with the points in $\mathbb{R}^2$ carefully graph their points to provide a visual check. Point out that using the dot product is usually the easier method for points in $\mathbb{R}^3$.

Sample triples:

$(-2,-1), (-2,8), (8,-1)$	$(3,4), (3,12), (6,5)$
$(0,0), (10,7), (-14,20)$	$(2,1,2), (3,3,1), (2,2,4)$
$(-1,-2,-3), (0,0,-4), (-1,-1,-1)$	$(2,3,6), (3,4,7), (3,3,6)$

Group Work 2: The Regular Hexagon

If the students have trouble with this one, copy the figure onto the blackboard. Then draw a point at its center, and draw lines from this point to every corner point. This modified figure should make the exercise more straightforward.

Group Work 3: Gravity's Rainbow

Depending upon the background of a class, a detailed introduction to this exercise may be necessary. Students with a good physics background could anticipate the final answer. The idea is that since the constant gravitational force points downward, work is done against gravity only while the student is moving upward. In other words, horizontal movement does *no* work against gravity, and vertical movement results in the maximum work against gravity. The students should discover that the work done in going from point A to point D is independent of the path taken.

Homework Problems

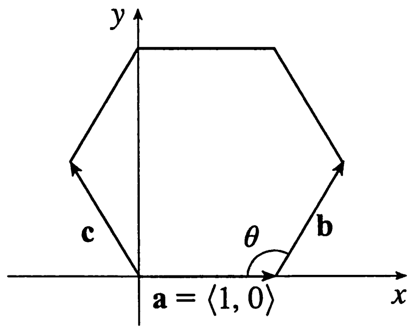
Core Exercises: 1, 3, 5, 9, 17(a), 17(d), 22, 26, 29

Sample Assignment: 1, 2, 3, 5, 6, 9, 10, 14, 17, 18, 22, 26, 29, 33, 35

Exercise	C	A	N	G	V
1	×				
2		×			
3–8		×			
9		×			×
10		×			×
14		×			
17		×			
18		×			
22		×			
26		×			×
29		×			
33	×	×			
35		×			×

Group Work 2, Section 9.3
The Regular Hexagon

Consider the following regular hexagon:



1. Compute $|\mathbf{a}|$, $|\mathbf{b}|$, and $|\mathbf{c}|$.

2. What is the angle θ?

3. What is $\mathbf{a} \cdot \mathbf{b}$?

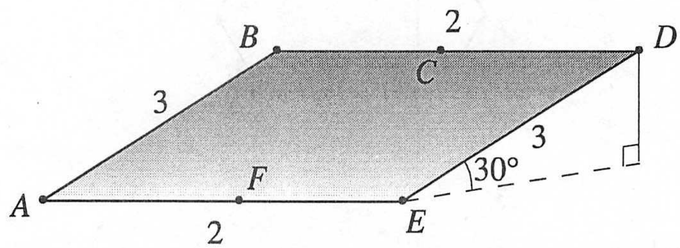
4. What is $\mathbf{a} \cdot \mathbf{c}$?

5. What are $\text{proj}_{\mathbf{a}}\, \mathbf{b}$ and $\text{proj}_{\mathbf{b}}\, \mathbf{c}$?

6. What is the x-component of $\mathbf{a} + \mathbf{b} + \mathbf{c}$?

Group Work 3, Section 9.3
Gravity's Rainbow

You are standing at the base of a hill. The hill is very broad and flat, and has a constant 30° incline. It is... Oh, what the heck, we won't lie to you, it's not a hill, it's just a plain old inclined plane. Let's start again. You are standing at the base of a big inclined plane, at the point marked A. You need to walk to a point D, two miles to the west, and three miles up the plane, as illustrated by the diagram below. The plane has a 30° incline.



We assume that gravity always points straight down, with a constant force G.

Now compute the amount of work done against gravity in walking from A to D along the following paths:

1. First walking up the hill to point B, and then walking across to point D.

2. First going across the plane to point E, and then up to point D.

3. First going halfway between A and E, to point F, then walking up to point C, and then over to point D.

4. First going to a point halfway between A and B, walking straight across, and then up to point D.

5. Walking along the diagonal between A and D. (Actually do the calculation.)

6. Walking along your own zany path going upward from A to D.

9.4 The Cross Product

Suggested Time and Emphasis

$\frac{3}{4}$–1 class   Essential material

Note: The scalar triple product comes up in the derivation of Kepler's Laws in Section 10.4, but can be omitted if this section is not to be covered.

Points to Stress

1. The cross product defined as a vector perpendicular to two given vectors, whose length is the area of the parallelogram determined by the vectors, and its algebraic interpretation.
2. The right-hand rule and properties of the cross product.
3. The scalar triple product as the volume of a parallelepiped.

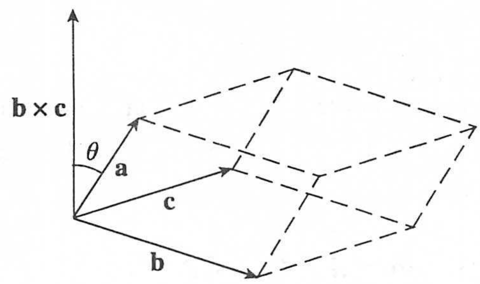
Text Discussion

- If $\mathbf{a} \perp \mathbf{b}$, what is the length of $\mathbf{a} \times \mathbf{b}$? What can we say about $(\mathbf{a} \times \mathbf{b}) + (\mathbf{b} \times \mathbf{a})$?
- Why is it that if $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0$ then $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are coplanar?

Materials for Lecture

- Point out that one can define $\mathbf{a} \times \mathbf{b}$ as $\mathbf{a} \times \mathbf{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$. From this definition, you can show directly that $\mathbf{a} \times \mathbf{b}$ is mutually orthogonal to $\mathbf{a}$ and $\mathbf{b}$, and that $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}| \sin\theta$.
- Pose the problem of finding a vector perpendicular to two given vectors (without considering the length of the resultant vector). Show that there is an obvious solution for $\langle 1,0,0 \rangle$ and $\langle 0,1,0 \rangle$. Then look at the two vectors $\langle 1,0,1 \rangle$ and $\langle 0,1,1 \rangle$. Repeat for $\langle -3,1,-7 \rangle$ and $\langle 0,-5,-5 \rangle$. Point out that in this case an alternate algebraic solution using dot products gives $-3x + y - 7z = -5y - 5z$, $y = -z$, $x = -\frac{8}{3}z$, and so $\mathbf{c} = \left\langle -\frac{8}{3}, -1, 1 \right\rangle$ works. Note that $\mathbf{c}$ is a scalar multiple of $\mathbf{a} \times \mathbf{b} = \langle -40, -15, -15 \rangle$. This suggests the general idea behind the cross product.
- Point out that while $\mathbf{a} \times \mathbf{b}$ and $\mathbf{b} \times \mathbf{a}$ have the same length, $\mathbf{b} \times \mathbf{a}$ points in the opposite direction to that of $\mathbf{a} \times \mathbf{b}$, by the right-hand rule. Thus $\mathbf{b} \times \mathbf{a} = -(\mathbf{a} \times \mathbf{b})$.
- If discussing torque, have a strong student stand on one side of a swinging door, while you stand on the other. Have the student stand relatively close to the hinge, while you stand towards the door edge. Have a pushing contest, and ask the students how a mere mathematics professor was able to best a mighty teenager. Alternatively, bring a bicycle wheel to class, and show how, when it is spinning, it is easy to translate, yet hard to twist. In the latter case, the twisting is acting against the torque that was set up by the spinning wheel.
- If $\mathbf{a} = \alpha\mathbf{b} + \beta\mathbf{c}$, with α and β constant, show that $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0$.

- Explain the geometry involved in computing the volume of a parallelepiped spanned by vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$:



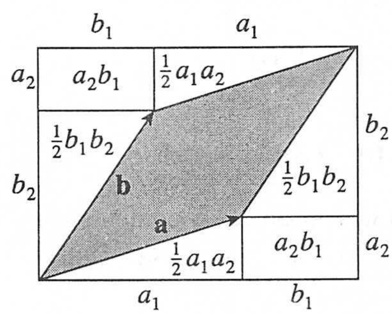
$$\begin{aligned} V &= (\text{area of the base}) \times (\text{height}) = |\mathbf{b} \times \mathbf{c}| \left|\operatorname{proj}_{\mathbf{b}\times\mathbf{c}}\, \mathbf{a}\right| \\ &= |\mathbf{b} \times \mathbf{c}|\, |\mathbf{a}|\, |\cos\theta| = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \end{aligned}$$

Conclude that $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are coplanar if and only if the triple product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0$.

- The solid discussed above was originally called a *parallelepipedon* from the Greek words: *para* (beside), *allele* (other), *epi* (upon), and *pedon* (ground). It meant that there was always a face that was parallel to the one on the ground.)

Workshop/Discussion

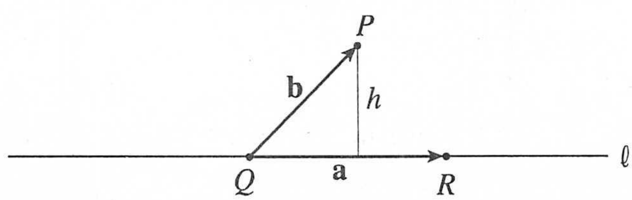
- Give a geometric proof (without using cross products) that the area of the parallelogram defined by $\mathbf{a} = \langle a_1, a_2, 0\rangle$ and $\mathbf{b} = \langle b_1, b_2, 0\rangle$ is $a_1b_2 - a_2b_1$, that is, $|\mathbf{a} \times \mathbf{b}|$.



Shaded Area $= (a_1 + b_1)(a_2 + b_2) - 2a_2b_1 - b_1b_2 - a_1a_2 = a_1b_2 - a_2b_1$

- Find the set of all position vectors mutually perpendicular to non-collinear vectors $\langle a_1, a_2, 0\rangle$ and $\langle b_1, b_2, 0\rangle$. Note that the resultant set of vectors determines a line in space (the z-axis, in fact). Then have the students try to determine what the set of all vectors perpendicular to the single vector $\langle 1, 2, 3\rangle$ will look like. (They can just try to visualize the answer.) Show how the set of all of these vectors from a given base point forms a plane in space.

- Discuss the distance from a point to a line (Exercise 27) this way:



Let h be the distance from P to $\overrightarrow{QR} = \mathbf{a}$. Pause to allow the students to find the area of the parallelogram determined by a and b in terms of h. ($A = h\,|\mathbf{a}|$) From that fact, derive that the distance from P to l is $\dfrac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{a}|}$.

- Show that if $\mathbf{a}\times\mathbf{b} = \mathbf{a}\times\mathbf{c}$, then a is parallel to $\mathbf{b}-\mathbf{c}$. Then use geometry to show that if a is parallel to $\mathbf{b}-\mathbf{c}$, the area of the parallelogram determined by a and b equals the area of the parallelogram determined by a and c.

Group Work 1: Messing with the Cross Product

Four different problems are given for the group work. Give each group one problem to solve. If a group finishes early, give them another one to work out. After every group has finished their original problem, give them a few minutes to practice, and then have them come up and state their problem and demonstrate their solution.

Note: Problem 4 is more difficult than the others; assign it to groups accordingly. It may be helpful to point out that $(\mathbf{b}-\mathbf{c}) \parallel \mathbf{a}$ implies that a lies in the plane determined by b and c.

Group Work 2: A Matter of Shading

Lab Project 1: Cross Product Properties

With a computer algebra system, have the students check the distributive property of cross products: $\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = (\mathbf{a}\times\mathbf{b}) + (\mathbf{a}\times\mathbf{c})$ and show the non-associativity of $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$ and the non-commutativity of $\mathbf{a}\times\mathbf{b}$ by writing the vectors as $\langle a_1, a_2, a_3\rangle$, $\langle b_1, b_2, b_3\rangle$, and $\langle c_1, c_2, c_3\rangle$ and having the system perform the relevant computations.

Lab Project 2: Exploring the Triple Product

For this project, you should use a CAS which can easily compute and graphically illustrate cross products. Consider the triple product $\mathbf{n} = \mathbf{a}\times(\mathbf{b}\times\mathbf{c})$. Use two fixed vectors $\mathbf{b} = \langle 1, -1, 2\rangle$ and $\mathbf{c} = \langle 2, 1, 3\rangle$.

1. Treating all vectors as position vectors, let the students check, using several different vectors a, that n is in the plane through $\langle 0, 0, 0\rangle$ generated by b and c.
2. Have the students explain why the fact illustrated in point 1 is true.
3. Let them explore how n changes with various conditions on a, b, and c, for example:

   (a) $\mathbf{a}\perp\mathbf{b}$ (Result: n is parallel to b)

   (b) $\mathbf{a}\parallel\mathbf{b}\times\mathbf{c}$ (Result: $\mathbf{n} = 0$)

   (c) $\mathbf{a} = \mathbf{b}$ (Result: n is parallel to $\operatorname{orth}_{\mathbf{b}}\mathbf{c}$)

   (d) $\mathbf{a}\parallel\mathbf{b}$ (Result: n is perpendicular to a and n is parallel to $\operatorname{orth}_{\mathbf{b}}\mathbf{c}$)

   Note: the orthogonal projection $\operatorname{orth}_{\mathbf{b}}\mathbf{a}$ is introduced in Exercise 25 of Section 9.3.

4. Have the students verify Formula 8 (page 673) for the vector triple product by checking that $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\,\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\,\mathbf{c}$ when $\mathbf{a} = \langle -2, -1, 2\rangle$, $\mathbf{b} = \langle 1, -1, 2\rangle$, and $\mathbf{c} = \langle 2, 1, 3\rangle$.

Homework Problems

Core Exercises: 1, 2, 5, 7, 13, 16, 17, 21, 24, 33

Sample Assignment: 1, 2, 5, 7, 11, 12, 13, 16, 17, 21, 24, 25, 28, 33

Exercise	C	A	N	G	V
1	×				
2		×			×
5		×			×
7		×			
11		×			
12		×			×
13		×			

Exercise	C	A	N	G	V
16		×			
17		×			
21		×			
24		×			
25		×			
28	×	×			×
33	×				×

Group Work 1, Section 9.4
Messing with the Cross Product

1. Consider the vector $\mathbf{c} = \mathbf{a} \times (\mathbf{a} \times \mathbf{b})$. Is $\mathbf{c} \perp \mathbf{a}$? Is $\mathbf{c} \perp \mathbf{b}$? Either prove or give a counterexample.

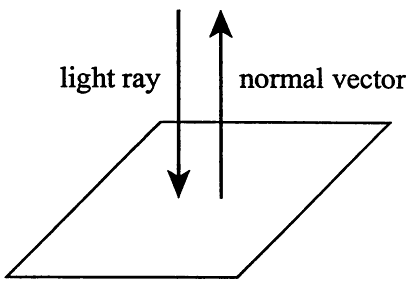
2. Suppose that $\mathbf{j} \times \mathbf{a} = \mathbf{i}$. Give two possible solutions for $\mathbf{a}$, and discuss other possible solutions.

3. Given $\mathbf{a} = \langle 1, 2, 3 \rangle$ and $\mathbf{b} = \langle 1, -1, -1 \rangle$, sketch the collection of all position vectors $\mathbf{c}$ satisfying $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$.

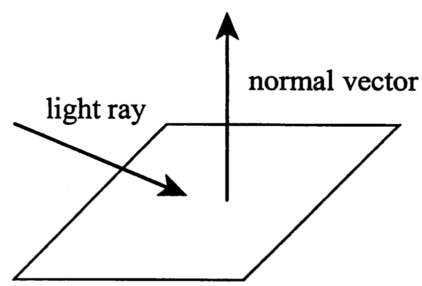
4. Suppose $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$. Show that $\mathbf{a}$ is perpendicular to $(\mathbf{b} - \mathbf{c}) \times (\mathbf{a} + \mathbf{b})$.
Hint: Why is $\mathbf{b} - \mathbf{c}$ parallel to $\mathbf{a}$?

Group Work 2, Section 9.4
A Matter of Shading

One way to accurately render three-dimensional objects on a computer screen involves using the dot and cross products. In order to determine how to shade a piece of a surface we need to determine the angle at which rays from the light source hit the surface. To determine this angle, we compute the dot product of the light vector with the vector perpendicular to the surface at the particular point, called the normal vector.
If the light ray hits the surface straight on, that is, has an angle of 0° with the normal, then this piece of the surface will appear bright. On the other hand, if the light comes in on an angle, this piece of the surface will not appear as bright.



This piece appears bright.



This piece appears dimmer.

Suppose the light source is placed directly above the xy-plane, so that the light rays come in parallel to the vector $\langle 0, 0, -1 \rangle$. At what angle (to the normal) do the light rays hit a triangle bounded by the points

1. $(3, 2, 4)$, $(2, 5, 3)$, and $(1, 2, 6)$?

2. $(3, 5, 2)$, $(3, 3, 1)$, and $(1, 3, 1)$?

3. Suppose we are standing above the light source looking down on the xy-plane. Which of these two regions will appear brighter to us?

Discovery Project: The Geometry of a Tetrahedron

The three parts of this project can be assigned independently of each other. The temptation will be for the instructor to start giving diagrams and hints too soon. Students will probably not be familiar with how to get started on this type of exercise, but the "getting started" process is one of the most important things they will learn from this project.

Note that three very different solutions to Problem 3 are given in the solutions manual, and only one of them uses the result of Problem 1.

9.5 Equations of Lines and Planes

Suggested Time and Emphasis

1 class Essential Material

Points to Stress

1. Three ways to describe a line:
   - Vector (parametric) equations (starting with point P_0 on the line and direction vector $\mathbf{d}$): $\mathbf{r} = \overrightarrow{OP_0} + t\mathbf{d}$.
   - Symmetric equations: $\dfrac{x - x_0}{a} = \dfrac{y - y_0}{b} = \dfrac{z - z_0}{c}$.
   - Two-point vector equation (starting with two points P_0 and Q_0 on the line): $\mathbf{r} = \overrightarrow{OP_0} + t\,\overrightarrow{P_0Q_0}$.
2. Three ways to describe a plane:
   - Vector equation (starting with point P_0 and normal vector $\mathbf{n}$): $\mathbf{n} \cdot \left(\mathbf{r} - \overrightarrow{OP_0}\right) = 0$.
   - Scalar equation (starting with point P_0): $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$ or $ax + by + cz + d = 0$.
   - Parametric equations (starting with point P_0 and two direction vectors $\mathbf{a}$ and $\mathbf{b}$): $\mathbf{r} = \overrightarrow{OP_0} + t\mathbf{a} + s\mathbf{b}$.

Text Discussion

- When specifying the equation of a line in space, the text says that you need a point on the line and a vector parallel to the line. Why can't you determine a line in space simply by using one vector?
- In two dimensions, we can specify a line just by writing the one equation $y = mx + b$. In three dimensions, we can specify a line by the three equations $x = x_0 + at$, $y = y_0 + bt$, and $z = z_0 + ct$. Is there a way that we can write two equations for a two-dimensional line?

Materials for Lecture

- An overall theme for this section could be that a "line" is determined by a point and a direction, and a "plane" is determined by a point and a normal vector, or a point and two directions.
- Review parametric representation of lines in $\mathbb{R}^2$, and then generalize to $\mathbb{R}^3$. Recall that in two dimensions, a line can be determined by a point and a slope. Ask for the slope of the line between the points $(0, 0)$ and $(1, 2)$. Start with the line $y = mx + b$ and write it parametrically as $x = t, y = mt + b$. Then write the vector equation with $\mathbf{r}_0 = \langle 0, b \rangle$ and $\mathbf{d} = \langle 1, m \rangle$, a vector whose direction has slope m. Next ask for the slope of the line between $(0, 0, 0)$ and $(1, 2, 3)$. Note that there is no answer — we lose the idea of "slope" when going from two to three dimensions. So a vector is our only way of specifying direction in three dimensions.
- Discuss the overdetermined system which comes up in Example 3. Note the possibilities:
  1. The lines are the same (infinitely many solutions)
  2. The lines intersect (one solution)
  3. The lines are parallel or skew (no solution)

- Discuss two ways to find the equation of a plane containing three non-collinear points P, Q, and R. C way is to form the normal vector $\mathbf{n} = \mathbf{a} \times \mathbf{b}$, where $\mathbf{a} = \overrightarrow{PQ}$ and $\mathbf{b} = \overrightarrow{PR}$, and use the vector equati Another approach is to use the parametric equation $\mathbf{r} = \overrightarrow{OP} + t\mathbf{a} + s\mathbf{b}$. This latter equation requires l computation. Verify that points given by the parametric equation also satisfy the vector equation.

- Redo Example 5 [find the plane passing through the points $P(1,3,2)$, $Q(3,-1,6)$, and $R(5,2,0)$] us parametric equations.

- Use Example 7 (page 680) to show how to solve algebraically for the line of intersection of two planes $y + z = 1 - x$, $-2y + 3z = 1 - x$, and then compute the vector equation of the line by setting $x = t$ solving the 2×2 system, which gives $x = t$, $y = \frac{2}{5} - \frac{2}{5}t$, $z = \frac{3}{5} - \frac{3}{5}t$.

Workshop/Discussion

- Review lines in two dimensions. For example, ask the students to draw the line passing through $(1,2)$ the direction of $\mathbf{i} - 2\mathbf{j}$ and then write parametric equations for this line. Stress that these equations are unique.

- Discuss intersecting lines, perpendicular lines, and parallel lines. Find the angle between a pair intersecting lines.

- Go through Example 8 carefully, emphasizing the geometry.

- Define the parametric equation of a plane through the origin generated by vectors $\mathbf{a}$ and $\mathbf{b}$: $\mathbf{r} = s\mathbf{a} +$ Verify that this is a plane by showing that $\mathbf{r} \cdot (\mathbf{a} \times \mathbf{b}) = 0$ for all $\mathbf{r}$. Parametrize the plane with equat $4x - y + 2z = 8$, by finding three points on the plane [for example, $(2,0,0)$, $(0,-8,0)$, and $(0,0,$ using the method of setting two of the coordinates equal to zero and computing the third] and obtain $\mathbf{r} = \langle 2,0,0\rangle + s\langle -2,-8,0\rangle + t\langle -2,0,4\rangle$. Alternatively, write the equation as $z = -2x + \frac{1}{2}y + 4$ show that a typical vector in this plane is $\mathbf{r} = \left\langle x, y, -2x + \frac{1}{2}y + 4\right\rangle = x\langle 1,0,-2\rangle + y\left\langle 0,1,\frac{1}{2}\right\rangle + \langle 0,0,$ This is another parametric equation, with parameters x and y.

- Give the students some parts of Exercise 1 to try in groups. Poll the groups before giving out any answe Ask students with conflicting opinions to try to explain their answers to each other.

Group Work 1: The Match Game

This is a pandemonium-inducing game. Each group is given two points on a straight line, parametric equati for a different line, a symmetric equation for a third line, and a vector equation for a fourth line. Each lin described in each of the four ways. The first group to find four descriptions for the same line wins some fo of prize.

The activity works best if the students can walk around, showing each other their descriptions and trying find matches.

For the convenience of the teacher, each row contains a winning combination. Make sure that each team st with descriptions from different rows.

Category A	**Category B**	**Category C**	**Category D**
The line between $(0,0,1)$ and $(1,2,1)$	$\mathbf{r} = \langle 2,4,1\rangle + t\langle 1,2,0\rangle$	$x = 2t$, $y = 4t$, $z = 1$	$\frac{x-1}{2} = \frac{y-2}{4}, z = 1$
The line between $(0,-3,3)$ and $(3,3,0)$	$\mathbf{r} = \langle 1,-1,2\rangle + t\langle 1,2,-1\rangle$	$x = 2+t$, $y = 1+2t$, $z = 1-t$	$\frac{x-1}{2} = \frac{y+1}{4} = \frac{z-2}{-2}$
The line between $(1,3,2)$ and $(1,-1,6)$	$\mathbf{r} = \langle 1,2,3\rangle + t\langle 0,-1,1\rangle$	$x = 1$, $y = -t$, $z = 5+t$	$x = 1, \frac{y-1}{-2} = \frac{z-4}{2}$
The line between $(0,0,4)$ and $(12,8,8)$	$\mathbf{r} = \langle 9,6,7\rangle + t\langle -3,-2,-1\rangle$	$x = 6-6t$, $y = 4-4t$, $z = 6-2t$	$\frac{x-3}{3} = \frac{y-2}{2} = \frac{z-5}{1}$
The line between $(5,0,7)$ and $(-2,-7,0)$	$\mathbf{r} = \langle 3,-2,5\rangle + t\langle -1,-1,-1\rangle$	$x = 2-2t$, $y = -3-2t$, $z = 4-2t$	$\frac{x}{2} = \frac{y+5}{2} = \frac{z-2}{2}$
The line between $(-3,3,-9)$ and $(3,-3,9)$	$\mathbf{r} = \langle 0,0,0\rangle + t\langle -1,1,-3\rangle$	$x = -1+t$, $y = 1-t$, $z = -3+3t$	$\frac{x+2}{-2} = \frac{y-2}{2} = \frac{z+6}{-6}$
The line between $(-4,2,1)$ and $(-11,1,-1)$	$\mathbf{r} = \langle 3,3,3\rangle + t\langle 7,1,2\rangle$	$x = 10+7t$, $y = 4+t$, $z = 5+2t$	$\frac{x+4}{14} = \frac{y-2}{2} = \frac{z-1}{4}$

▲ Group Work 2: Da Planes! Da Planes!

For this activity, each group gets a different point in space. The groups have to find equations of planes containing this point perpendicular to various directions, such as the x-axis, or the line $y = x$. As a follow-up question, have them try to determine if two given lines are skew.

▲ Group Work 3: Planes from Points

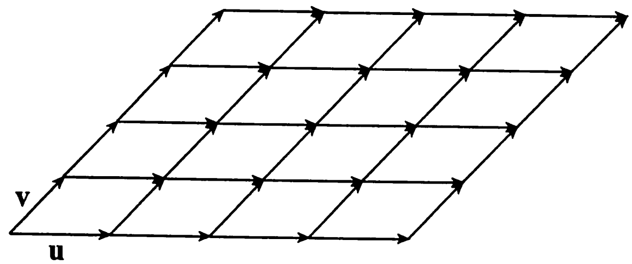
Give each group two sets of three points each, one non-collinear set and one collinear set. Ask the students to give a parametric equation of the unique plane containing the points. For the second set of points this is a trick question, since collinear points do not determine a plane.

Group Work 4: The Moving Plane

Group Work 5: The Spanning Set

The purpose of this activity is to give the students a sense of how two non-parallel vectors in two dimensions span the entire xy-plane.

Start by giving each student or group of students a sheet of regular graph paper and a transparent grid of parallelograms formed by two vectors **u** and **v**.



Next give each student a point (x, y) in the plane. By placing the grid over the graph paper they should estimate values of r and s such that $\langle x, y\rangle = r\mathbf{u} + s\mathbf{v}$. Repeat for several other points (including some for which one or both of r and s will be negative) until the students have convinced themselves that every point in the plane can be expressed in this manner.

Now repeat the activity with different vectors **u** and **v**, perhaps using the same points as before.

As a wrap-up, give the students specific vectors **u** and **v**, such as $\mathbf{u} = \langle 3, 1\rangle$ and $\mathbf{v} = \langle -1, -2\rangle$, and have them determine values of r and s for several points. See if they can find general formulas for r and s in terms of the point (x, y). What goes wrong algebraically if **u** and **v** are parallel?

Lab Project: Equations of Planes

Use a computer graphing program (for example, Maple, Mathematica, or Matlab) for this lab.

Goal: By the end of this lab, students should be able to find and visualize the plane given by three points, or by a point and the normal to the plane. Along the way they will also learn about parametrically represented planes and implicitly defined graphs. This is a good chance for the students to learn how to use their software effectively. Make sure they graph using boxed axes.

1. Give the students three non-collinear points in space. They should choose one of the points as a base point, and from that point find two direction vectors which lie in the plane.
2. The students should use the base point and direction vectors to define the same plane parametrically. They should also graph the plane on the computer as a parametrically defined surface.
3. Now have the students use the two direction vectors to find a vector normal to the plane. They should then find a linear equation of the plane, and plot the plane as the graph of a surface.
4. As a discussion question, give the students three points which each have the same y-coordinate k, so the plane $y = k$ is not the graph of a function of x and y. Discuss how to graph this plane, noting that some graphing programs can graph it implicitly, while others cannot. Make the point that the parametric equation for the plane is really no different than for any other plane. This can lead to a discussion of the many different equivalent representations of the same plane.

Homework Problems

Core Exercises: 1, 2, 8, 17, 20, 24, 34, 37

Sample Assignment: 1, 2, 8, 12, 17, 20, 22, 24, 26, 30, 34, 35, 37, 47, 52, 54

Note: Problem 3 of Focus on Problem Solving (page 703) would make a good, challenging project for motivated students.

Exercise	C	A	N	G	V
1					×
2		×			
8		×			
12		×			
17		×			
19–28		×			
30		×			

Exercise	C	A	N	G	V
34		×			
35		×			
37		×			×
47		×			
52		×			
54					×

Group Work 2, Section 9.5
Da Planes! Da Planes!

Consider the point __________ in $\mathbb{R}^3$.

1. Find an equation of a line that contains your point and the origin.

2. Find an equation of a line that contains your point and the point $(1, -1, 1)$.

3. Now find an equation of the plane that contains the two lines you've just found.

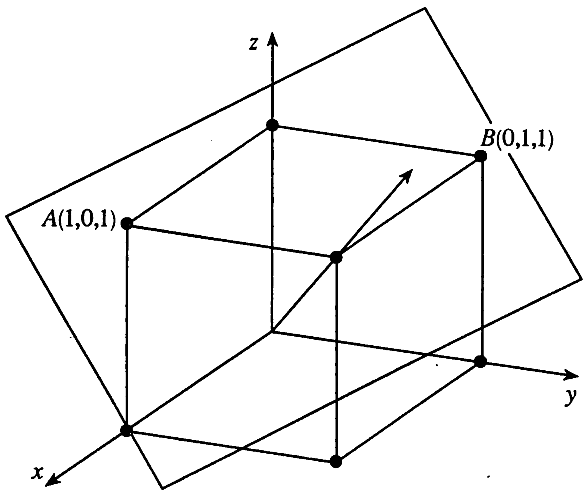
4. Find an equation of the plane that contains your point and is perpendicular to the x-axis.

5. Find an equation of the plane that contains your point and is perpendicular to the line $y = x$ in the xy-plane.

6. Finally, find an equation of a plane that does *not* contain your point.

Group Work 4, Section 9.5
The Moving Plane

Consider the unit box $\{(x, y, z) \mid 0 \leq x, y, z \leq 1\}$ and a plane which is perpendicular to the diagonal line from $(0, 0, 0)$ to $(1, 1, 1)$.



Start moving the plane along the diagonal until it hits the points $A\,(1, 0, 1)$ and $B\,(0, 1, 1)$ simultaneously.

1. What is the equation of the plane P, where P is perpendicular to the diagonal line from $(0, 0, 0)$ to $(1, 1, 1)$ and contains the points $A\,(1, 0, 1)$ and $B\,(0, 1, 1)$?

2. Show that the plane P goes through the point $C\,(1, 1, 0)$.

3. Describe the figure made by the intersection of P and the unit box.

4. Compute the area of the figure found in Problem 3.

9.6 Functions and Surfaces

Suggested Time and Emphasis

$\frac{3}{4}$–1 class   Essential material

Points to Stress

1. A function of two variables as a rule assigning a real number to every point in its domain, and the definition and shape of the domain of such a function.

2. The representation of graphs of functions of two variables as surfaces in $\mathbb{R}^3$, and the uses of horizontal traces to describe these surfaces.

3. Quadric surfaces as the graphs of second-degree polynomials in x, y, and z.

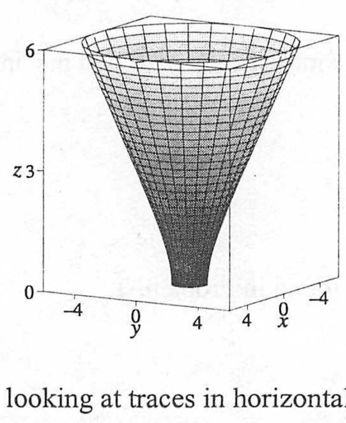
Text Discussion

- If f is a function of two variables and $f(3,4) = -1$, give the coordinates of a point on the graph of f.
- What are the vertical traces of the surface $z = 4x^2 + y^2$? What are the horizontal traces for $z > 0$? For $z < 0$?
- Why is the quadric surface $x^2 + \frac{1}{9}y^2 + \frac{1}{4}z^2 = 1$ not the graph of a function $z = f(x,y)$?

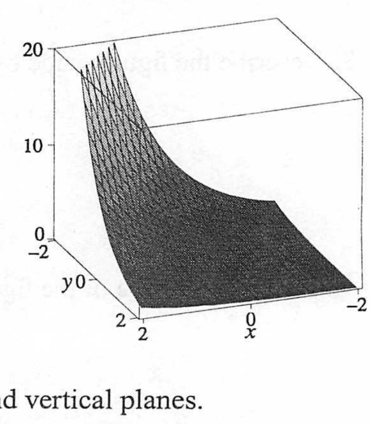
Materials for Lecture

- Revisit the function $f(x,y) = x \ln(y^2 - x)$. Sketch the domain, as done in Figure 2 (page 686) of the text. Then go on to sketch the set of points (x,y) where $f(x,y) = 0$, $f(x,y) > 0$, $f(x,y) < 0$.
- Use mathematical reasoning and traces to describe the domains and the graphs of the functions at right, perhaps later putting up transparencies of the solutions to verify the reasoning.

$f(x,y) = \sqrt{x^2 + y^2 - 1}$

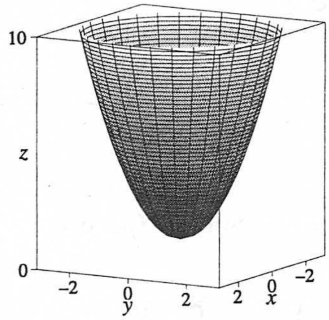


$f(x,y) = e^{x-y}$

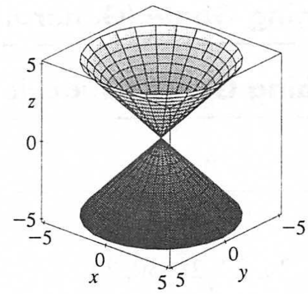


- Sketch the surface $x^2 + y^2 - z^2 = 1$ by looking at traces in horizontal and vertical planes.
- Use mathematical reasoning and traces to describe the following quadric surfaces:

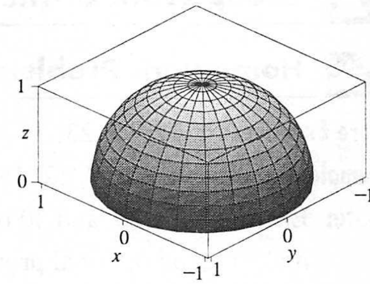
1. $z = x^2 + y^2 + 1$, a paraboloid with vertex at $(0, 0, -1)$



2. $z^2 = x^2 + y^2$, a cone



3. $z = \sqrt{1 - x^2 - y^2}$, the top hemisphere of the sphere $x^2 + y^2 + z^2 = 1$



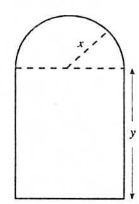
Workshop/Discussion

- Domain calculations often involve solving inequalities (sometimes nontrivial ones), and thus are usually not so simple for the students. Go over some examples, such as calculating domains for the following functions:

$$f(x, y) = \frac{\sqrt{x^2 + y^3}}{x^2 + 3x - 8} \qquad f(x, y) = -2\cos 2x + y$$

$$f(x, y) = \sin\left(\sqrt{1 - (x^2 + y^2)}\right) \qquad f(x, y) = \exp\left(\frac{x + y}{xy}\right)$$

- Let A be the area of the Norman window shown at right. Lead the students to see that A can be expressed as a function of two variables x and y. Have them figure out the domain of A and use level curves to determine what the graph of A looks like.



- Examine the hyperboloid of two sheets $-x^2 - y^2 + \frac{1}{2}z^2 = 1$. Show what conditions are needed on z to ensure that there are horizontal traces. Show that the horizontal trace at $z = k \geq \sqrt{2}$ is a circle of radius $\sqrt{\frac{1}{2}z^2 - 1}$, and that the vertical traces are hyperbolas.
- Compare the graphs of the surfaces $z^2 = x^2 + y^2$ and $z = x^2 + y^2$. Once they see the difference visually, ask the students to use vertical traces to illustrate this difference algebraically.
- Examine the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. Ask students what conditions on a, b, and c ensure that all horizontal traces are circles. What conditions ensure that all vertical traces are circles? If both horizontal and vertical traces are circles, what does that say about the ellipsoid? If time permits, have the students vary a, b, and c in the ellipsoid equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. Make sure they try varying the relative magnitudes of a, b, and c ($a < b < c$, $a < c < b$, $a = b < c$, and so on). Also make sure to point out that when $a = b = c$, we have a sphere of radius a.

Group Work 1: Staying Cool

Group Work 2: The Matching Game (General Functions)

Group Work 3: The Matching Game (Quadric Surfaces)

Homework Problems

Core Exercises: 4, 12, 15, 23, 24, 27

Sample Assignment: 1, 4, 7, 12, 15, 23, 24, 26, 27, 28, 30, 33

Note: Exercises 27, 28, and 30 require a CAS. Problem 5 from Focus on Problem Solving (page 703) would make a good optional project to assign to motivated students.

Exercise	C	A	N	G	V
1	×				
4		×		×	
7		×		×	
12				×	
15					×
23					×

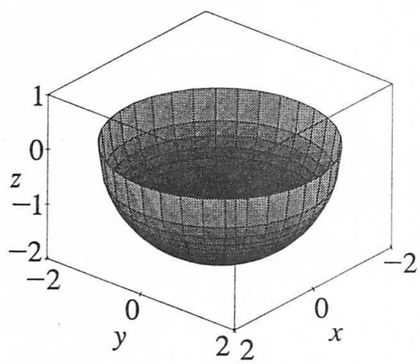
Exercise	C	A	N	G	V
24		×			×
26		×			×
27					×
28				×	
30				×	
33	×	×			

Group Work 3, Section 9.6
The Matching Game (Quadric Surfaces)

Match each function with its graph. Give reasons for your choices.

1. $x^2 + y^2 + \frac{1}{4}z^2 = 1$ **2.** $z = -\sqrt{4 - x^2 - y^2}$ **3.** $y^2 + \frac{1}{4}z^2 = 1$

4. $\frac{1}{9}z^2 - \frac{1}{4}y^2 = 1$ **5.** $\frac{1}{4}x^2 - y^2 - z^2 = 1$ **6.** $|z| = \sqrt{x^2 + y^2}$



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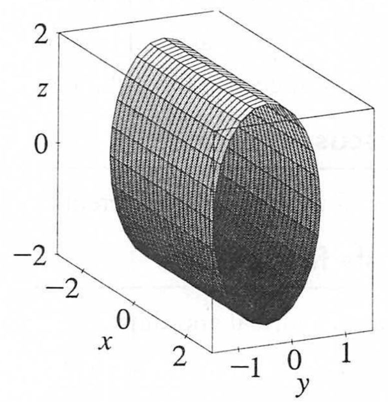

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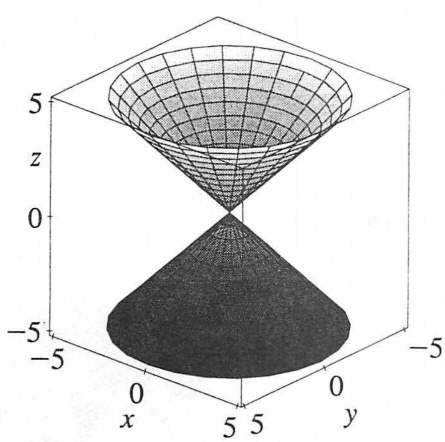

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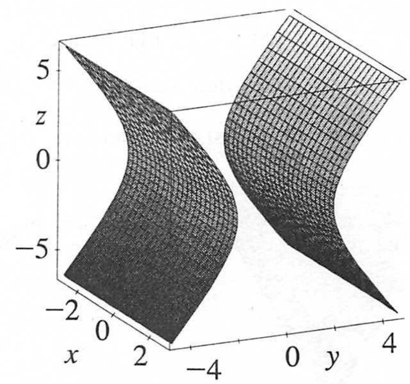

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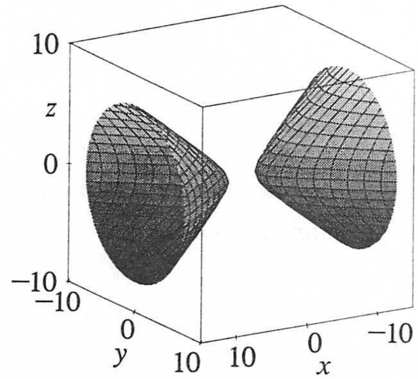

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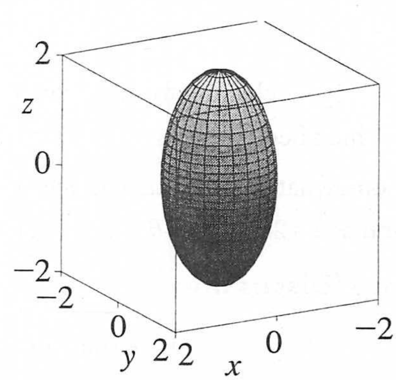


VI

9.7 Cylindrical and Spherical Coordinates

Suggested Time and Emphasis

1 class Essential Material

Points to Stress

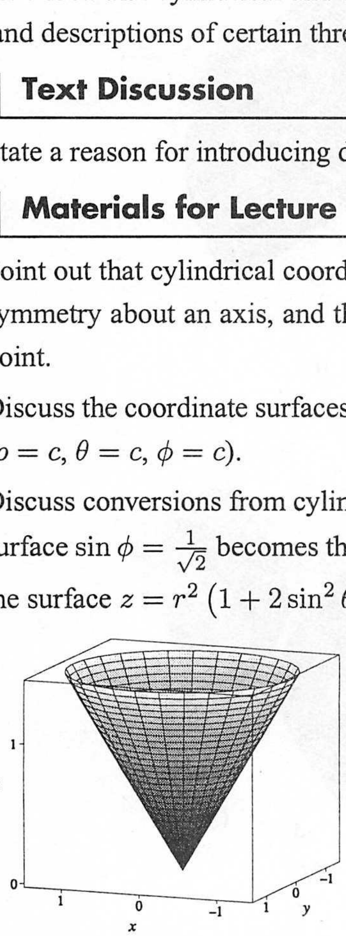
1. The basic formulas for cylindrical and spherical coordinates as extensions of polar coordinates in $\mathbb{R}^2$.
2. The idea that cylindrical and spherical coordinate systems are tools that can be used to make equations and descriptions of certain three-dimensional surfaces and solids simpler.

Text Discussion

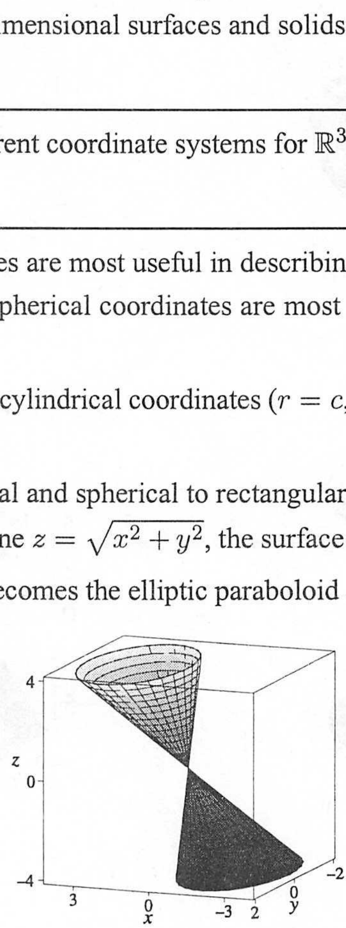
- State a reason for introducing different coordinate systems for $\mathbb{R}^3$.

Materials for Lecture

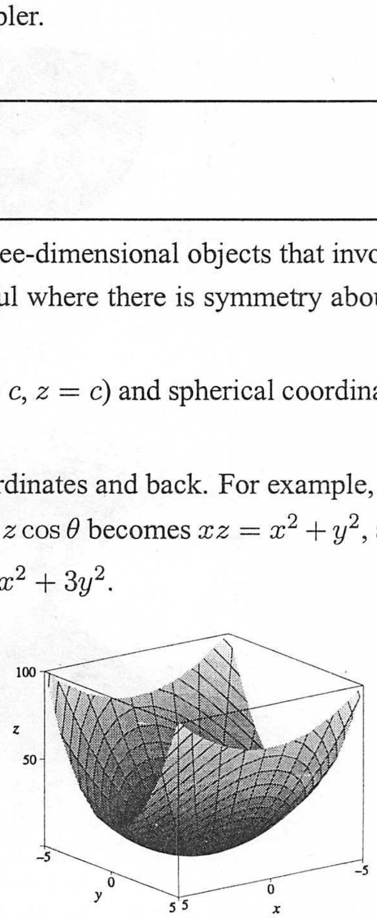
- Point out that cylindrical coordinates are most useful in describing three-dimensional objects that involve symmetry about an axis, and that spherical coordinates are most useful where there is symmetry about a point.
- Discuss the coordinate surfaces for cylindrical coordinates ($r = c$, $\theta = c$, $z = c$) and spherical coordinates ($\rho = c$, $\theta = c$, $\phi = c$).
- Discuss conversions from cylindrical and spherical to rectangular coordinates and back. For example, the surface $\sin\phi = \frac{1}{\sqrt{2}}$ becomes the cone $z = \sqrt{x^2 + y^2}$, the surface $r = z\cos\theta$ becomes $xz = x^2 + y^2$, and the surface $z = r^2\left(1 + 2\sin^2\theta\right)$ becomes the elliptic paraboloid $z = x^2 + 3y^2$.



$\sin\phi = \frac{1}{\sqrt{2}}$



$r = z\cos\theta$



$z = r^2\left(1 - \sin^2\theta\right)$

- As an example of a change from rectangular to spherical coordinates, use the circular paraboloid $z = x^2 + y^2$, which becomes $\cos\phi = \rho\sin^2\phi$.
- Identify the somewhat mysterious surface $\rho\cos\phi = \rho^2\sin^2\phi\cos 2\theta$ given in spherical coordinates by using the formula $\cos 2\theta = \cos^2\theta - \sin^2\theta$ and changing to rectangular coordinates.

Workshop/Discussion

- Describe in terms of cylindrical coordinates the surface of rotation formed by rotating the cone $z = 1/x$ about the z-axis, noting that there is an axis of symmetry. Point out that the equations come from simply replacing x by r.

Group Work 1, Section 9.6
Staying Cool

Let $T(x, y)$ be the temperature in a 10 ft $\times$ 10 ft room on a winter night, where one corner of the room is at $(0, 0)$ and the opposite corner is at $(10, 10)$ For each of the following functions T,

(a) Draw or describe in words a graph of the temperature function.

(b) Describe the likely floor locations of the heat vents.

(c) Suppose you like to sleep with a temperature of 70° or less. Where would you put the bed?

1. $T(x, y) = 78 - \frac{1}{10}\left[x^2 + (y-5)^2\right]$

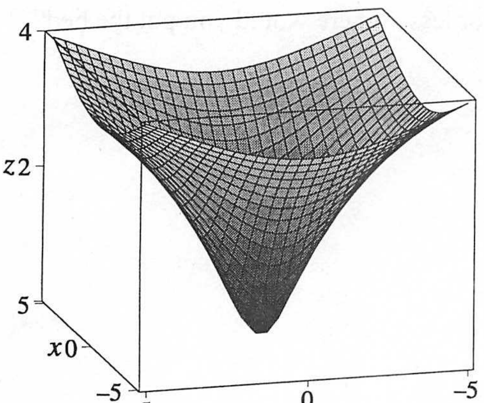
2. $T(x, y) = \frac{1}{2}x - y + 75$

Group Work 2, Section 9.6
The Matching Game (General Functions)

Match each function with its graph. Give reasons for your choices.

1. $f(x,y) = \dfrac{1}{x+1} + \sin y$ **2.** $f(x,y) = \sqrt{4 - x^2 - y^2}$ **3.** $f(x,y) = \cos\left(x + y^2\right)$

4. $f(x,y) = \ln\left(x^2 + y^2 + 1\right)$ **5.** $f(x,y) = x^2\sqrt{y}$ **6.** $f(x,y) = x^3 y$



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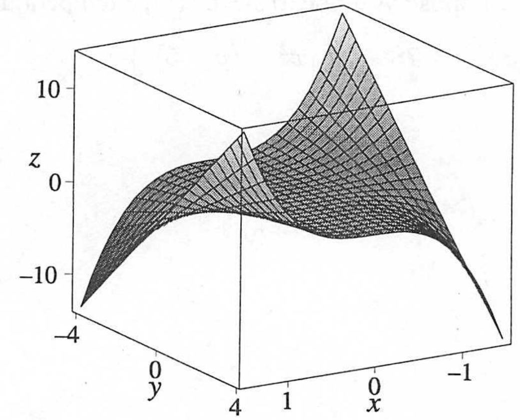

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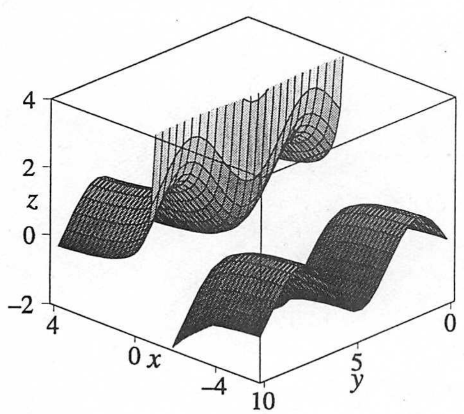

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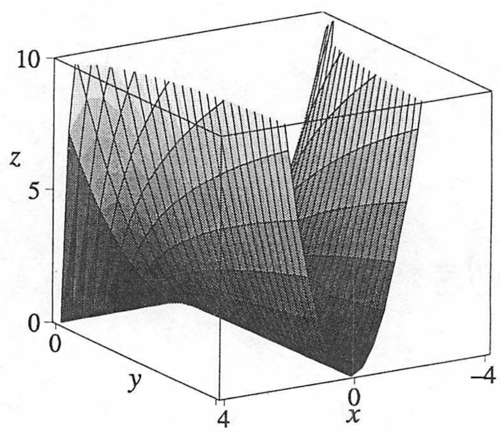

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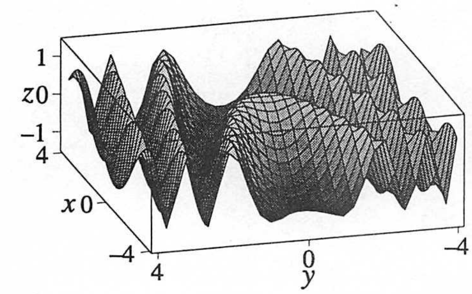

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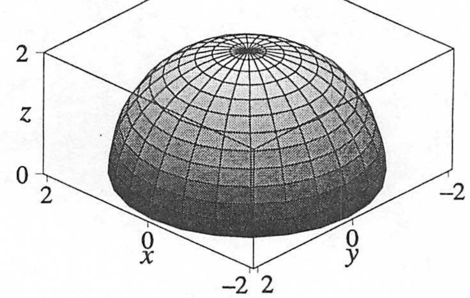
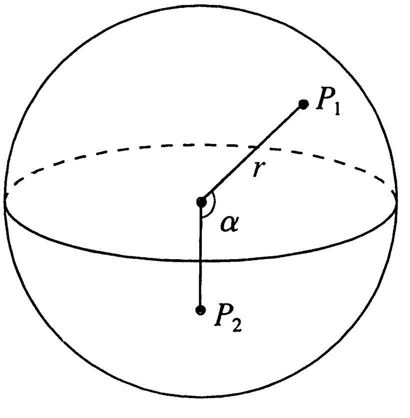


VI

- Calculate the intersection of the surfaces $z = x^2 + y^2$ and $z = x$, first in rectangular coordinates, then in cylindrical coordinates. (Here is a case where the rectangular coordinates are the easiest to visualize, even though there is an $x^2 + y^2$ term.)
- Describe the intensity from a point light source in terms of spherical coordinates.
- Describe the coordinates of all position vectors with length 3 in each of the three coordinate systems. Repeat for the coordinates of all points on a circular cylinder of radius 2 with central axis the z-axis, and the coordinates of all points on the line through the origin with direction vector $\mathbf{i} + \mathbf{j} + \mathbf{k}$. Explain which coordinate system is best suited to represent each object.
- Elaborate on Exercise 34 in the following way (this idea can also be used as a group work):
  1. Have the students read the exercise.
  2. Show one approach to the solution:
     (a) Note that the distance along an arc of a circle is given by the formula $d = r\alpha$, where r is the radius and α is the angle which subtends the arc.

     (b) Draw the following picture:

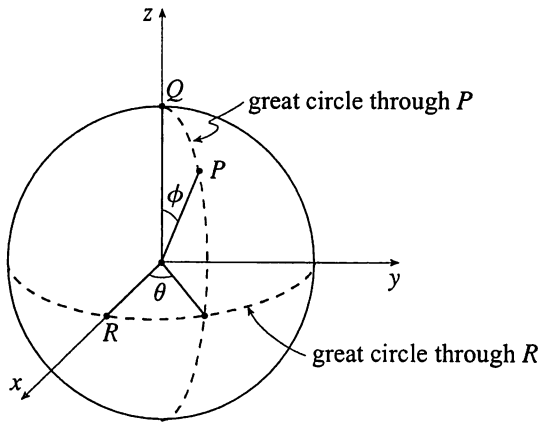


     (c) Assuming that $r = 1$, write the points P_1 and P_2 in spherical coordinates $[(1, \theta_1, \phi_1)$ and $(1, \theta_2, \phi_2)]$, and derive the rectangular representations $P_1 = (\sin\phi_1 \cos\theta_1, \sin\phi_1 \sin\theta_1, \cos\phi_1)$ and $P_2 = (\sin\phi_2 \cos\theta_2, \sin\phi_2 \sin\theta_2, \cos\phi_2)$.

     (d) Conclude that $\cos\alpha = \overrightarrow{OP_1} \cdot \overrightarrow{OP_2}$.

     (e) Use the formula in part (d) to find the answer to Exercise 34.
  3. Compute the Great Circle distance between the students' current location and a city with the same latitude on another continent.
  4. Discuss how latitude and longitude can be translated to θ and ϕ.

- Bring a beach ball and a tape measure to class. Set up the following problem: given an arbitrary orientation of the coordinate axes and a point P on the beach ball, find its spherical coordinates.



The students should quickly see that we cannot measure ϕ and θ directly, because the ball is opaque. However, given points Q and R where the positive z- and x-axes intersect the sphere, we can calculate these quantities by measuring along great circles (as indicated in the diagram).

Group Work 1: Surfaces

Group Work 2: Describe Me!

Homework Problems

Core Exercises: 3, 6, 8, 10, 13, 18, 23, 26

Sample Assignment: 1, 2, 3, 4, 6, 7, 8, 10, 13, 14, 18, 19, 23, 26

Exercise	C	A	N	G	V
1	×				
2	×				
3		×			×
4		×			×
6		×			
7		×			×
8		×			×

Exercise	C	A	N	G	V
10		×			
13					×
14					×
18		×			×
19		×			×
23		×			
26					×

Group Work 1, Section 9.7
Surfaces

1. Using inequalities in some coordinate system, describe the solid that remains when a hole of radius 1 is drilled parallel to the z-axis through the cylinder $x^2 + y^2 = 4$.

2. Consider the piece of surface cut from the paraboloid $z = x^2 + y^2$ by the plane $z = x$.

(a) Describe the bounding curve in rectangular coordinates. What does the curve look like when projected onto the xy-plane?

(b) Show (by translating into cylindrical coordinates) that the projection of the bounding curve in the plane $z = 0$ is $r^2 = r\cos\theta$. (That is, $r = 0$ or $r = \cos\theta$.)

(c) Show that a parametrization of the bounding curve can be given in rectangular coordinates by $\langle \cos^2\theta, \cos\theta\sin\theta, \cos^2\theta \rangle$, $0 \le \theta \le \pi$.

Group Work 2, Section 9.7
Describe Me!

Sketch, or describe in words, the following surfaces whose equations are given in cylindrical coordinates:

1. $r = \theta$

2. $z = r$

3. $z = \theta$

Sketch, or describe in words, the following surfaces whose equations are given in spherical coordinates:

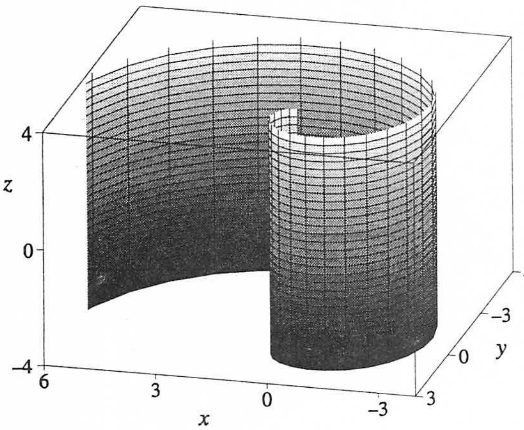
4. $\theta = \frac{\pi}{4}$

5. $\phi = \frac{\pi}{4}$

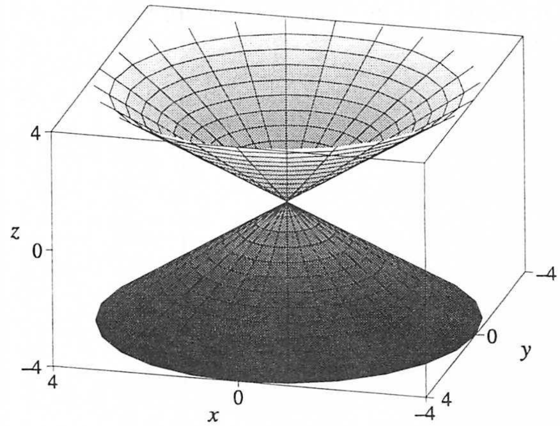
6. $\rho = \phi$

Group Work 2, Section 9.7
Describe Me! (Solutions)

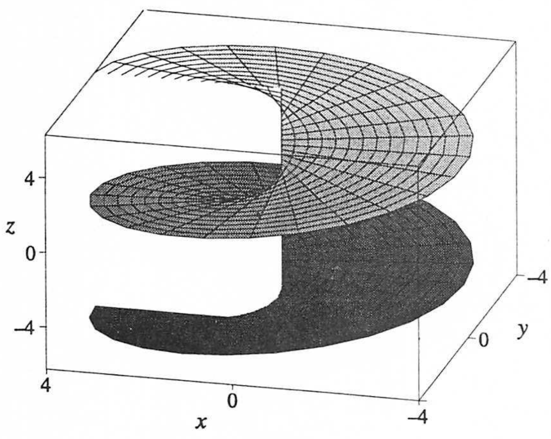
1. $r = \theta$



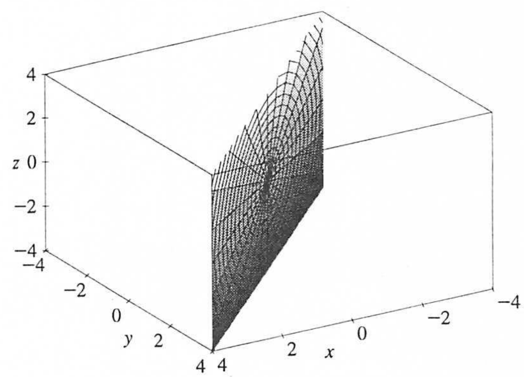
2. $z = r$



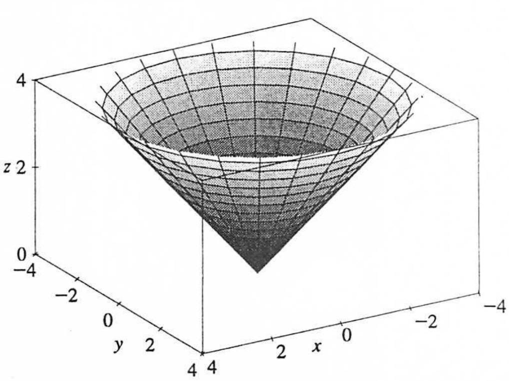
3. $z = \theta$



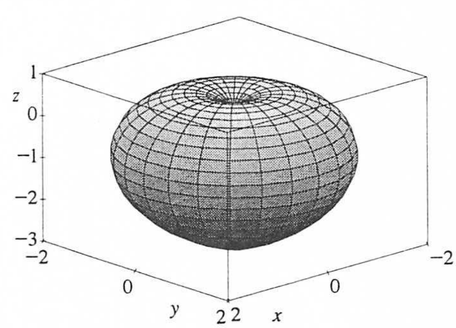
4. $\theta = \frac{\pi}{4}$



5. $\phi = \frac{\pi}{4}$



6. $\rho = \phi$



Laboratory Project: Families of Surfaces

This project serves as an introduction both to the variety of shapes that can be obtained by varying the parameters in a family of surfaces, and to the use of a computer to investigate these surfaces. If this project is assigned, it is recommended that the third part be assigned to all students, and then the first or second parts also assigned if there is time and interest.

9 Sample Exam

Problems marked with an asterisk (*) are particularly challenging and should be given careful consideration.

1. (a) Consider the two surfaces $\rho = 3\csc\phi$ (given in spherical coordinates) and $r = 3$ (given in cylindrical coordinates). Are they the same surface, or are they different surfaces? Explain your answer.

(b) Consider the two surfaces $\sin\phi = \cos\phi$ (given in spherical coordinates) and $z = \sqrt{r^2}$ (given in cylindrical coordinates). Are they the same surface, or are they different surfaces? Explain your answer.

2. Describe the following in rectangular coordinates:

(a) The intersection of the surfaces given in cylindrical coordinates by $\theta = \frac{\pi}{4}$ and $\theta = \frac{2\pi}{3}$

(b) The intersection of the surfaces given in spherical coordinates by $\rho = 1$ and $\theta = \frac{\pi}{2}$

3. Find a pair of values x, y with $0 < x < 1$, $0 < y < 1$ such that $f(x, y) \geq 8$, or show that no such values x and y can exist, for each of the following functions of two variables:

(a) $f(x,y) = \dfrac{1}{x} + \dfrac{1}{y}$

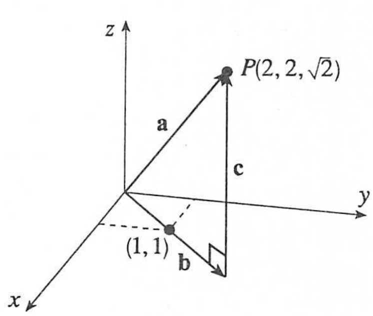
(b) $f(x,y) = 2^{xy+2.5}$

(c) $f(x,y) = \dfrac{1}{x^2 + y^2 + 1}$

(d) $f(x,y) = \cos\left(2x + 3y + \ln\left(x^2 + y^2\right)\right)$

(e) $f(x,y) = \dfrac{1}{x - y + 1}$

4. Let $\mathbf{a} = \overrightarrow{OP}$, where P is the point $\left(2, 2, \sqrt{2}\right)$. Compute the vectors $\mathbf{b}$ and $\mathbf{c}$.



5. Let $\mathbf{a} = (x + y)\,\mathbf{i} + 2\mathbf{j} + y\mathbf{k}$ and $\mathbf{b} = 3\mathbf{i} + (4x + y + 1)\,\mathbf{j} + 4\mathbf{k}$.

(a) Find values of x and y such that $\mathbf{a} \perp \mathbf{b}$.

(b) Find values of x and y such that $\mathbf{a} \parallel \mathbf{b}$. (*Hint:* Assume that $c\mathbf{a} = \mathbf{b}$ for some value c.)

6. Let $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ be three vectors in the plane $3x - 5y + 6z = 7$. Compute $(-\mathbf{a} + 4\mathbf{b} - 7\mathbf{c}) \cdot (-3\mathbf{i} + 5\mathbf{j} - 6\mathbf{k})$.

7. Let r, s, and t be distinct non-zero vectors in space. Which of the following must be true, which might be true, and which cannot be true? Justify your answers.

(a) If $\mathbf{r} \parallel \mathbf{s}$ and $\mathbf{s} \parallel \mathbf{t}$, then $\mathbf{r} \parallel \mathbf{t}$.

(b) If $\mathbf{r} \perp \mathbf{s}$ and $\mathbf{s} \perp \mathbf{t}$, then $\mathbf{r} \perp \mathbf{t}$.

(c) If $\mathbf{r} \times (\mathbf{s} \times \mathbf{t}) = \mathbf{0}$ and $\mathbf{s} \times \mathbf{t} \neq \mathbf{0}$, then $\mathbf{r} \perp (\mathbf{s} + \mathbf{t})$.

(d) If $\mathbf{r} \cdot (\mathbf{s} \times \mathbf{t}) = 0$ and $\mathbf{s} \times \mathbf{t} \neq \mathbf{0}$, then $\mathbf{r} \perp (\mathbf{s} + \mathbf{t})$.

8. Suppose we have three distinct unit vectors a, b, and c which satisfy the following conditions:

(i) $\mathbf{b} \times \mathbf{c} \neq \mathbf{0}$ (ii) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{0}$

Which of the following must be true, which might be true, and which cannot be true? Justify your answers.

(a) b is perpendicular to c.

(b) a is perpendicular to c.

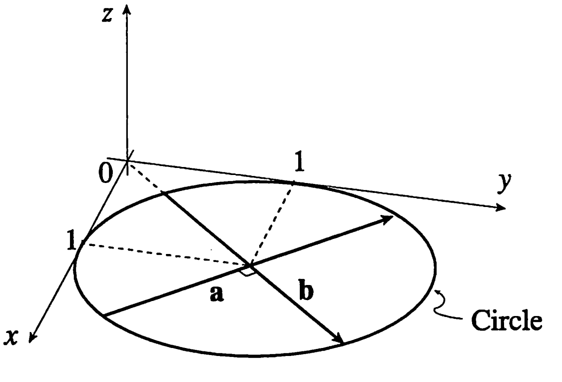
(c) a is perpendicular to b.

9. Describe and sketch the surfaces in space defined by the following equations.

(a) $y = -z + 1$

(b) $x^2 + y^2 = 3$

10. Referring to the diagram below, give the component representation of each vector.



(a) a

(b) b

(c) $\mathbf{a} \times \mathbf{b}$

(d) $\mathbf{a} + \mathbf{b}$

(e) $(\mathbf{a} + \mathbf{b}) \times (\mathbf{a} \times \mathbf{b})$

11. Let $\mathbf{N} = \mathbf{i} - \mathbf{j} + \mathbf{k}$.

(a) What is the equation of the plane P containing $(0, 0, 0)$ with normal vector N?

(b) Find two unit vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ in the plane P which are not parallel to one another.

(c) What is the relationship between N and $\mathbf{u}_1 \times \mathbf{u}_2$?

12. (a) Show that the line given by $x = t$, $y = 3t - 2$, $z = -t$ intersects the plane $x + y + z = 1$

(b) Find a point of intersection.

13. Consider the plane $x + y + z = 0$.

(a) Give three distinct points with integer coordinates that lie on this plane.

(b) Find the area of the triangle formed by those three points.

14. A particle moves in such a way that its path traces out the circle $x^2 + y^2 = 4$, $z = 3$.

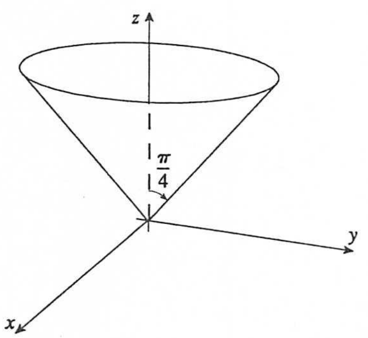
(a) Write an equation of the curve traced out by the particle in cylindrical coordinates.

(b) Write an equation of the curve traced out by the particle in spherical coordinates.

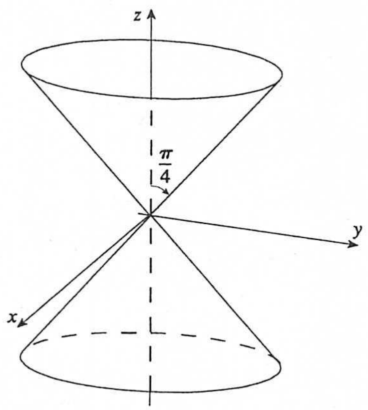
9 Sample Exam Solutions

1. (a) $\rho = 3\csc\phi \Rightarrow \rho\sin\phi = 3$ or $r = 3$ in cylindrical coordinates. These are the same surface, a cylinder of radius 3.

(b) The surfaces are different. The surface $\cos\phi = \frac{1}{\sqrt{2}}$, $\rho \geq 0$ is a single cone, and the surface $z = r$ is a double cone.



$\cos\phi = \frac{1}{\sqrt{2}}, \rho \geq 0$



$z = r$

2. (a) The intersection of two half-planes, namely, the z-axis

(b) The circle $y^2 + z^2 = 1$ in the yz-plane ($x = 0$)

3. (a) $x = 0.25$, $y = 0.25$ gives $f(x, y) = 8$.

(b) $x = 0.9$, $y = 0.9$ gives $f(x, y) \approx 9.9$.

(c) $\frac{1}{2} \leq \frac{1}{x^2 + y^2 + 1} \leq 1$

(d) $|\cos w| \leq 1$ for any w

(e) $x = 0.09$, $y = 0.99$ gives $f(x, y) = 10$

4. $\mathbf{b} = \langle 2, 2, 0 \rangle$, $\mathbf{c} = \langle 0, 0, \sqrt{2} \rangle$

5. (a) $x = -1$, $y = 1$ (among others) (b) $x = -2$, $y = 8$

6. 0

7. (a) True

(b) Might be true; false if $\mathbf{r} = \mathbf{t}$

(c) True: $\mathbf{r}$ is parallel to $\mathbf{s} \times \mathbf{t}$

(d) False: $\mathbf{r} = \mathbf{s} + \mathbf{t} \perp \mathbf{s} \times \mathbf{t}$, but $\mathbf{r} \cdot (\mathbf{s} + \mathbf{t}) \neq 0$

8. (a) Might be true

(b) True

(c) True

9. (a) $y + z = 1$, a plane

(b) $x^2 + y^2 = 3$, a cylinder of radius $\sqrt{3}$

10. (a) $\mathbf{a} = -\sqrt{2}\,\mathbf{i} + \sqrt{2}\,\mathbf{j}$

(b) $\mathbf{b} = \sqrt{2}\,\mathbf{i} + \sqrt{2}\,\mathbf{j}$

(c) $\mathbf{a} \times \mathbf{b} = -4\mathbf{k}$

(d) $\mathbf{a} + \mathbf{b} = 2\sqrt{2}\,\mathbf{j}$

(e) $(\mathbf{a} + \mathbf{b}) \times (\mathbf{a} \times \mathbf{b}) = 8\sqrt{2}\,\mathbf{i}$

11. (a) $x - y + z = 0$

(b) $\mathbf{u}_1 = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$, $\mathbf{u}_2 = \frac{1}{\sqrt{2}}\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k}$

(c) $\mathbf{v} \parallel \mathbf{u}_1 \times \mathbf{u}_2$

12. (a) The line is L: $\mathbf{r}(t) = \langle 0, -2, 0 \rangle + t\langle 1, 3, -1 \rangle$ and the normal to the plane is $\langle 1, 1, 1 \rangle$. Since $\langle 1, 3, -1 \rangle \cdot \langle 1, 1, 1 \rangle = 3 \neq 0$, these vectors are not perpendicular and thus the line intersects the plane.

(b) When $t = 1$, $\mathbf{r}(1) = \langle 1, 1, -1 \rangle$ is in the plane $x + y + z = 1$.

13. (a) $P_1 = (1, -1, 0)$, $P_2 = (1, 0, -1)$, $P_3 = (0, -1, 1)$

(b) If $\mathbf{a} = \overrightarrow{P_1P_2} = \mathbf{j} - \mathbf{k}$ and $\mathbf{b} = \overrightarrow{P_1P_3} = -\mathbf{i} + \mathbf{k}$, then the area of the triangle is $\frac{1}{2}|\mathbf{a} \times \mathbf{b}| = \frac{\sqrt{3}}{2}$

14. (a) Cylindrical coordinates: $\mathbf{s}(t) = \langle 2, t, 3 \rangle$, $0 \leq t \leq 2\pi$

(b) Spherical coordinates: $\mathbf{w}(t) = \left\langle \sqrt{13}, t, \arccos \frac{3}{\sqrt{13}} \right\rangle$, $0 \leq t \leq 2\pi$

10 Vector Functions

10.1 Vector Functions and Space Curves

Suggested Time and Emphasis

1 class   Essential Material

Transparencies Available

- Transparency 36 (Exercises 5–10, graphs I–VI, page 710)

Points to Stress

1. The connection between space curves and ranges of vector functions.
2. Matching vector equations with their curves.
3. Parametrizations of curves in space are not unique.
4. Visualization of curves in three dimensions.

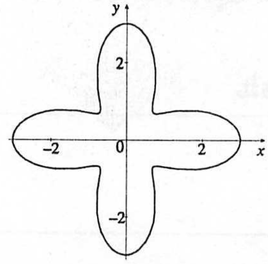
Text Discussion

- Give an example of two vector functions whose space curves lie in the plane $y + z = 2$.

Materials for Lecture

- Explain the differences and similarities between vector functions and parametric equations for a space curve.
- Point out that in going from a vector function $\mathbf{r}(t)$ to a space curve described by a variable point $P(t)$, we are picking the origin as the base of the vectors (position vectors) and the tip of $\mathbf{r}(t)$ traces out the curve: $\mathbf{r}(t) = \overrightarrow{OP}(t)$. For example, writing $\mathbf{r}(t) = \langle t, t^2 \rangle$ really means $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j}$.
- Consider a circle with radius 2. Parametrize this circle several ways: using the angle θ made with the positive x axis, using the angle $-\theta$, using the arc length starting from the point $(2, 0)$, and using the parametric equations $x(t) = -2\cos t$, $y(t) = -2\sin t$.
- Look at the vector function $\mathbf{v}(t) = \langle 3\sin t\cos t, 2\cos^2 t, \sin t \rangle$. Point out that since $x = 3\sin t\cos t$, $y = 2\cos^2 t$, and $z = \sin t$, we have $\frac{1}{9}x^2 + \frac{1}{4}y^2 = \cos^2 t$ and so $\frac{1}{9}x^2 + \frac{1}{4}y^2 + z^2 = 1$. Thus, the space curve lies on the ellipsoid $\frac{1}{9}x^2 + \frac{1}{4}y^2 + z^2 = 1$.

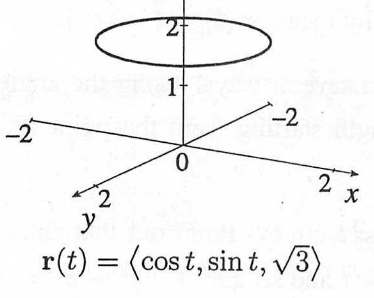
- Find the intersections of the sphere $x^2 + y^2 + z^2 = 1$ and the plane $y = z$: Solving these equations simultaneously gives $x^2 + 2y^2 = 1$. This describes an ellipse in the xy-plane with parametric equations $x = \cos t$, $y = \frac{1}{\sqrt{2}} \sin t$, $0 \le t \le 2\pi$. Thus, the space curve $x = \cos t$, $y = z = \frac{1}{\sqrt{2}} \sin t$, $0 \le t \le 2\pi$, describes the space curve which is the intersection of the two surfaces.
- Sketch the parametric curve described in polar coordinates (r, θ) by the equations $r = 2 + \cos 4t$, $\theta = t$, $0 \le t \le 2\pi$.



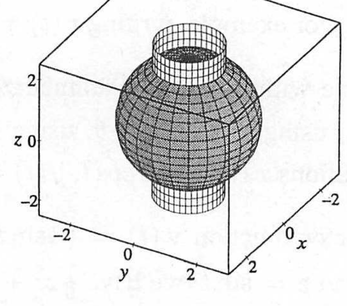
Indicate on the curve the points closest to and farthest from the origin, and determine the values of t that give these points. Then determine the Cartesian coordinates $(x(\theta), y(\theta))$ of a point on this curve as a function of θ.

Workshop/Discussion

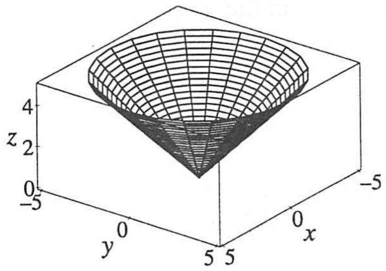
- Have the students explain why the vector functions $\langle \sin t, \cos t \rangle$ and $\langle \cos t, \sin t \rangle$ describe the same set of points. Then have them think about the three-dimensional functions $\langle \sin t, \cos t, t \rangle$ and $\langle \cos t, \sin t, t \rangle$, and notice that this time the sets of points described are different.
- This is a good opportunity to solidify the students' knowledge of $\mathbb{R}^3$. For example, they should be able to describe the curve $\langle \cos t, \sin t, t \rangle$ at this point in the course without too much difficulty.
- Describe the difference between the parametrizations $\langle \sin t, \cos t \rangle$ and $\langle \sin t^2, \cos t^2 \rangle$ graphically, in terms of both the domain of the parameter needed for one full revolution ($0 \le t \le 2\pi$ in the first case, $0 \le t \le \sqrt{2\pi}$ in the second case) and the "speed" with which the curve is traced out.
- Ask students to show that the vector function $\mathbf{v}(t) = \langle 2t + 1, 3t + 2, -5t \rangle$ describes a line, and that this line lies in the plane $x + y - z = 5$.
- Show that the vector function $\mathbf{r}(t) = \langle \cos t, \sin t, \sqrt{3} \rangle$ lies on the intersection of the sphere $x^2+y^2+z^2 = 4$ and the cylinder $x^2+y^2 = 1$. Describe this intersection, and then have the students attempt to give another vector function which parametrizes the other piece of the intersection.



$\mathbf{r}(t) = \langle \cos t, \sin t, \sqrt{3} \rangle$



- Consider the helix $x = \cos t, y = \sin t, z = t^2$. Sketch all three coordinate planar projections of this curve (that is, the projections onto the xy-plane, the yz-plane, and the xz-plane). Help the students visualize the entire curve by looking at the projections.
- Consider the cone $z = \sqrt{x^2 + y^2}$. Show that $\langle t\cos t, t\sin t, t\rangle$ parametrizes a curve that spirals around the cone, and that there are several other choices for other spiral curves.



Group Work 1: Many Paths

Ask the students to parametrize three curves on the unit sphere which connect the North Pole $(0, 0, 1)$ to the South Pole $(0, 0, -1)$. The first one should be an arc of a great circle that lies in one of the coordinate planes, the second should be an arc of a great circle that does *not* lie in one of the coordinate planes, and the third should be a curve that spirals once around the sphere. After some time, give the hint that they should use spherical coordinates.

If a group finishes early, have the students replace $(0, 0, -1)$ with $(1, 0, 0)$ and parametrize the path of shortest length between these two points.

A good way to illustrate these curves is to draw them on the surface of a ball or balloon with a felt-tip pen.

Group Work 2: Intersections and Curves

Make sure that students doing the first problem obtain both curves: $x^2 + y^2 = 1$ and $z = \pm\frac{3\sqrt{3}}{2}$.

Students doing the second problem should check their answers with you, or present their answers to the class.

Group Work 3: Visualizing Curves from their Projections

This group work attempts to give students a better understanding of the relationships between a space curve and its various two-dimensional projections.

Homework Problems

Core Exercises: 1, 5, 6, 7, 8, 9, 10, 15, 19, 29

Sample Assignment: 1, 4, 5, 6, 7, 8, 9, 10, 12, 15, 19, 27, 29, 34

Note: If three-dimensional graphics are available, add Exercises 21–24 and 25.

Exercise	C	A	N	G	V
1		×			
4			×		
5–10				×	×
12					×
15					×
19		×			×

Exercise	C	A	N	G	V
19		×			×
21–24				×	
25		×		×	
27		×			
29		×			
34		×		×	×

Group Work 2, Section 10.1
Intersections and Curves

1. In this group work, you are given two surfaces whose intersection is two distinct curves. We want to describe these curves.

(a) Describe, intuitively, the two curves in the intersection of the ellipsoid $\frac{9}{4}x^2 + \frac{9}{4}y^2 + z^2 = 9$ and the cylinder $x^2 + y^2 = 1$.

(b) Parametrize the two curves that make up this intersection.

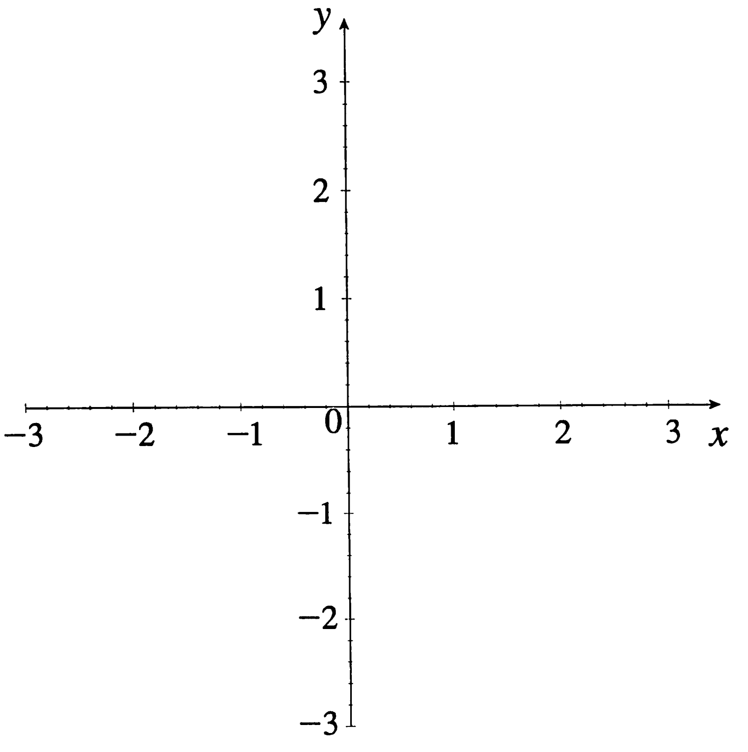
2. (a) Show that the curve $\mathbf{r} = \langle \sin t, \sin t, \cos t \rangle$ lies on the intersection of the two cylinders $x^2 + z^2 = 1$, $y^2 + z^2 = 1$.

(b) Show the same for $\mathbf{s} = \langle \cos t, \cos t, \sin t \rangle$. Do they describe the same set of points? If not, what is the difference?

Group Work 3, Section 10.1
Visualizing Curves from their Projections

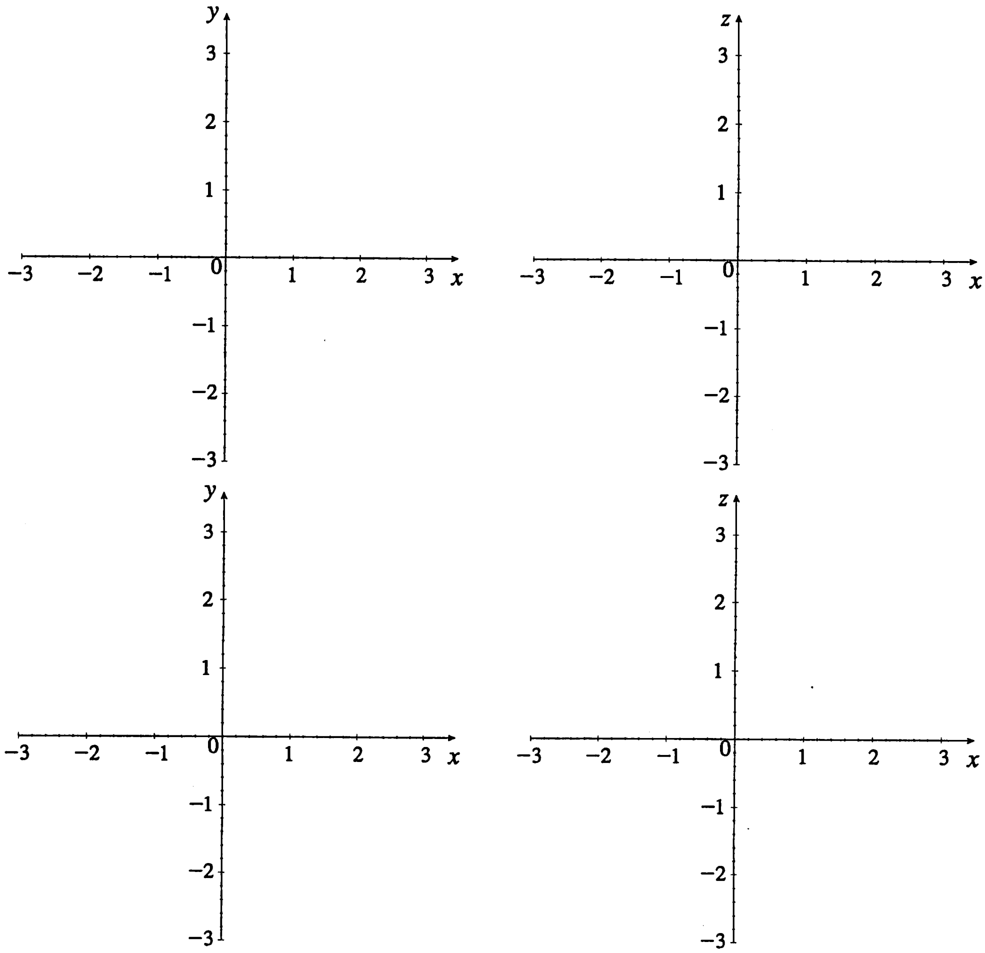
1. Consider the conical spiral $x = t\cos t$, $y = t\sin t$, $z = t$. Sketch all three planar projections. Can you visualize the entire curve by looking at the projections? Describe this curve in words.

2. Consider the trefoil knot as parametrized in Exercise 34. Sketch the projection of the trefoil knot onto the xy-plane.



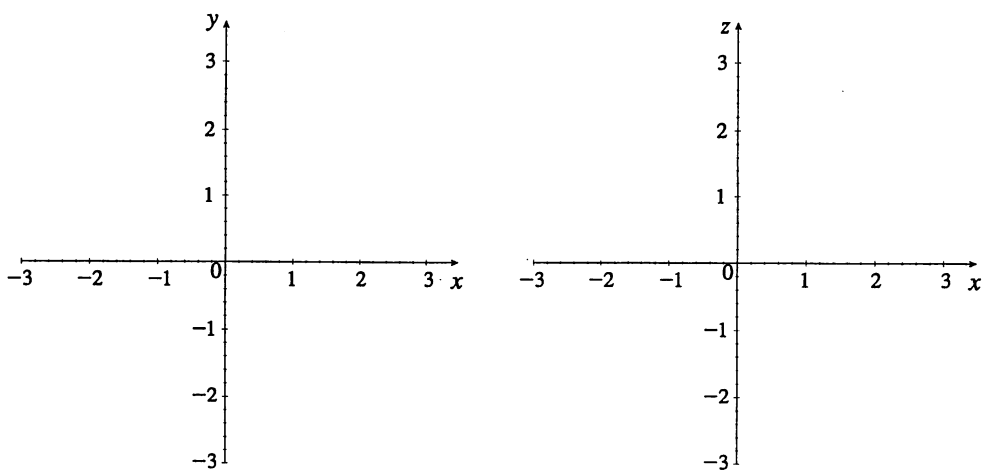
3. Consider the vector functions $\mathbf{r}(t) = \langle t, t^2, \sin t \rangle$ and $\mathbf{s}(t) = \langle t, \sin t, t^2 \rangle$.

(a) Sketch the projections of both functions in the xy- and xz-planes.



(b) Describe in words the differences between these two functions.

(c) Sketch the xy- and xz-projections for the function $\mathbf{w}(t) = \langle \sin t, t^2, t \rangle$ and describe the differences in this case.



10.2 Derivatives and Integrals of Vector Functions

Suggested Time and Emphasis

$\frac{3}{4}$ class Essential Material: Vector derivatives and unit tangent vectors.
Optional Material (recommended if time permits): Integrals of vector functions.

Points to Stress

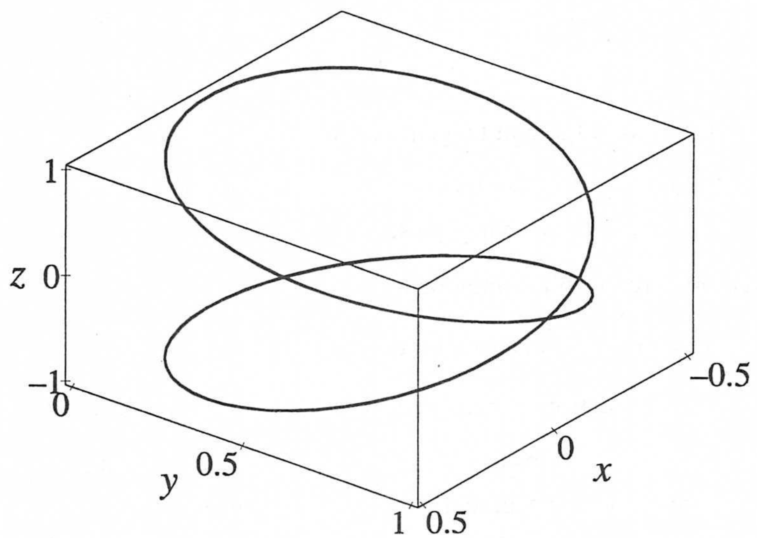
1. The vector derivative and the unit tangent vector.
2. The definition of the tangent line to a space curve.
3. The geometric interpretation of the tangent vector and smooth curves.
4. Integrals of vector functions.

Text Discussion

- Give an example of a space curve that has a cusp. Sketch one if you can.

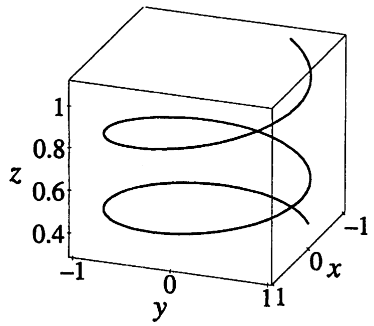
Materials for Lecture

- Elaborate on Example 5 as follows: Consider the curve $\mathbf{r}(t) = \langle \sin t \cos t, \cos^2 t, \sin t \rangle$. Show that $\mathbf{r}'$ is perpendicular to $\mathbf{r}$ by computing $|\mathbf{r}|$ and showing that it is constant. Explain why this guarantees that $\mathbf{r}'$ is perpendicular to $\mathbf{r}$. Point out that this problem could also be done analytically by computing $\mathbf{r}' \cdot \mathbf{r}$, but that the technique of showing that $|\mathbf{r}|$ is constant is often simpler.

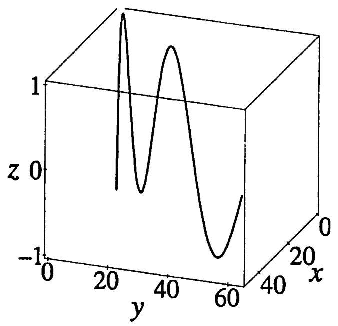


$\mathbf{r}(t) = \langle \sin t \cos t, \cos^2 t, \sin t \rangle$

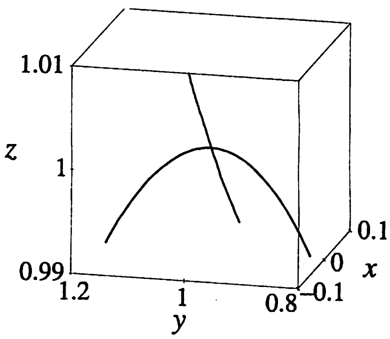
- Consider the "decaying spiral" $\mathbf{r}(t) = \langle \sin t, \cos t, e^{-0.1t} \rangle$, $t \geq -1$. Examine the curve traced out by this function, and explain why its name is appropriate. Show that the unit tangent vector at time t is $\dfrac{1}{\sqrt{1 + 0.01e^{-0.2t}}} \langle \cos t, -\sin t, -0.1e^{-0.1t} \rangle$. Next, let $\mathbf{q}(s) = \left\langle (s-1)^2, s^2, \sin \frac{\pi}{2} s \right\rangle$, and show that $\mathbf{r} \perp \mathbf{q}$ at their point of intersection $(0, 1, 1)$. If graphing software is available, have the students graph the two curves as shown below to verify the conclusion.



$\mathbf{r}(t) = \langle \sin t, \cos t, e^{-0.1t} \rangle, t \geq -1$



$\mathbf{q}(s) = \left\langle (s-1)^2, s^2, \sin \frac{\pi}{2} s \right\rangle$



$\mathbf{r} \perp \mathbf{q}$ at $\langle 0, 1, 1 \rangle$

- Since the unit tangent $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$ has constant length 1, the only quantity that changes over time is its direction. Illustrate this fact using $\mathbf{r}(t) = t^3\,\mathbf{i} + t^6\,\mathbf{j}$, $t > 0$. Hence, $|\mathbf{T}'(t)|$ measures the rate of change of direction of a unit tangent vector. Also, since $|\mathbf{T}(t)| = 1$, we know that $\mathbf{T}'(t) \perp \mathbf{T}(t)$. Check this fact analytically for $\mathbf{r}(t) = t^3\,\mathbf{i} + t^6\,\mathbf{j}$.
- Point out that $\mathbf{r}(t) = \langle t^3, t^6 \rangle$ and $\mathbf{s}(t) = \langle t^6, t^3 \rangle$ satisfy $\mathbf{r}'(0) = \mathbf{s}'(0) = \mathbf{0}$, and hence neither is smooth for $-1 \leq t \leq 1$. Note that $\mathbf{s}(t)$ has a cusp when $t = 0$ at $(0,0)$, while $\mathbf{r}(t)$ does not have a cusp.
- The main idea to convey about definite vector integrals is that the students should not think of them as areas under curves, but rather as vectors.

Workshop/Discussion

- Find the tangent line to $\mathbf{r}(t) = \langle \sin(e^t \pi), \cos(e^t \pi)\, e^t \rangle$ at $t = 0$. (Answer: $\langle 0,1,1 \rangle + t\langle -\pi, 0, 1 \rangle$)
- Compute the tangent vectors *and* the unit tangent vectors to the two curves $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + \frac{\sqrt{5}}{3}\,\mathbf{k}$ and $\mathbf{s}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} - \frac{\sqrt{5}}{3}\,\mathbf{k}$ which form the intersections of the ellipsoid $\frac{4}{9}x^2 + \frac{4}{9}y^2 + z^2 = 1$ and the cylinder $x^2 + y^2 = 1$. Ask the following questions:
  1. Why do these vectors have zero $\mathbf{k}$-components?
  2. Why are these vectors the same on each curve?
  3. What angular change in direction is measured by $\mathbf{T}'(t)$ for both of these curves?
- Point out that different parametrizations of the same curve C can lead to quite different vector integrals of C. As an example, consider the line segment $y = x$ from $(0,0)$ to $(1,1)$. Compute and interpret vector integrals of different parametrizations of the same segment, to show that the integrals of vector functions do *not* generally have an easy geometric representation or any relationships among themselves. Possible parametrizations to try are $\int_0^1 \langle t, t \rangle\, dt$, $\int_0^{\pi/2} \langle \cos t, \cos t \rangle\, dt$, $\int_0^{\pi} \langle \sin \frac{1}{2}t, \sin \frac{1}{2}t \rangle\, dt$, $\int_0^1 \langle t^2, t^2 \rangle\, dt$.

Group Work 1: Velocity Vectors

See if the students recognize that Problem 3 is identical to Example 5 (page 715) from the text.

Group Work 2: Many Parametrizations

This is an open-ended group work. The word "explore" is deliberately left undefined. Closure is very important in an exercise of this nature; the students should have a chance to share their discoveries with others.

Group Work 3: Whom to Believe?

Group Work 4: The Grim Reaper

This exercise allows students to practice working with tangent vectors, using a function that has some surprising properties. For example, the parameter turns out to be the angle that the unit tangent makes with the x-axis.

Homework Problems

Core Exercises: 1, 4, 9, 17, 22, 25, 28, 32

Sample Assignment: 1, 2, 3, 4, 7, 8, 9, 11, 17, 22, 24, 25, 26, 28, 32, 43

Note: Exercises 24 and 26 require a CAS or calculator with three-dimensional graphing capability.

Exercise	C	A	N	G	V
1		×		×	
2		×		×	
3–8		×		×	
9		×			
11		×			
17		×			
22		×			

Exercise	C	A	N	G	V
24		×		×	
25		×			
26		×		×	
28		×			
32		×			
43		×			

Group Work 1, Section 10.2
Velocity Vectors

1. Consider a vector function $\mathbf{v} = \langle f(t), g(t), h(t)\rangle$. Is it true that $|\mathbf{v}'(t)| = |\mathbf{v}(t)|'$? Why or why not?

2. Show that the vector derivative for the helix $\mathbf{h}(t) = \langle a\cos t, a\sin t, bt\rangle$ makes a constant angle with the z-axis. What is this angle if $\mathbf{h}(t) = \left\langle \cos t, \sin t, \sqrt{3}t\right\rangle$?

3. Suppose a point moves along the surface of a sphere. Show that its velocity vector is always perpendicular to its position vector.

Group Work 2, Section 10.2
Many Parametrizations

1. Which of the following vector functions parametrize the helix $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$, $t \geq 0$, presented in Example 4 of Section 10.1?

(a) $\langle \cos 2t, \sin 2t, 2t \rangle$, $t \geq 0$

(b) $\langle \cos t^3, \sin t^3, t^3 \rangle$, $t \geq 0$

(c) $-\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \left(t + \frac{\pi}{2}\right)\mathbf{k}$, $t \geq -\frac{\pi}{2}$

2. Check that the following curves parametrize the unit circle. For each curve, compute the unit tangent vector $\mathbf{T}(t)$.

(a) $\langle \sin t, \cos t \rangle$, $0 \leq t \leq 2\pi$

(b) $\cos t\,\mathbf{i} + \sin t\,\mathbf{j}$, $\pi \leq t \leq 3\pi$

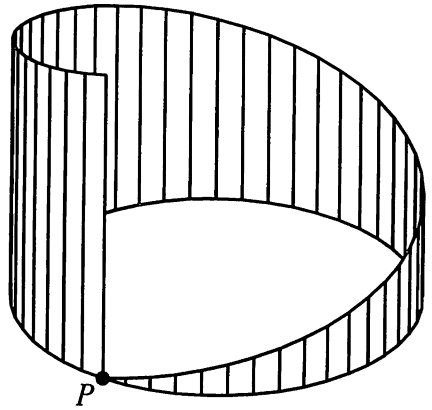
(c) $\langle \sin 2t, \cos 2t \rangle$, $0 \leq t \leq \pi$

(d) $\sin t^2\,\mathbf{i} + \cos t^2\,\mathbf{j}$, $0 \leq t \leq \sqrt{2\pi}$

(e) $\left\langle \frac{2t}{1+t^2}, \frac{t^2-1}{1+t^2} \right\rangle$, $-\infty < t < \infty$

Group Work 3, Section 10.2
Whom to Believe?

Sally Dart, the great sculptor, had just finished designing her newest masterstroke of minimalism, "The Spiral to the Stars." It was to be a rising, curving thin wall, the top of which would follow the curve $\mathbf{r}(t) = \langle \cos \pi t, \sin \pi t, t \rangle$ from $t = 0$ to $t = 2$ as shown.



In order to help her calculate her costs, she wanted to know the area of this shape, in order to determine how much stainless Italian sheet steel she would need. To answer this question, she visited noted textbook author James Stewart.

Suddenly there is a crashing sound in the hallway! The door slams open and in walks Professor Stewart's evil twin sister Onad! She looks at the problems, laughs, and then faces Sally Dart. "My brother is noted for making things complicated for no good reason. What you really want is the area under a curve, right? And just like in single-variable calculus, the area under a parametric curve is given simply by the definite integral $\left| \int_0^2 \mathbf{r}(t)\, dt \right| = \left| \int_0^2 \langle \cos \pi t, \sin \pi t, t \rangle\, dt \right|$."

1. Following Onad's advice, derive the area of the spiral surface.

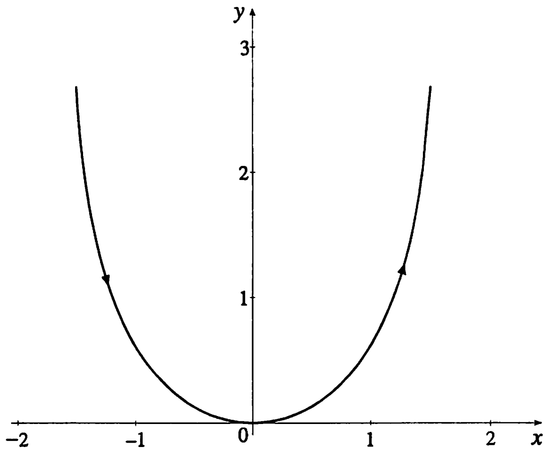
Professor Stewart smiles and says, "There is a very simple solution. Take a pair of bolt cutters and snip your piece of steel at point P. Then flatten it out and *voila*, we have a right triangle. So the area is now really easy to compute."

2. Following Professor Stewart's advice, find the area of the spiral surface.

3. Sally looks from brother to sister and back again. They have given her two different answers! Which of the two answers is correct, and why?

Group Work 4, Section 10.2
The Grim Reaper

Consider the curve $\mathbf{r}(w) = w\,\mathbf{i} - \ln(\cos w)\,\mathbf{j}$ on the interval $-\frac{\pi}{2} < w < \frac{\pi}{2}$.



1. Compute the tangent vector $\mathbf{r}'(w)$. Sketch tangent vectors corresponding to $w = -\frac{\pi}{3}$, $w = -\frac{\pi}{4}$, $w = -\frac{\pi}{6}$, $w = 0$, $w = \frac{\pi}{6}$, $w = \frac{\pi}{4}$, and $w = \frac{\pi}{3}$.

2. For each value of w, what is the length of the tangent vector $\mathbf{r}'(w)$? Find an equation for the unit tangent vector $\mathbf{T}(w)$.

3. For each value of w, what angle does the unit tangent vector $\mathbf{T}(w)$ make with the x-axis?

4. Find a vector $\mathbf{N}(w)$ perpendicular to $\mathbf{T}(w)$ and pointing away from the curve $\mathbf{r}(w)$.

10.3 Arc Length and Curvature

Suggested Time and Emphasis

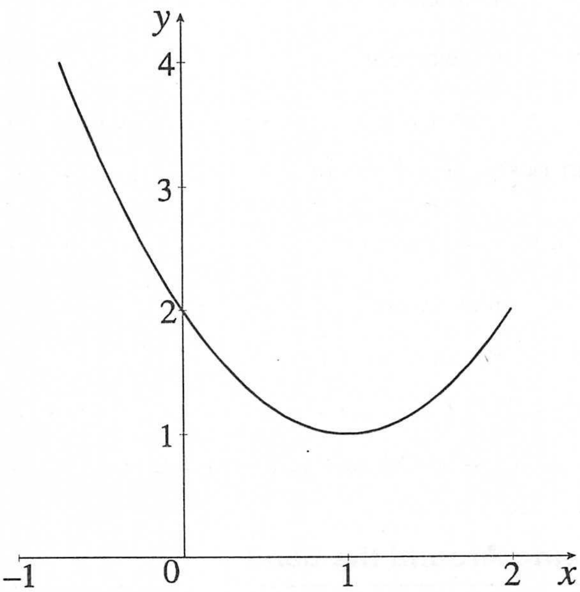
$\frac{3}{4}$–1 class   Essential Material: Arc length and the basic ideas of curvature.
Optional Material: The TNB frame.

Points to Stress

1. The arc length and curvature formulas.
2. The independence of arc length and parametrization.
3. The geometric definition of curvature.
4. The TNB frame.

Text Discussion

- Why do we need to assume that the curve C is traversed exactly once as t increases by the vector function $\mathbf{r} = \mathbf{r}(t)$ in order to define the arc-length function $s = s(t)$?
- For the following curve, sketch in an approximation of the osculating circle at the point $(1, 1)$.



Materials for Lecture

- Find a condition on $\mathbf{d}$ so that the straight line segment $\mathbf{r}(t) = t\,\mathbf{d} + \mathbf{b}$, $0 \le t \le 2$ is parametrized by arc length. (Answer: $|\mathbf{r}'(t)| = |\mathbf{d}|$, $s = \int_0^t |\mathbf{d}|\,du = |\mathbf{d}|\,t$, $t = \frac{s}{|\mathbf{d}|}$, and $\mathbf{r}(t(s)) = \frac{s}{|\mathbf{d}|}\mathbf{d} + \mathbf{b} = s\frac{\mathbf{d}}{|\mathbf{d}|} + \mathbf{b}$. So $\mathbf{d}$ needs to be a unit vector.) Show why this answer makes sense geometrically.
- Compute the length of the curve traced out by $\mathbf{r}(t) = \frac{1}{2}e^t\,\mathbf{i} + \cos e^t\,\mathbf{j} + \sin e^t\,\mathbf{k}$ as t goes from 0 to 1. [Answer: $L = \int_0^1 |\mathbf{r}'(t)|\,dt = \int_0^1 \sqrt{e^{2t} + \frac{1}{4}e^{2t}}\,dt = \frac{\sqrt{5}}{2}(e-1)$]

- Point out that since the unit tangent vector satisfies $|\mathbf{T}(t)| = 1$, we have $\mathbf{T}'(t) \perp \mathbf{T}(t)$ and so $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$ is perpendicular to $\mathbf{T}(t)$. Hence $\mathbf{N}(t)$ is a unit normal.
- Use the following method to intuitively describe the curvature $\kappa = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$: Since the length of $\mathbf{T}$ is constant, $|\mathbf{T}'(t)|$ is the rate of change of direction, or turning, of the unit tangent, and $|\mathbf{r}'(t)|$ measures the speed along the curve. So κ is essentially the rate of turning of the unit tangent $\mathbf{T}$ divided by the speed along the curve. For a circle parametrized in the standard way, $\mathbf{r}(t) = \langle a\cos t, a\sin t\rangle$, the angle of the unit tangent will change by 2π as we go around the circle, and so the rate of turning is 1, and the speed a is the radius of the circle. Therefore $\kappa = \dfrac{1}{\text{radius}}$. For a straight line, since the unit tangent never turns, κ is zero. One can also conclude that κ is zero by noting that in the case of a straight line, the radius is "infinite".

Workshop/Discussion

- Consider the curve $\mathbf{r}(t) = \sin t\,\mathbf{i} + t\,\mathbf{j} + \cos t\,\mathbf{k}$, $0 \le t \le 2\pi$. Use this example to show that arc length is independent of parametrization by computing the arc length for various parametrizations:

$$\begin{aligned} L &= \int_0^{2\pi} |\mathbf{r}'(t)|\,dt \\ L_1 &= \int_0^{\sqrt{2\pi}} |\mathbf{s}'(t)|\,dt, \text{ where } \mathbf{s}(t) = \left\langle \sin t^2, t^2, \cos t^2 \right\rangle, 0 \le t \le \sqrt{2\pi}, \text{ and} \\ L_2 &= \int_{\pi/2}^{5\pi/2} |\mathbf{v}'(s)|\,ds, \text{ where } \mathbf{v}(s) = \left\langle \cos s, s + \tfrac{\pi}{2}, -\sin s \right\rangle, \ -\tfrac{\pi}{2} \le s \le \tfrac{3\pi}{2} \end{aligned}$$

- Talk about the curvature of the helix $\mathbf{r}(t) = \langle \cos t, \sin t, t\rangle$. Mention that it is possible to first find the arc length parameterization $\mathbf{r}(s) = \cos\left(\frac{1}{\sqrt{2}}s\right)\mathbf{i} + \sin\left(\frac{1}{\sqrt{2}}s\right)\mathbf{j} + \frac{1}{\sqrt{2}}s\,\mathbf{k}$, then compute $\mathbf{T}(s) = \mathbf{r}'(s)$, and finally, $\left|\dfrac{d\mathbf{T}}{ds}\right|$. But a direct computation using one of the formulas for curvature with the given parametrization gives $\kappa = \frac{1}{2}$, with less work.
- Compute $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ for a simple curve such as $\left\langle t^2, -t, t\right\rangle$ at $t = 1$.
- Consider the curve $\mathbf{r}(t) = \left\langle e^{-t}\cos t, e^{-t}\sin t, e^{-t}\right\rangle$, $t \ge 0$. Explain why this curve spirals down to the origin along the surface $z^2 = x^2 + y^2$. Show that the curvature is $\kappa(t) = \frac{\sqrt{2}}{3}e^t$, and explain what this means about the curve as $t \to \infty$.

Group Work 1: Going Around the Bend

There are two ways to do Problem 1, either intuitively by thinking about osculating circles, or computationally, as a max/min problem. We recommend instructing the students to first make an intuitive guess about the correct answer, and then use computations to verify their guess. Problem 2 answers a question raised in the workshop/discussion material.

Group Work 2: The Length of the Reaper

Parts 1–4 are straightforward. Parts 5–7 further develop the arc-length parametrization.

Group Work 3: T, N, B

Extended Group Work 4: The Evolute of a Plane Curve

iis project can be assigned to a group of students who wish to explore the material more deeply in an teresting context. It should take between one and three hours to complete.

Lab Project: Osculating Circles

se a CAS to plot the plane curve and its osculating circle at various points:

. $y = x^3$

. $y = \sin x$

. $y = e^x$

Extended Lab Project: Estimating Arc Length

Homework Problems

re Exercises: 1, 9, 14, 17, 27, 33, 37

mple Assignment: 1, 5, 9, 10, 14, 17, 24, 26, 27, 28, 33, 35, 37, 39, 42, 44

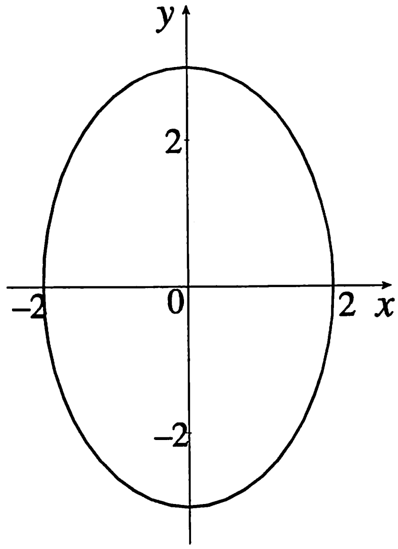
te:
- Exercise 42 requires a CAS.
- Problems 8 from Focus on Problem Solving (page 747) is very challenging and would make a good project.

Exercise	C	A	N	G	V
1		×			
5				×	
9	×	×			
10		×			
14		×			
17		×			
24		×			
26		×			

Exercise	C	A	N	G	V
27					×
28		×		×	
33		×			
35		×			
37		×			
39		×		×	
42		×			
44	×	×			

Group Work 1, Section 10.3
Going Around the Bend

1. Consider the ellipse $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 3\sin t\,\mathbf{j}$.



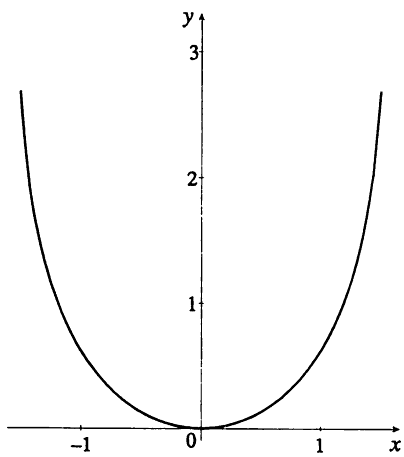
(a) Where is the curvature maximal? Give reasons for your answer.

(b) Where is the curvature minimal? Give reasons for your answer.

2. Consider the curve $\mathbf{r}(t) = \langle e^{-t}\cos t, e^{-t}\sin t, e^{-t}\rangle$, $t \geq 0$. Show that the distance traveled by a particle along this curve over *all* time is $\sqrt{3}$. (**Hint:** What is the appropriate improper integral to evaluate?)

Group Work 2, Section 10.3
The Length of the Reaper

Consider the graph of $y = -\ln(\cos x)$.



1. Fill in the blank: This graph can be written as a parametrized curve by $\mathbf{r}(t) = \langle t, \underline{\qquad\qquad} \rangle$, $-\frac{\pi}{2} < t < \frac{\pi}{2}$.

2. Compute $\mathbf{r}'(t)$ and $\kappa(t)$.

3. Find the unit tangent vector $\mathbf{T}(t)$ and the unit normal vector $\mathbf{N}(t)$.

4. Find the arc length function $s(t)$ by solving $s(t) = \int_0^t |\mathbf{r}'(u)|\, du$

5. Solve for $\sec t$ in terms of s. (*Hint:* You'll have to solve the arc length function you found in the previous part for $\tan t$, and then use a trigonometric identity.)

6. Solve for $\cos t$ in terms of s.

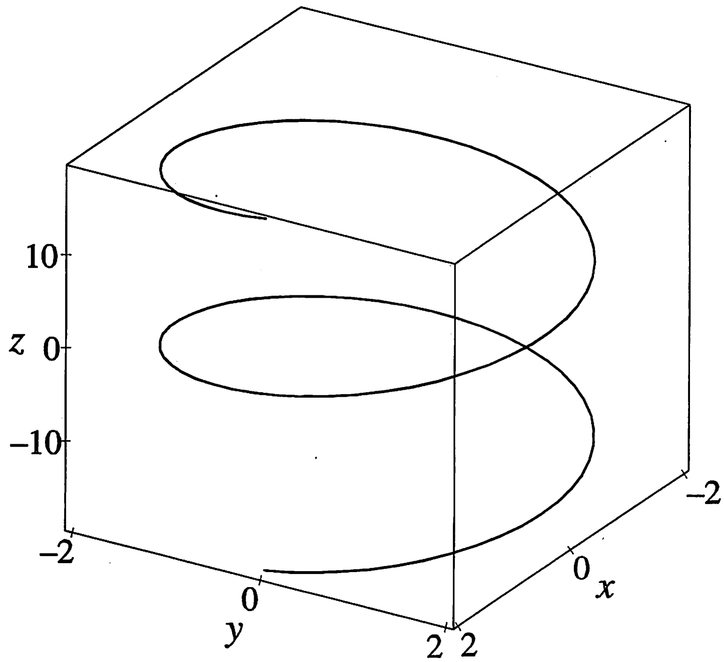
7. Parametrize $\mathbf{r}(t)$ with respect to arc length. That is, find $\mathbf{r}(s)$.

Notice that even though we now have parametrized the Grim Reaper with respect to arc length, these computations in terms of s are very awkward. Would you like to, say, find r when $s = 1$ using the equation you derived above? We wouldn't either. In theory, we can compute $|\mathbf{r}'(s)| = 1$ for all s and $\kappa(s) = |\mathbf{r}''(s)|$, but these computations are often messy.

Group Work 3, Section 10.3

T, N, B

If $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + 3t\,\mathbf{k}$, find $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$. Sketch $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ when $t = \frac{3\pi}{2}$.



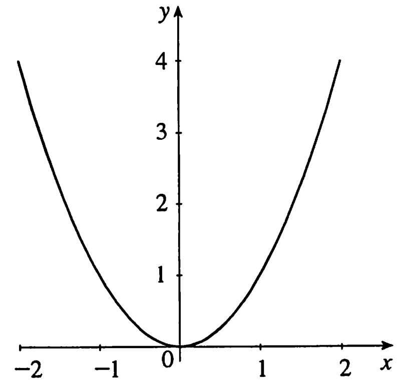
Extended Group Work 4, Section 10.3
The Evolute of a Plane Curve

Suppose the vector function $\boldsymbol{\alpha}(t)$ describes a plane curve, and $\mathbf{N}(t)$ is the unit normal vector as defininED in the text. Now define a different vector function $\boldsymbol{\beta}(t) = \boldsymbol{\alpha}(t) + \frac{1}{\kappa(t)}\mathbf{N}(t)$, where $\kappa(t)$ is the curvature to $\boldsymbol{\alpha}$. This function $\boldsymbol{\beta}$ is called the **evolute** of $\boldsymbol{\alpha}$. In this exercise, we will figure out exactly what makes the evolute so special.

1. Let $\boldsymbol{\alpha}(t) = \langle t, t^2 \rangle$. Compute $\mathbf{N}(t)$ and show that $\kappa(t) = \dfrac{2}{(1+4t^2)^{3/2}}$.

2. Use Problem 1 to compute the evolute $\boldsymbol{\beta}(t)$ of $\boldsymbol{\alpha}(t)$.

3. Sketch the curve given by $\boldsymbol{\beta}(t)$ on the axes below.



$\boldsymbol{\alpha}(t)$

4. Show that $|\boldsymbol{\alpha}(t) - \boldsymbol{\beta}(t)|$ is the radius of the osculating circle for the above curve.

5. Show that the endpoint of the position vector $\boldsymbol{\beta}(t)$ is the center of the osculating circle at $\boldsymbol{\alpha}(t)$ for the above curve.

6. Verify your solutions to the problems above by specifically computing the curvature of $\boldsymbol{\alpha}(t)$ at $t = 0$ and $t = 1$.

10.4 Motion in Space

Suggested Time and Emphasis

$\frac{3}{4}$–1 class Essential Material: Velocity and acceleration
Optional Material: Kepler's Laws

Points to Stress

1. Definitions of velocity and acceleration as vector functions.
2. How to derive velocity from acceleration and position from velocity.
3. Tangential and normal components of acceleration.

Text Discussion

- What is the relationship between the velocity vector and the tangent vector?
- How is ds/dt, the rate of change of distance along the direction of motion with respect to time, related to the velocity vector $\mathbf{v}(t)$?
- Why must the acceleration vector $\mathbf{a}$ always lie in the osculating plane at a point on a curve?

Materials for Lecture

- Emphasize that the velocity vector $\mathbf{v}$ gives the direction and speed with which a particle would travel if it flew off the curve at that instant. Also, the speed $|\mathbf{v}(t)| = ds/dt$ is the rate of change of distance along the direction of motion with respect to time.
- Reinforce Example 4 by reminding students that $\theta(t)$ is the angle that $\mathbf{r}(t)$ makes with the positive x-axis, and that the angular speed $d\theta/dt$ is the rate of change of θ with respect to time.
- Point out that the model described in Figure 5 is not a good description of the way the Earth moves about the Sun. The Earth moves around the Sun in an elliptical orbit with the Sun at a focus. Note that the acceleration of the Earth points toward the Sun, not toward the center of our elliptical orbit.
- Emphasize that the acceleration vector $\mathbf{a}(t)$ is not generally perpendicular to $\mathbf{v}(t)$. (Use $\mathbf{r}(t) = \langle t, e^t, e^{-t} \rangle$ as an example.) However, if $|\mathbf{v}(t)|$ is constant, then $\mathbf{a}(t) \perp \mathbf{v}(t)$.
- Remind students that since the unit tangent vector $\mathbf{T}$ satisfies $|\mathbf{T}| = 1$, then $\mathbf{T}$ is orthogonal to $\mathbf{T}'$ and hence $\mathbf{N} = \frac{1}{|\mathbf{T}'|}\mathbf{T}'$. This is important in deriving the expressions for the components of acceleration $a_{\mathbf{N}}$ and $a_{\mathbf{T}}$.

Workshop/Discussion

- Compute velocity and acceleration vectors for $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$ and $\mathbf{s}(t) = \langle \cos t, \sin t, t^2 \rangle$. Point out that $\mathbf{r}$ has no z-component of acceleration, while $\mathbf{s}$ has a constant z-component of acceleration. Then note that the length of the acceleration vector is still constant for both curves.
- Compute the tangential and normal accelerations in Example 5.

- Give an example to show how different vector functions whose ranges are the same can have the same speed but different velocities (for example, $\mathbf{q} = \langle \sin t, \cos t \rangle$ and $\mathbf{r} = \langle \cos t, \sin t \rangle$).
- Graph and analyze an example of three-dimensional centripetal force. (*Centripetal* means "directed from the outside towards the center.") For example, the function $\mathbf{r}(t) = \langle 5 \sin 2t, 4 \cos 2t, 3 \cos 2t \rangle$ gives $\mathbf{r}'' = -4\mathbf{r}$, while $\mathbf{r} = \langle \cos t, \sin t, \cos t + \sin t \rangle$ gives $\mathbf{r}'' = -\mathbf{r}$.
- Describe the motion of $\mathbf{w}(t) = \langle 10e^{-2t}, 0.2e^{2t}, 3e^{-2t} \rangle$. Show that $\mathbf{w}'' = 4\mathbf{w}$ and explain why this force is called centrifugal (directed from the center to the outside.)

Group Work 1: Checking Out the Action

[he equation of motion in Problem 2 is $\mathbf{r}(t) = \frac{1}{k}\left(e^{kt} - 1\right)\mathbf{v}_0 + \mathbf{r}_0$, and this curve lies along the line $(s) = s\mathbf{v}_0 + \mathbf{r}_0$.

Group Work 2: Back to Start

[he second question is impossible to solve as stated, for it would violate the Mean Value Theorem. The idea s to illustrate a fundamental difference between graphs of ordinary functions and curves in $\mathbb{R}^2$ or $\mathbb{R}^3$. After illowing the students to try the second question for awhile, announce that an acceptable answer would be a eason why it cannot be solved.

Group Work 3: Find the Error

[his group work has a subtle solution, and the students may require some guidance. The main flaw in the easoning presented is that the argument doesn't take into account that the base points of the vectors are not he same. Hence, the vector labelled $\mathbf{v}(t + \Delta t) - \mathbf{v}(t)$ is really $\mathbf{v}(t + \Delta t) + \mathbf{r}(t + \Delta t) - (\mathbf{v}(t) + \mathbf{r}(t))$ and he limit vector is $\mathbf{a}(t) + \mathbf{v}(t)$. Notice that our *formula* is correct, but the *picture* gives the wrong acceleration 'ector.

Homework Problems

ore Exercises: 1, 2, 5, 7, 14, 17, 32

ample Assignment: 1, 2, 5, 7, 12, 14, 16, 17, 23, 26, 27, 32

lote: For a physical extension of this material, Problems 3 and 4 from Focus on Problem Solving (page 746) can be assigned. The computations for these problems are quite challenging.

Exercise	C	A	N	G	V
1			×		
2				×	
5		×		×	
7		×		×	
12		×			
14		×			

Exercise	C	A	N	G	V
16		×		×	
17		×			
23		×			
26		×			
27		×			
32		×		×	×

Group Work 1, Section 10.4
Checking Out the Action

1. Using the tangential and normal components of acceleration, describe in words those curves $\mathbf{r}$ for which

(a) $\mathbf{a} \parallel \mathbf{v}$

(b) $\mathbf{a} \perp \mathbf{v}$

2. We now look at the case $\mathbf{a} \parallel \mathbf{v}$ in a little more detail. Suppose $\mathbf{a} \parallel \mathbf{v}$; for example, say $\mathbf{a} = k\mathbf{v}$, k a non-zero constant. Assuming initial position $\mathbf{r}_0$ and initial velocity $\mathbf{v}_0$,

(a) Find an equation for the velocity function $\mathbf{v}(t)$ in terms of $\mathbf{v}_0$.

Hint: Remember that $\mathbf{a} = \dfrac{d\mathbf{v}}{dt}$.

(b) Find an equation for the position function $\mathbf{r}(t)$ in terms of $\mathbf{v}_0$ and $\mathbf{r}_0$.

(c) Describe the line which contains the range of the position function $\mathbf{r}(t)$.

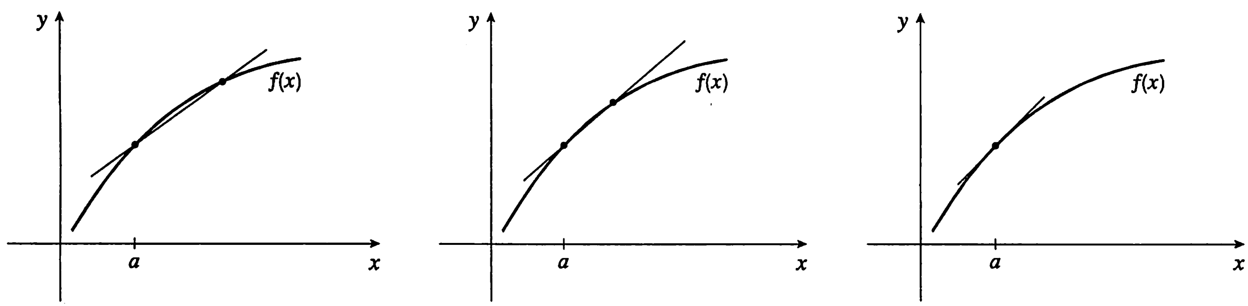
Group Work 2, Section 10.4
Back to Start

1. Give an example of a curve in $\mathbb{R}^2$ or $\mathbb{R}^3$ for which the position at time $t = 10$ is the same as the position at time $t = 0$, yet the speed is never zero.

2. Is it possible to find a differentiable function of a single variable with this property, that is, a function $y = f(x)$ for which $f(0) = f(10)$ and $f'(x)$ is never zero for $x \in [0, 10]$? Why or why not?
Hint: Think of some important theorems about derivatives.

Group Work 3, Section 10.4
Find the Error (Part 1)

As you recall, one way to figure out the derivative of a function $f(x)$ at a point was to compute the slope of the line tangent to f at the point x by taking the limit of secant lines as follows:

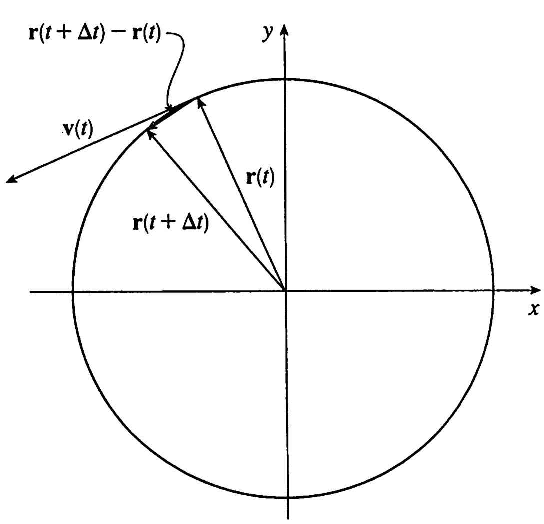


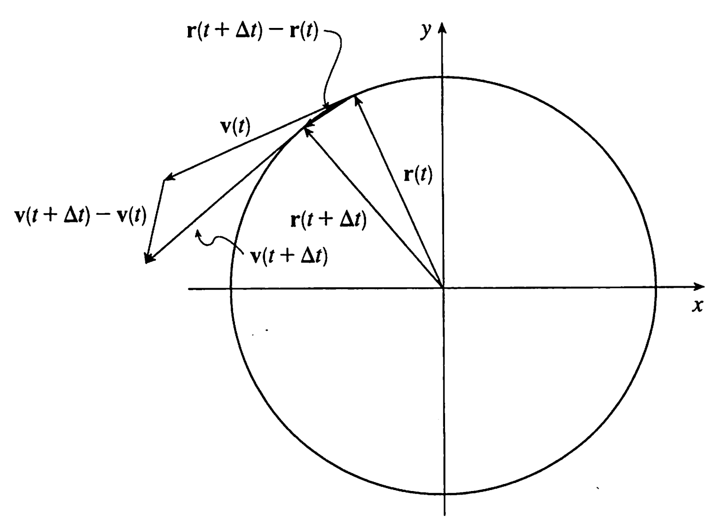
We found that $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

At first glance, it may seem reasonable to try to compute the vector acceleration of a function using that same technique. In this case, however, something goes wrong. We present the false argument below:

Consider the circle parametrized by $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$, $0 \le t \le 2\pi$. The direction of $\mathbf{r}(t + \Delta t) - \mathbf{r}(t)$ should be a good approximation for the direction of $\mathbf{v}(t)$ since $\mathbf{v}(t) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t}$ (see the diagram on the left).

Now consider the same circle. The direction of $\mathbf{v}(t + \Delta t) - \mathbf{v}(t)$ should be a good approximation to the direction of $\mathbf{a}(t)$ since $\mathbf{a}(t) = \lim_{\Delta t \to 0} \frac{\mathbf{v}(t+\Delta t) - \mathbf{v}(t)}{\Delta t}$ (see the diagram on the right).





Since $\mathbf{a}(t) = \langle -\cos t, -\sin t \rangle$, we see that $\mathbf{a}(t)$ should point toward the origin, however the direction of $\mathbf{v}(t + \Delta t) - \mathbf{v}(t)$ is not the same as that of $\mathbf{a}(t)$.

Find the Error (Part 1)

1. Why should $\mathbf{a}(t)$ point towards the origin?

2. What, then, is wrong with our analysis above?

Applied Project: Kepler's Laws

This is a project that is best assigned in its entirety, although Problem 3 or Problem 4 can be omitted if there is a time crunch. (We recommend keeping Problem 4.) Make sure to impart to the students how amazing the results of this project really are: Given *only* that

1. $\mathbf{F} = m\mathbf{a}$ and
2. the gravitational force between two objects is proportional to the product of their masses and inversely proportional to the square of the distance between them,

it is possible to use calculus to deduce Kepler's three laws without even looking out of the window to make a measurement!

10.5 Parametric Surfaces

Suggested Time and Emphasis

$\frac{3}{4}$–1 classes   Essential Material, but it can be delayed until just before Section 13.6.

Transparencies Available

- Transparency 37 (Figure 5, page 737)
- Transparency 38 (Exercises 11–16, graphs 1–VI, page 741)

Points to Stress

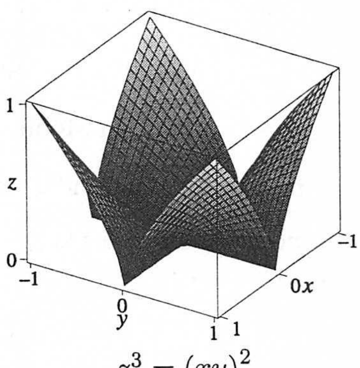
1. Parametric surfaces and the role of grid curves in studying these surfaces.
2. How the form and/or symmetry of a surface helps one in choosing a parametrization.
3. Different parametrizations for surfaces.

Text Discussion

- Why parametrize a surface?

Materials for Lecture

- Revisit the discussion from Section 9.5 on parametrizing a plane.
- Present an example of how to choose a parametrization for a surface using form or symmetry. A good example is the top half of the ellipsoid $\frac{x^2}{4} + y^2 + \frac{z^2}{4} = 1$, $y \geq 0$. Notice that $x^2 + z^2 = 4\left(1 - y^2\right)$, so if we let $u = 2\sqrt{1-y^2}$, then $x^2 + z^2 = u^2$. So we let $x = u\cos v$, $z = u \sin v$, and we have the parametrization $\mathbf{r}(u,v) = \left\langle u\cos v, \sqrt{1 - \frac{1}{4}u^2}, u \sin v \right\rangle$ for $0 \leq u \leq 2$, $0 \leq v \leq 2\pi$. Note that now the surface is the graph of a parametric function, and that for the same values of u and v, $\mathbf{s}(u,v) = \left\langle u\cos v, -\sqrt{1 - \frac{1}{4}u^2}, u\sin v\right\rangle$ parametrizes the bottom half of the ellipsoid.
- Present examples of how to determine what a surface looks like from its parametrization. Perhaps start with the example $q(s,t) = \langle s, t, st^2\rangle$, which parametrizes the surface $z = xy^2$. Then look at the parametrization $\mathbf{w}(s,t) = \langle st^2, s^2t, s^2t^2\rangle$, $x = st^2$, $y = s^2t$, and $z = s^2t^2$. We have $xy = s^3t^3$ and $(xy)^2 = s^6t^6 = \left(s^2t^2\right)^3 = z^3$, so an equation of the surface is $z^3 = (xy)^2$. This surface can be visualized by setting $y = 1$ and thinking of the plane curve $z = x^{2/3}$ or setting $x = 1$ and thinking of the plane curve $z = y^{2/3}$.



$z^3 = (xy)^2$

- Point out that surface of revolution S formed by rotating $y = f(x)$ about the x-axis satisfies $y^2 + z^2 = [f(x)]^2\left(\cos^2\theta + \sin^2\theta\right) = [f(x)]^2$, which describes the surface directly as an equation in x, y, and z. Similarly, the surface T formed by rotating $z = g(y)$ about the y-axis has equation $x^2 + z^2 = [g(y)]^2$ or parametric equations $x = g(y)\cos\theta$, $y = y$, $z = g(y)\sin\theta$.

Workshop/Discussion

- Parametrize the elliptic paraboloid $x^2 + y^2 - z^2 = 1$. (Answer: $\mathbf{r}(u,v) = \left\langle u\cos v, u\sin v, \pm\sqrt{u^2-1}\right\rangle$)
- Identify the surfaces parametrized by $\mathbf{r}(s,t) = \langle s\sin t, s\cos t, s\rangle$ if

  1. $0 \le s \le 1, 0 \le t \le \pi$ **2.** $0 \le s \le 4, 0 \le t \le 2\pi$ **3.** $0 \le s, 0 \le t \le 2\pi$
- Give two different parametrizations of a cone, one in which the grid curves meet at right angles and one in which one set of grid curves spirals up the cone.
  [Examples: $\mathbf{A}(s,t) = \langle s\cos t, s\sin t, s\rangle$ and $\mathbf{B}(s,t) = \langle st\cos t, st\sin t, st\rangle$]
- Find two parametrizations for the parabolic surface $z = 16 - x^2 - y^2$ that lies above the xy-plane, first by using the Cartesian parametrization ($x = x$, $y = y$, $z = 16 - x^2 - y^2$, $z \ge 0$) and then identifying z as a surface of revolution for the curve $y = \sqrt{16-z}$, $z \ge 0$ (which gives the parametrization $x = \sqrt{16-z}\cos\theta$, $y = \sqrt{16-z}\sin\theta$, $z = z$, $z \ge 0$). Note that the grid curves are parabolas in the first case and parabolas/circles in the second. A third approach is to use cylindrical coordinates.

Group Work 1: The Propeller Problem

Be sure that the students understand that $\mathbf{r}(\theta, z)$ is being given to them in cylindrical coordinates (r, θ, z). For instance, $2 + \cos(4(\theta - z))$ is the value of r at angle θ and height z above the xy-plane.
If the parameter z is imagined to represent time, this surface represents a spinning propeller. Note that the $z = 0$ grid curve of this surface appeared in Group Work 2: Intersections and Curves, from Section 10.1.

Group Work 2: Bagels, Bagels, Bagels!

This is Exercise 32 from the text. There are two versions of this group work included. The second requires more independent thought on the part of the students.

Group Work 3: More With Möbius Strips

This group work extends Exercise 30 and requires technology.

The surface with parametric equation $\mathbf{r}(\theta,t) = \left\langle 2\cos\theta + t\cos\frac{1}{2}\theta, 2\sin\theta + t\cos\frac{1}{2}\theta, t\sin\frac{\theta}{2}\right\rangle$, $0 \le \theta \le 2\pi$, $-\frac{1}{2} \le t \le \frac{1}{2}$ is called a Möbius strip.

1. Graph this surface and consider several viewpoints. What is unusual about it?

2. Find the coordinates (x, y, z) corresponding to

(a) $\theta = 0, t = \frac{1}{2}$ $\theta = \pi, t = \frac{1}{2}$ $\theta = 2\pi, t = \frac{1}{2}$ $\theta = 3\pi, t = \frac{1}{2}$ $\theta = 4\pi, t = \frac{1}{2}$

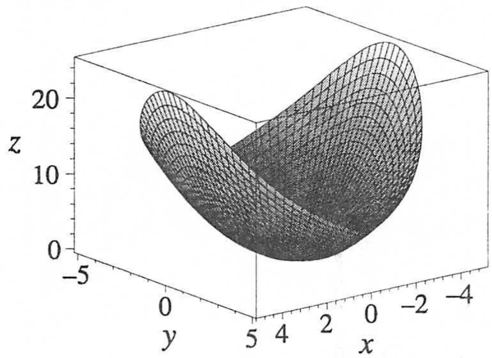
(b) $\theta = 0, t = -\frac{1}{2}$ $\theta = \pi, t = -\frac{1}{2}$ $\theta = 2\pi, t = -\frac{1}{2}\theta = 3\pi, t = -\frac{1}{2}$ $\theta = 4\pi, t = -\frac{1}{2}$

3. Graph the grid curves corresponding to $t = \frac{1}{2}$ and $t = -\frac{1}{2}$ and note that they are the same set of points. Also note that each curve makes two circuits about the z-axis and then closes up at $\theta = 4\pi$. However, the complete Möbius strip is created when $0 \le \theta \le 2\pi$. How can this be?

Lab Project: A Difficult Plot

1. Have the students plot the surface described by $x^2 + y^2 = z\left[1 + \left(\tan^{-1}\frac{x}{y}\right)^2\right]$. Allow them to notice that this is very hard to do, even with a sophisticated graphing package like Maple or Mathematica. They may fail to get a picture of this surface that makes sense. This is okay; let them fail this time.

2. Now have them draw the parametric surface $\mathbf{r}(s,t) = \left\langle s\cos t, s\sin t, \frac{s^2}{1+t^2}\right\rangle$, $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$. This one should be straightforward. They should get a picture like this:



3. Now have them discover, using algebra, that the two surfaces are the same. When they do this, discuss how important it is to be able to parametrize surfaces.

4. Let them investigate the surface for $t \in \mathbb{R}$. Ask what the relationship is between the two intersecting surfaces which result.

Homework Problems

Core Exercises: 1, 4, 11–16, 27, 32(a)

Sample Assignment: 1, 4, 6, 9, 11–16, 19, 21, 27, 32

Note:
- Problems 6, 9, 27, and 32 require a CAS.
- Problem 32 should be done in class, as group work, or as part of the assignment.

Exercise	C	A	N	G	V
1		×			×
4		×			×
6				×	
9				×	
11–16					×

Exercise	C	A	N	G	V
19		×			×
21		×			×
27		×		×	
32		×		×	

Group Work 1, Section 10.5
The Propeller Problem

Consider the parametrized surface $\mathbf{r}(\theta, z) = \langle 2 + \cos(4(\theta - z)), \theta, z \rangle$, where $\langle r, \theta, z \rangle$ is the cylindrical representation of the position vector $\mathbf{r}(\theta, z)$.

1. Sketch grid curves for $(\theta, 0)$, $(\theta, 1)$, and $(\theta, 2)$.

2. Describe this surface and sketch it if you can.

3. If the parameter z is imagined to represent time, that is, increasing time is represented by increasing the z-coordinate, what can this surface represent?

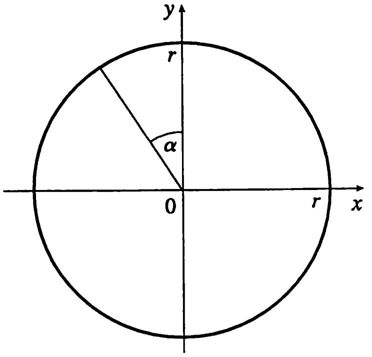
Group Work 2, Section 10.5

Bagels! Bagels! Bagels! (Version 1)

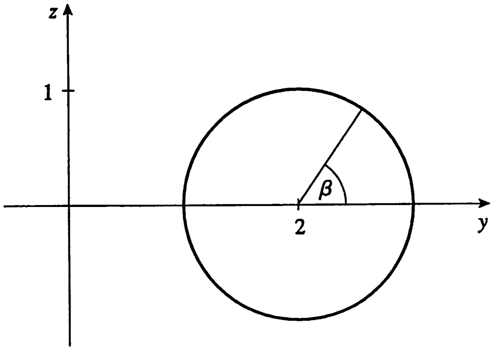
There are many ways to create bagels. Some people prefer boiling them, some prefer baking them, and some prefer defrosting then toasting them. Today we shall be creating bagels by the method of parametrizing them.

In single-variable calculus, it was shown that one can compute the volume of a torus, or doughnut shape, by thinking of it as a circle rotated about a horizontal or vertical line.

1. Parametrize a circle of radius r centered at the origin in the xy-plane starting at $(0, r)$. Let α be the angle between the position vector and the y-axis.



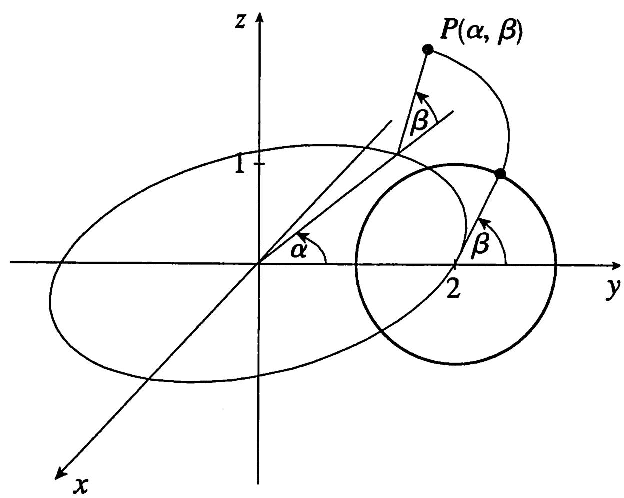
2. Parametrize a circle of radius 1 in the yz-plane with center $(2, 0)$ starting at $(0, 3)$. Let β be the angle between the position vector and the positive y-axis.



3. Now we want to characterize a typical point on our bagel, so we can write a vector function s (α, β) whose range is the entire breakfast treat. To find any specific point we

(a) Move α radians along the horizontal curve; then

(b) Rotate β radians along the vertical curve.



Now express the bagel as the range of a vector function s (α, β).

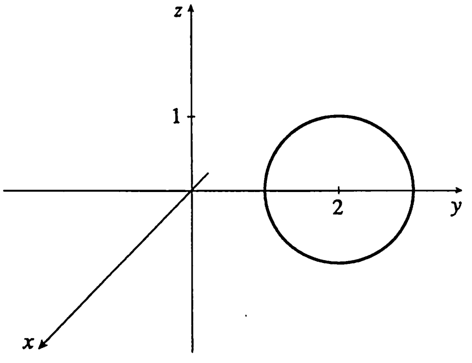
Group Work 2, Section 10.5

Bagels! Bagels! Bagels! (Version 2)

There are many ways to create bagels. Some people prefer boiling them, some prefer baking them, and some prefer defrosting and then toasting them. Today we shall be creating bagels by the method of parametrizing them.

We are going to parametrize a bagel obtained by rotating the circle $(y-2)^2+z^2=1$ about the z-axis.

1. Find a parametrization for the circle below.



2. Write a parametrization for the circle when it has been rotated about the z-axis through angles of

(a) 0

(b) $\frac{\pi}{2}$

(c) π

(d) $\frac{3\pi}{2}$

(e) 2π

3. When the above circle has been rotated through an angle of $\frac{\pi}{4}$ about the z-axis, in what plane does the circle lie? What is a parametrization of the circle that lies in the plane?

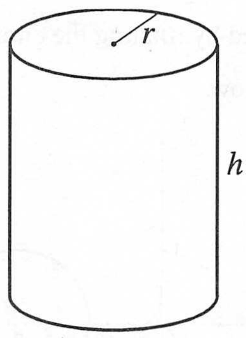
4. What is a parametrization of the circle when it has been rotated through an angle of α about the z-axis?

5. If you have a graphics program, graph the surface described by your parametrization in Problem 4.

10 Sample Exam

Problems marked with an asterisk (*) are particularly challenging and should be given careful consideration.

1. The volume of a can of base radius r and height h is a function of the variables r and h.



(a) Write a formula for the volume V of a can of radius r and height h.

(b) What is the domain of $V(r, h)$?

(c) Graph the traces $r = 2$ and $h = 3$.

(d) Graph the level curve $V = 4\pi$.

2. Describe a non-zero vector function $\mathbf{r}(t)$ in $\mathbb{R}^3$ whose acceleration vector $\mathbf{a}(t)$ satisfies $\mathbf{a}(t) = -\mathbf{r}(t)$ for all t.

3. Let $\mathbf{r}(s)$ be the arc-length parametrization of a circle of radius R.

(a) What is the domain of $\mathbf{r}$?

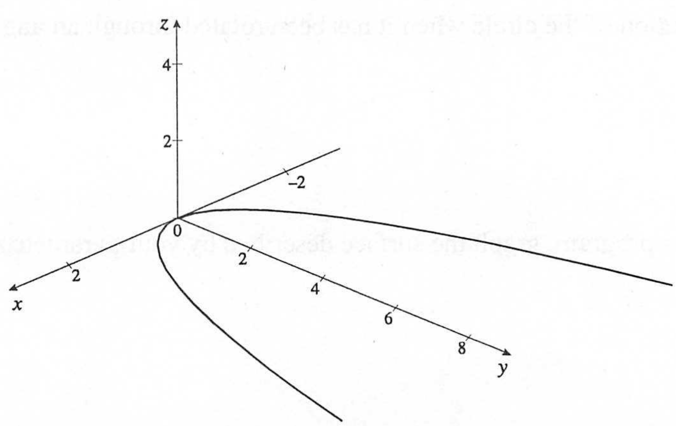
(b) Find a value of $s \neq 0$ such that $\mathbf{r}(s) = \mathbf{r}(0)$, or explain why no such value exists.

4. Let C be the circle of radius 3 centered at the point $(2, 5)$ in the xy-plane.

(a) Give the arc-length parametrization $\mathbf{r}(s)$ of this curve, starting at the point $(2, 8)$.

(b) Verify that the curvature of C is constant.

5. Let $y = x^2$ be a parabola in the xy-plane parametrized by $\mathbf{r}(t) = \langle t, t^2, 0\rangle$. What are the vectors $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ at the origin?



6. Match the equations with their graphs. Give reasons for your choices.

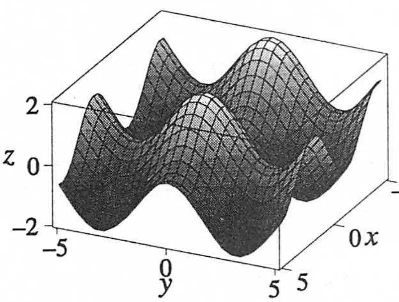
(a) $8x + 2y + 3z = 0$

(b) $z = \sin x + \cos y$

(c) $z = \sin\left(\dfrac{\pi}{2 + x^2 + y^2}\right)$

(d) $z = e^y$

I



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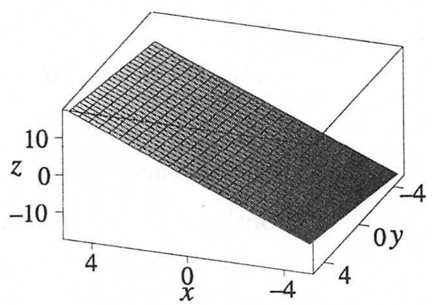
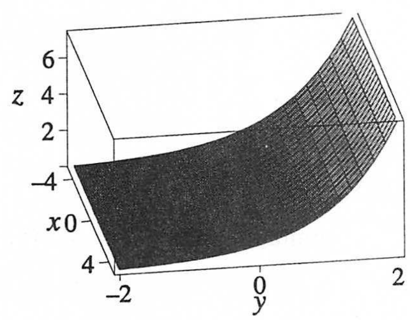
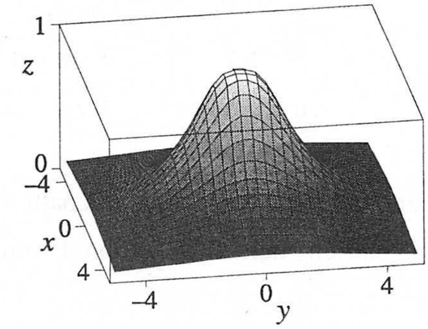


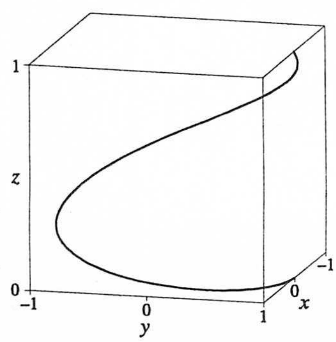
III



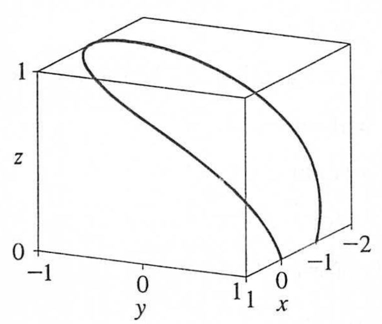
IV



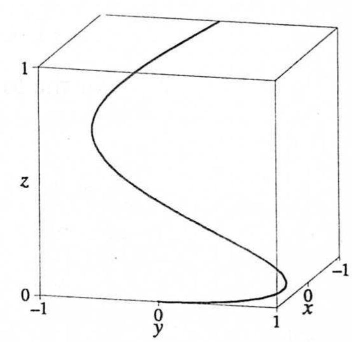
7. Which curve below is the path traced out by $\mathbf{r}(t) = \left\langle \sin \pi t, \cos \pi t, \frac{1}{4}t^2 \right\rangle$, $0 \le t \le 2$? Justify your answer.



Graph 1

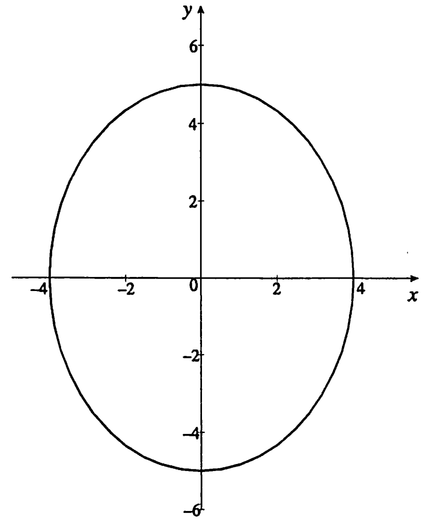


Graph 2



Graph 3

8. Consider the ellipse $\frac{x^2}{16} + \frac{y^2}{25} = 1$.



(a) Find two different parametrizations of this ellipse.

(b) Find a point where the curvature is minimal. Give a reason for your answer.

9. Consider the curve $\mathbf{w}(t) = \langle 3e^{t/2}, 4e^{-t/2} \rangle$.

(a) Compute the velocity and acceleration vectors $\mathbf{v}(t)$ and $\mathbf{a}(t)$.

(b) What is the relationship between $\mathbf{w}(t)$ and $\mathbf{a}(t)$?

(c) Is the unit tangent vector at 0 orthogonal to the acceleration vector at 0?

10. Consider the intersection of the paraboloid $z = x^2 + y^2$ with the plane $x - 2y = 0$. Find a parametrization of the curve of intersection and verify that it lies in each surface.

11. (a) Find a parametrization of the circular path along a circle of radius 2 going counterclockwise from $(0, 2)$ to $(0, -2)$.

(b) Give another parametrization of a path from $(0, 2)$ to $(0, -2)$.

12. A crazed ostrich named Rhomboid runs along a mountain path with coordinates given by $\mathbf{r}(t) = \langle e^t, e^{-t}, \sqrt{2}t \rangle$.

(a) What is the change in Rhomboid's altitude from $t = 0$ to $t = 10$?

(b) What is Rhomboid's speed in the x-direction when $t = 4$?

(c) What is Rhomboid's speed in the y-direction when $t = 4$?

(d) Find a formula for the total distance travelled by Rhomboid the Crazed Ostrich from $t = 0$ to $t = 4$.

13. A point moves along a circle with position vector given by $\mathbf{r}(t) = \langle \cos t^2, \sin t^2 \rangle$.

(a) Write $\mathbf{a}(t)$ as a combination of $\mathbf{r}(t)$ and $\mathbf{v}(t)$, for $t \neq 0$.

(b) Find the unit tangent vector $\mathbf{T}(t)$ for $t \neq 0$.

(c) Find the unit normal vector $\mathbf{N}(t)$ for $t \neq 0$.

(d) Find the tangential and normal components of the acceleration vector $\mathbf{a}(t)$.

14. A point moves with position given by $\mathbf{p}(t) = \langle 1, \cos 2t, t^2 + 4 \rangle$. Find the osculating plane at $t = 4$.

15. Suppose that the function given by $z = f(x, y)$ below represents height above sea level in kilometers.

y \ x	−3	−2	−1	0	1	2	3
−3	0.1	0.2	0.5	1	0.5	0.2	0.1
−2	0.2	1	4	5	4	1	0.2
−1	0.5	4	8	9	8	4	0.5
0	1	5	9	10	9	5	1
1	0.5	4	8	9	8	4	0.5
2	0.2	1	4	5	4	1	0.2
3	0.1	0.2	0.5	1	0.5	0.2	0.1

(a) What is the change in height from

(i) $(1, 0)$ to $(1, 1)$?

(ii) $(-1, -1)$ to $(0, 0)$?

(b) What is the average change in height if you walk from $(1, 0)$ to $(1, 1)$? From $(-1, -1)$ to $(0, 0)$?

(c) Approximately where are the points of steepest descent?

16. The formula for curvature for a function $y = f(x)$ is given by

$$\kappa(x) = \frac{|f''(x)|}{\left[1 + (f'(x))^2\right]^{3/2}}$$

(a) Using this formula, find the points in $[0, 2\pi]$ where the curvature of $y = \sin x$ is maximal.

(b) Using this formula, find the points in $[0, 2\pi]$ where the curvature of $y = \sin x$ is minimal.

(c) Give a geometric interpretation of your results from (a) and (b).

17. Find the length of the curve $\left\langle \sqrt{1+t^3}, \sqrt{1+t^3} \right\rangle$ between $t = 0$ and $t = 2$.

18. Give a parametric representation for the intersection of the cylinder $x^2 + y^2 = 2$ and

(a) the plane $z = 2$.

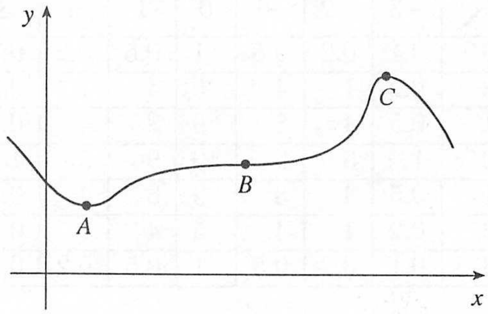
(b) the plane $x - 2z = 2$.

19. Let $\mathbf{r}(t) = \langle \sin 2t, 3t, \cos 2t \rangle$, $-\pi \le t \le \pi$ be the position vector of a particle at time t.

(a) Show that the velocity and acceleration vectors are always perpendicular.

(b) Is there any time t for which $\mathbf{r}(t)$ and the velocity vector are perpendicular? If so, find all s of t.

20. Consider the vector function $\mathbf{r}(t)$ describing the curve shown below. Put the curvatures of $\mathbf{r}$ at A, B, and C in order from smallest to largest.



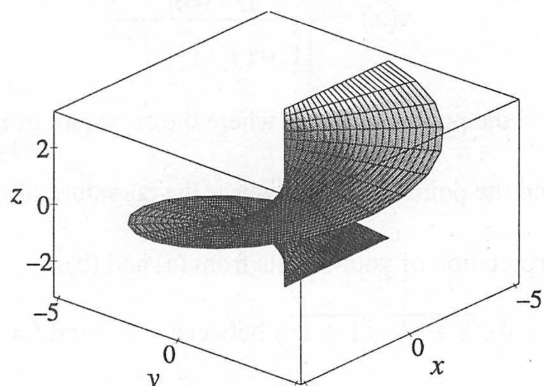
21. Show that the surface parametrization given by $\mathbf{r}(s,t) = \left\langle 2\cos t \sin s, \sin t \sin s, \frac{1}{\sqrt{2}}\cos s \right\rangle$, where $0 \le t \le 2\pi$, $0 \le s \le \pi$, describes the ellipsoid $\frac{1}{4}x^2 + y^2 + 2z^2 = 1$.

22. Find the equations for the following parametrized surfaces in rectangular coordinates, and describe them in words.

(a) $\left\langle t, \sqrt{1-t^2}\sin s, \sqrt{1-t^2}\cos s \right\rangle$

(b) $\langle t^2, s^2, s^2 + t^2 \rangle$

23. Find a parametric representation for the surface $z = \theta$ described in cylindrical coordinates.

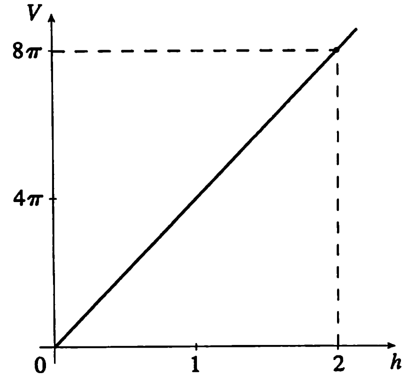


10 Sample Exam Solutions

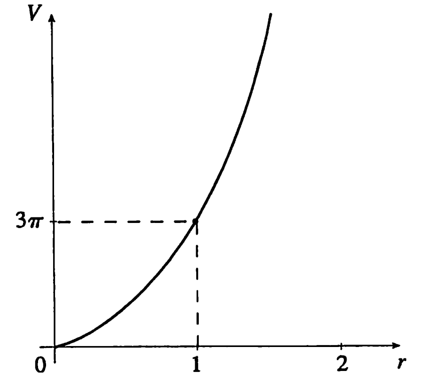
1. (a) $V = \pi r^2 h$

(b) $r > 0$, $h > 0$

(c)

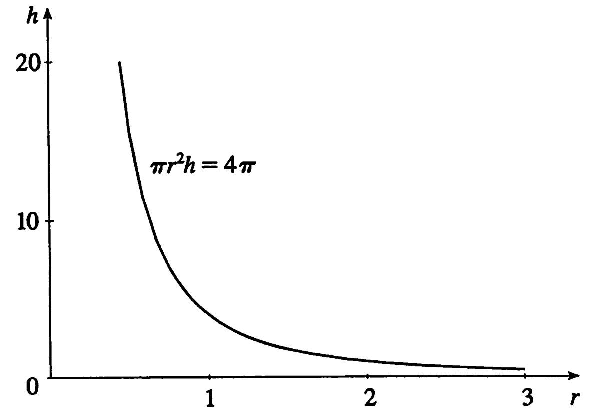


$r = 2 \Rightarrow V = 4\pi h$



$h = 3 \Rightarrow V = 3\pi r^2$

(d)



2. $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + \cos t\,\mathbf{k}$, $\mathbf{a}(t) = \mathbf{r}''(t) = -\mathbf{r}(t)$

3. (a) $[0, 2\pi R]$

(b) $s = 2\pi R$, the circumference of the circle

4. (a) A parametrization is $x = 3\cos t + 2$, $y = 3\sin t + 5$, $0 \le t \le 2\pi$. If s is the arc length, then $s = 3t$ or $t = s/3$. So the answer is $x = 3\cos(s/3) + 2$, $y = 3\sin(s/3) + 5$, $0 \le s \le 6\pi$.

(b) $\kappa = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \dfrac{|-\cos t\,\mathbf{i} + \sin t\,\mathbf{j}|}{3} = \dfrac{1}{3}$

5. $\mathbf{r}'(t) = \langle 1, 2t, 0\rangle$, so $\mathbf{T}(t) = \dfrac{\mathbf{i}}{\sqrt{1+4t^2}} + \dfrac{2t\,\mathbf{j}}{\sqrt{1+4t^2}}$, $\mathbf{T}(0) = \mathbf{i}$, $\mathbf{N}(0) = \mathbf{j}$, and hence $\mathbf{B}(0) = \mathbf{k}$.

6. (a) II (b) I (c) IV (d) III

7. Graph I $[\mathbf{r}(0) = \langle 0,1,0\rangle$ and $\mathbf{r}(2) = \langle 0,1,1\rangle]$

8. (a) $\mathbf{r}_1(t) = 4\cos t\,\mathbf{i} + 5\sin t\,\mathbf{j}$. $\mathbf{r}_2(t) = 4\sin t\,\mathbf{i} + 5\cos t\,\mathbf{j}$

(b) The curve is most "flat" (minimal curvature) at $(\pm 4, 0)$.

9. (a) $\mathbf{v}(t) = \langle \frac{3}{2}e^{t/2}, -2e^{-t/2}\rangle$, $\mathbf{a}(t) = \langle \frac{3}{4}e^{t/2}, e^{-t/2}\rangle$

(b) $\mathbf{a}(t) = \frac{1}{4}\mathbf{w}(t) \parallel \mathbf{w}(t)$

(c) $\mathbf{v}(0) = \langle \frac{3}{2}, -2\rangle$, which is not perpendicular to $\langle \frac{3}{4}, 1\rangle = \mathbf{a}(0)$, so $\mathbf{T}(0)$ is not perpendic

10. $\mathbf{r}(t) = \langle 2t, t, 5t^2 \rangle$

11. (a) $\mathbf{r}(t) = \langle -2\sin t, 2\cos t \rangle, 0 \le t \le \pi$

(b) $\mathbf{r}(t) = \langle 0, 2-t \rangle, 0 \le t \le 4$

12. (a) The change is $z(10) - z(0) = 10\sqrt{2}$.

(b) $\mathbf{r}'(t) = \langle e^t, -e^{-t}, \sqrt{2} \rangle$, so the change in x is e^4.

(c) e^{-4}

(d) $L = \int_0^4 |\mathbf{r}'(t)|\,dt = \int_0^4 \sqrt{e^{2t} + e^{-2t} + 2}\,dt = \int_0^4 \sqrt{(e^t + e^{-t})^2}\,dt = \int_0^4 (e^t + e^{-t})\,dt$
$= \left[e^t - e^{-t}\right]_0^4 = e^4 - e^{-4}$

13. (a) $\mathbf{v}(t) = \langle -2t\sin t^2, 2t\cos t^2 \rangle$, $\mathbf{a}(t) = \langle -2\sin t^2 - 4t\cos t^2, 2\cos t^2 - 4t^2\sin t^2 \rangle = \frac{1}{t}\mathbf{v} - 4t^2\mathbf{r}(t)$ for $t \ne 0$.

(b) $\mathbf{T}(t) = \frac{\mathbf{v}}{2t}, t \ne 0$

(c) $\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{\langle -\cos t^2, -\sin t^2 \rangle}{1} = -\mathbf{r}(t), t \ne 0$

(d) $\mathbf{a}(t) = \frac{1}{t}\mathbf{v} - 4t^2\mathbf{r}(t) = 2\mathbf{T} + 4t^2\mathbf{N}$, so $a_\mathbf{N} = 4t^2$, $a_\mathbf{T} = 2$.

14. Since $\mathbf{p}(t) = \langle 1, \cos 2t, t^2 + 4 \rangle$ lies in the plane $x = 1$, so does the osculating plane at $t = 4$.

15. (a) (i) -1 (ii) 2

(b) Change in height from $(1,0)$ to $(1,1)$ is -1; from $(-1,-1)$ to $(0,0)$ is $\sqrt{2}$

(c) The four points $(\pm 1, \pm 1)$

16. (a) $\kappa(x) = \frac{|\sin x|}{(1 + \cos^2 x)^{3/2}}$. κ is maximal when $|\sin x| = 1$ and $\cos x = 0$, that is, when $x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$. The maximum value is $\kappa = 1$.

(b) κ is minimal when $|\sin x| = 0$, or $x = 0$ or π. The minimum value is $\kappa = 0$.

(c) The minimum value occurs where $\sin x$ is "flat", and the maximum value occurs at maxima and minima of $\sin x$.

17. $\mathbf{v} = \sqrt{1+t^3}\,\langle 1, 1 \rangle = \sqrt{1+t^3}\,(\mathbf{i} + \mathbf{j})$ is a line with direction vector $\mathbf{i} + \mathbf{j}$. Since $\mathbf{v}(0) = 1$, $\mathbf{v}(2) = 3$, the length is $2\,|\mathbf{i} + \mathbf{j}| = 2\sqrt{2}$.

18. (a) $\mathbf{v}(t) = \langle \sqrt{2}\cos t, \sqrt{2}\sin t, 2 \rangle$

(b) Since $z = \frac{1}{2}(x - 2)$,
$\mathbf{v} = \langle \sqrt{2}\cos t, \sqrt{2}\sin t, \frac{1}{2}(\sqrt{2}\cos t - 2) \rangle = \left\langle \sqrt{2}\cos t, \sqrt{2}\sin t, \frac{1}{\sqrt{2}}(\cos t - \sqrt{2}) \right\rangle$.

19. (a) $\mathbf{v}(t) = \langle 2\cos 2t, 3, -2\sin 2t \rangle$, $\mathbf{a}(t) = \langle -4\sin 2t, 0, -4\cos 2t \rangle$.
$\mathbf{v}(t) \cdot \mathbf{a}(t) = -8\sin 2t\cos 2t + 8\sin 2t\cos 2t = 0$ for all t.

(b) $\mathbf{r}(t) \cdot \mathbf{v}(t) = 2\sin 2t\cos 2t + 9t - 2\sin 2t\cos 2t = 0$ when $t = 0$, the only time at which $\mathbf{r}(t) \perp \mathbf{v}(t)$.

20. B, A, C

21. If $x = 2\cos t \sin s$, $y = \sin t \sin s$, $z = \frac{1}{\sqrt{2}} \cos s$, then

$$\tfrac{1}{4}x^2 + y^2 = \tfrac{1}{4}\left(4\cos^2 t \sin^2 s\right) + \sin^2 t \sin^2 s = \sin^2 s \left(\cos^2 t + \sin^2 t\right) = \sin^2 s$$

and so

$$\tfrac{1}{4}x^2 + y^2 + 2z^2 = \sin^2 s + 2\left(\frac{1}{\sqrt{2}} \cos s\right)^2 = \sin^2 s + 2\left(\tfrac{1}{2}\cos^2 s\right) = \sin^2 s + \cos^2 s = 1$$

22. (a) $x = t$, $y = \sqrt{1-t^2}\sin s$, and $z = \sqrt{1-t^2}\cos s$ gives $y^2+z^2 = 1-t^2 = 1-x^2$, or $x^2+y^2+z^2 = 1$, a sphere of radius 1.

(b) $x = t^2$, $y = x^2$, and $z = s^2 + t^2 = y + x$, $x \geq 0$, $y \geq 0$, part of a plane above the first quadrant.

23. If $z = \theta$, then $x = r\cos\theta$, $y = r\sin\theta$, $z = \theta$ and $\mathbf{R}(r,\theta) = r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j} + \theta\mathbf{k}$, $r \geq 0$, $0 \leq \theta \leq 2\pi$ is a parametrization.

11 Partial Derivatives

11.1 Functions of Several Variables

Suggested Time and Emphasis

1 class Essential material

Transparencies Available

- Transparency 39 (Figure 6, page 752)
- Transparency 40 (Figure 10, page 754)
- Transparency 41 (Figure 12, page 755)
- Transparency 42 (Exercises 31–36, Graphs A–F and I–VI, page 759)

Points to Stress

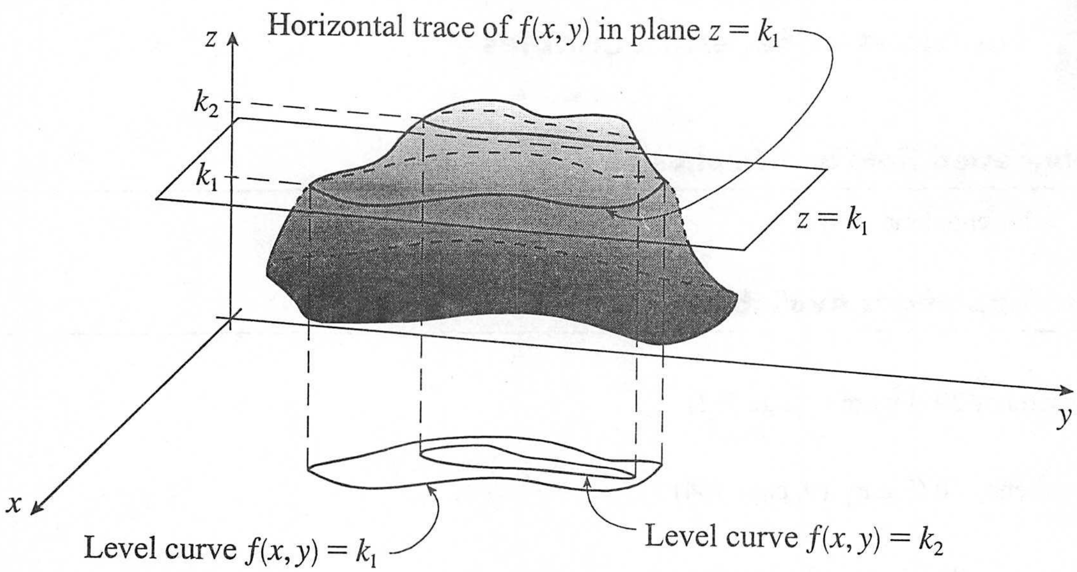
1. The notions of level curves (contour lines) and level surfaces.
2. Functions of two variables can be represented as surfaces, and can be described in two dimensions by contour maps and horizontal traces.
3. Functions of three variables can be described by level surfaces, and are generally more diffi visualize than functions of two variables.

Text Discussion

- Why is the domain of the function in Example 3 (page 750), $g(x, y) = \sqrt{9 - x^2 - y^2}$, sh
- Why are the level curves for $f(x, y) = 6 - 3x - 2y$ straight lines with the same slop y-intercept of these lines change as a function of k, if $f(x, y) = k$?

Materials for Lecture

- Discuss contour or level curves carefully, invoking the image of a moving plane slicing a three-dimensional surface. A picture like the following should help the students visualize the situation:



- Discuss the level curves for $f(x, y) = e^{x-y}$. Point out that they reduce to the simple equations $y = x - \ln k$ for $z = k > 0$.

- Describe the domain and range of $f(x, y, z) = \ln\left(z - \sqrt{xy}\right)$. Point out why $f(1, 2, 1)$ is not defined, and why points (x, y, z) in the domain must have (x, y) in the first or third quadrants.

- One way to describe functions of two variables $f(x, y)$ is to have the students think of the contour curve $f(x, y) = t$ as existing at time t. With this way of looking at things, a sphere is a point that becomes a circle, grows, and then shrinks back to a point. This approach then makes it easier to describe functions of three variables. A function of three variables can be thought of a level surface that changes with time. Example 12 (page 756) can be revisited in this context. $f(x, y, z) = x^2 + y^2 + z^2$ can be pictured as a point at time $t = 0$, a sphere of radius 1 at time $t = 1$, a sphere of radius $\sqrt{2}$ at $t = 2$, and so on. In other words, $f(x, y, z) = x^2 + y^2 + z^2$ can be pictured as a growing sphere, and the "level surfaces" of the function as snapshots of the process. Another good function to describe with this method is $f(x, y, z) = x^2 + y^2 - z$.

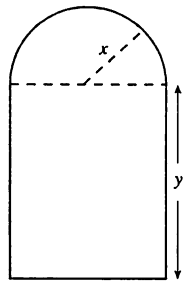
- An alternate way to approach the subject is to think of one dimension as "color". A surface such as $f(x, y) = \sin x + \cos y$ could be then drawn on a sheet of graph paper, with red representing the contour curve $f(x, y) = -2$, violet representing the contour curve $f(x, y) = 2$, and any number in between represented by the appropriate color. Some software packages represent functions of three variables using a method similar to this one.

Workshop/Discussion

- Calculate the domain and range for each of the following functions:

$$f(x,y) = \sin\left(\sqrt{1-(x^2+y^2)}\right) \qquad f(x,y) = \exp\left(\frac{x+y}{xy}\right) \qquad f(x,y,z) = \frac{\sqrt{z^2-3}}{\sqrt{2x^2+3y^2-4}}$$

- Let A be the area of the Norman window shown below. Revisit the formula for A as a function of two variables x and y. Have them use level curves to determine what the graph of A looks like.



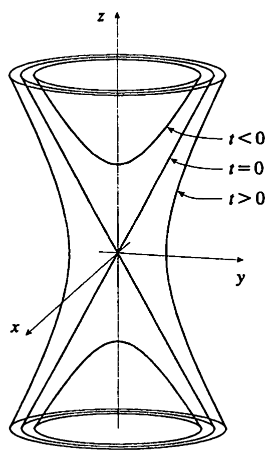
- Pass around some interesting solid figures, and have the students attempt to sketch the appropriate level curves for the solids.
- Sketch the ellipsoidal level surfaces for $f(x,y,z) = 2x^2 + y^2 + z^2$.
- Give the students a start on Group Work 5: The M. R. Project, doing either the two-dimensional problems or one of the three-dimensional problems.

Group Work 1: Dali's Target

This activity is designed for students who are having some difficulty with the concept of level curves.

Group Work 2: Level Surfaces

Part 2 of Problem 4 requires that the students have some familiarity with conic sections, or have some type of graphing software. The solution to Part 4 (and hence Parts 1–3) is given by the picture below:



Group Work 3: The M. R. Project

This exercise involves looking at some interesting functions given a particular domain in the xy-plane or in space. Each group should get the same worksheet, but a different domain. (There is a blank space on the sheet in which to write the assigned domain.) Possible domains to give the students are:

$$\text{(Two-dimensional)} \quad 0 \le x, y \le 1,\ z = 0$$
$$\text{(Two-dimensional)} \quad x^2 + y^2 \le 1,\ z = 0$$
$$0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1$$
$$x^2 + y^2 \le 1, 0 \le z \le 1$$
$$x^2 + y^2 + z^2 = 1$$
$$x^2 + y^2 = 1, |z| \le 1$$

If a group finishes early, they could be given another domain to do, or instructed to prepare a presentation about their solution to give to the class. Ideally, each group should solve the problem themselves for at least one domain, and see a discussion of at least two other domains.

In some classes, it may be most appropriate to go through the domain $0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1$ with the students as an example.

If the students are able to do the last problem well, make sure to point out that what they are really doing is trying to comprehend a four-dimensional object, with the fourth dimension being time.

Group Work 4: Applied Contour Maps

Give the students a list of today's temperature in various cities, along with a map of the country. The temperatures are available in many newspapers. Have the students draw a contour map showing curves of constant temperature. Then give them a copy of the temperature contour map from a newspaper to compare with their map to see how they did. Discuss what a three-dimensional representation of today's weather would be like. You can have the students cut the contours out of corrugated cardboard and make actual three-dimensional weather maps.

Homework Problems

Core Exercises: 8, 9, 10, 11, 12, 16, 31–36, 37, 40

Sample Assignment: 1, 2, 4, 8, 9, 10, 11, 12, 16, 20, 28, 29, 31–36, 37, 40

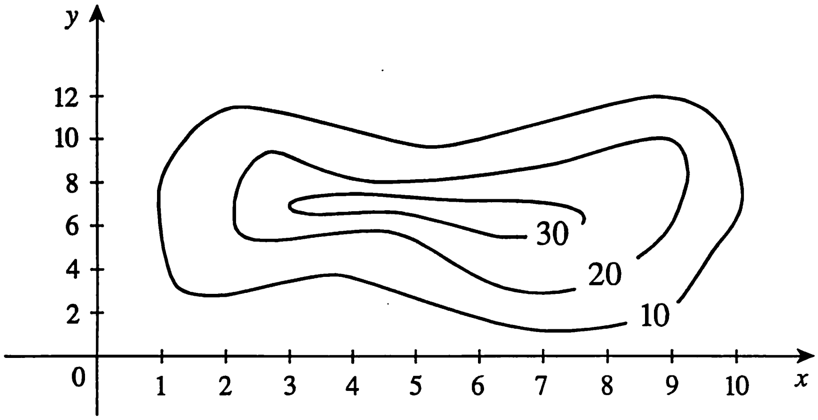
Note: Exercises 28 and 29 require a CAS.

Exercise	C	A	N	G	V
1	×				
2	×		×		
4			×		
8		×		×	
9				×	×
10					×
11					×
12					×

Exercise	C	A	N	G	V
16				×	
20				×	
28				×	
29				×	
31–36					×
37					×
40					×

Group Work 1, Section 11.1
Dali's Target

Consider the following contour map of a continuous function $f(x, y)$:



1. For approximately what values of y is it true that $10 \le f(5, y) \le 30$?

2. What can you estimate $f(2, 4)$ to be, and why?

3. Do we have any good estimates for $f(5, 8)$? Explain.

4. How many values y satisfy $f(7, y) = 20$?

5. How many values of x satisfy $f(x, 9) = 20$?

Group Work 2, Section 11.1
Level Surfaces

It can be difficult to visualize functions of three variables. One way to do it is by thinking of each level surface as representing a different point in time. As we let t vary in the equation for the level surface $f(x, y, z) = t$ we can think of the function $f(x, y, z)$ as a surface whose shape and size vary as time changes.

Consider the function $f(x, y, z) = x^2 + y^2 - z^2$.

1. What is the level surface $f(x, y, z) = 0$?

2. What is the level surface $f(x, y, z) = 1$?

3. For $t > 0$, what do the level surfaces $f(x, y, z) = t$ look like?

4. What is the level surface $f(x, y, z) = -1$?

5. Describe all the level surfaces $f(x, y, z) = t$.

Group Work 3, Section 11.1
The M.R. Project

Consider the region ______________________.

1. Sketch or describe this region.

We are now going to describe some functions of three variables for which the region in Part 1 is the domain. In other words, every point in your domain will have a function value for the functions below. The functions are:

$M(x, y, z) = \max(x, y, z)$ $\qquad m(x, y, z) = \min(x, y, z)$ $\qquad R(x, y, z) = x + y + z$

2. Evaluate M, m, and R at several different points in your domain. The first line in the following table is an example for you to look at.

Point	$M(x,y,z)$	$m(x,y,z)$	$R(x,y,z)$
$\left(\frac{1}{5}, \frac{1}{3}, \frac{1}{2}\right)$	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{31}{30}$
(, ,)			
(, ,)			
(, ,)			

3. Find the maximum values of M, m, and R on your domain.

4. Sketch the level surfaces $M = \frac{1}{2}$, $R = \frac{1}{2}$, $R = 0$, and $m = \frac{1}{2}$ for your domain.

5. For an extra challenge, try to describe the level surfaces $M = t$, $R = t$, and $m = t$, for $0 \le t \le 2$. If we let t stand for time, and make a movie of the level surface changing as t goes from 0 to 2, what would the movie look like?

11.2 Limits and Continuity

Suggested Time and Emphasis

1 class Recommended material

Note: This material can be covered from a variety of perspectives, and at a variety of depths. (For example, the nonexistence of certain limits can be de-emphasized.) The instructor should feel especially free to pick and choose from the suggestions below.

Points to Stress

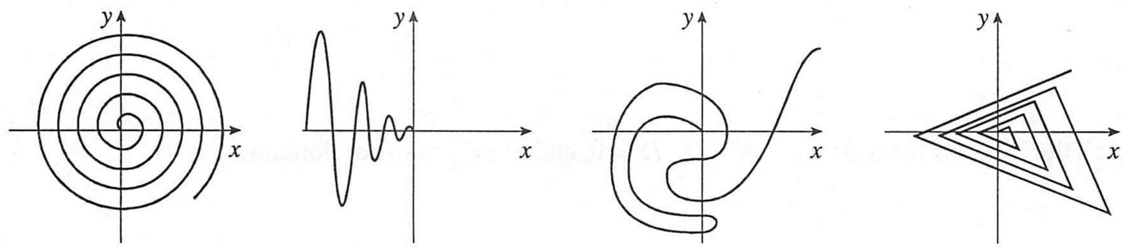
1. While the definitions of limits and continuity for multivariable functions are nearly identical to those of their single variable counterparts, very different behavior can take place in the multivariable case.
2. The idea of points being "close" in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Text Discussion

- When talking about limits for functions of several variables, why isn't it sufficient to say, "$\lim_{(x,y)\to(0,0)} f(x,y) = L$ if $f(x,y)$ gets close to L as we approach $(0,0)$ along the x-axis ($y = 0$) and along the y-axis ($x = 0$)"?

Materials for Lecture

- Stress that $f(x,y) \to L$ as $(x,y) \to (a,b)$ means that we can make $f(x,y)$ as close to L as we like by taking (x,y) close to (a,b) in distance, *regardless of path*. Give examples of exotic paths to $(0,0)$ such as the following:



- This is a rich example of a limit that exists:

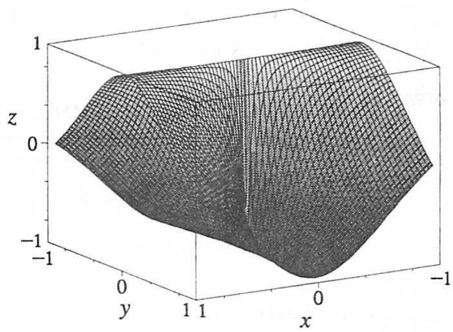
$$\lim_{(x,y)\to(0,0)} \frac{\sin\left(x^2+y^2\right)}{x^2+y^2} = 1$$

Since $\dfrac{\sin\left(x^2+y^2\right)}{x^2+y^2}$ is constant on circles centered at the origin, we want to look at the distance w between (x,y) and $(0,0)$: $w = \sqrt{x^2+y^2}$. Computationally, it is best to look at what happens when $w^2 \to 0$. In this case, $\dfrac{\sin\left(x^2+y^2\right)}{x^2+y^2} = \dfrac{\sin w^2}{w^2}$, and single-variable calculus gives us that $\lim_{w\to 0} \dfrac{\sin w^2}{w^2} = 1$. Stress that in general it does *not* suffice to just let $x = 0$ or $y = 0$ and then compute the limit.

- This is a good example of a limit that does not exist:

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$$

The text shows this fact in an interesting way: If we let $x = 0$, then we get $\lim_{y\to 0} \frac{-y^2}{y^2} = -1$, but if we let $y = 0$ then we get $\lim_{x\to 0} \frac{x^2}{x^2} = 1$. So if we approach the origin by one radial path, we get a different limit than we do if we go by a different radial path. In fact, assume we go to the origin by a straight line $y = mx$. Then $\frac{x^2 - y^2}{x^2 + y^2} = \frac{x^2(1 - m^2)}{x^2(1 + m^2)}$ and $\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2} = \frac{1 - m^2}{1 + m^2}$. So this limit can take any value from -1 to 1 if we approach the origin by a straight line. For example, if we use the line $y = \frac{1}{\sqrt{3}}x$, we get $\frac{1}{2}$ as a limit.

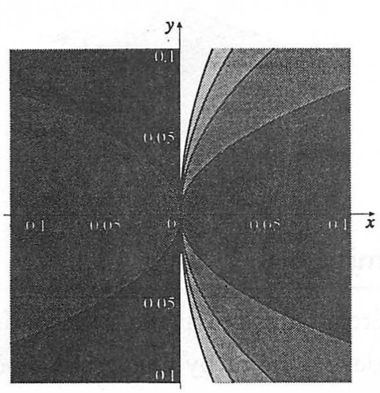


- Note that just because a function has a limit at a point doesn't imply that it is continuous there. For example,

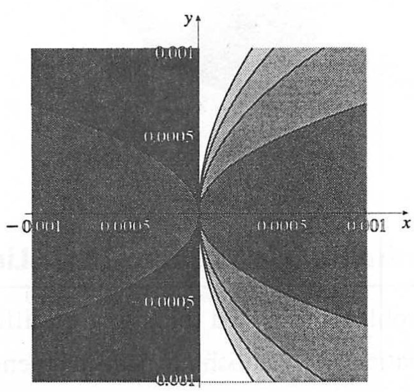
$$f(x, y) = \begin{cases} \dfrac{\sin(x^2 + y^2)}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

is discontinuous at the origin even though $\lim_{(x,y)\to(0,0)} f(x, y)$ exists.

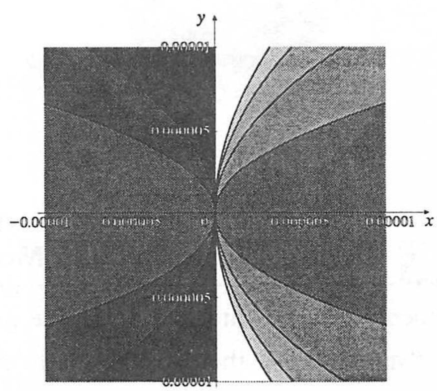
- Expand the explanation of Example 3 (page 762), where $f(x, y) = \frac{xy^2}{x^2 + y^4}$, using a visual approach, perhaps using figures like the following or using algebra to compute the limit along the general parabola $x = my^2$. The figures show progressively smaller viewing rectangles centered at the origin. The black regions correspond to larger negative values of $f(x, y) = \frac{xy^2}{x^2 + y^4}$, and the white regions correspond to larger positive values. Notice that when travelling along any straight line $y = mx$, the color of the points on the path eventually becomes gray [at points where $f(x, y) = 0$] as the origin is approached. This effect is best observed (even if m is large) using the later pictures.



$-0.1 \le x \le 0.1$



$-0.001 \le x \le 0.001$



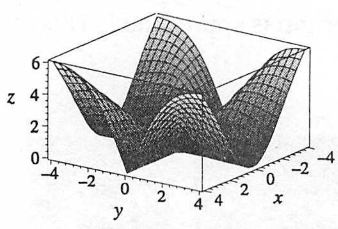
$-0.00001 \le x \le 0.00001$

However, when approaching the origin on a parabolic path, $x = my^2$, the color of the points on the path always stays the same! This phenomenon is best illustrated by the earlier pictures. Therefore, this set of plots illustrates how the limit as $(x, y) \to (0, 0)$ of $f(x, y) = \dfrac{xy^2}{x^2 + y^4}$ does not exist, although one would erroneously believe it to be 0 if one looked only at the "obvious" linear paths.

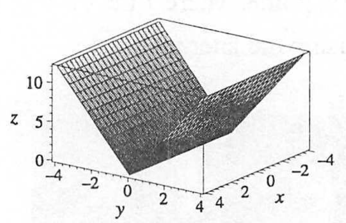
- Point out that while $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ does not exist, $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2+y^2+z^2}$ does exist. To see this, use cylindrical coordinates to get $\dfrac{xyz}{x^2+y^2+z^2} = \dfrac{r^2 z \sin\theta\cos\theta}{r^2+z^2}$, and compare to $\dfrac{r^2 z}{r^2+z^2}$, which approaches 0 as in Example 8.

Workshop/Discussion

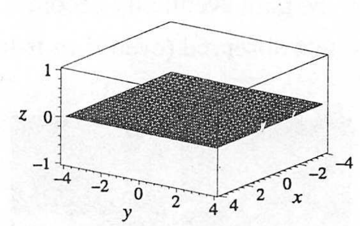
- Try to compute $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x+y}{x^2+y^2}$. Show that the function is unbounded when restricted to the y-axis $(x = 0)$, so there is no limit.
- Introduce the use of polar coordinates by trying to compute $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^2 y}{\sqrt{x^2+y^2}}$. Point out how intractable the problem looks at first glance. But notice that in polar coordinates, the statement "$(x, y) \to (0, 0)$" translates to the much simpler "$r \to 0$", so the limit can be rewritten as $\displaystyle\lim_{r\to 0} \frac{3(r\cos\theta)^2 r\sin\theta}{r}$, which simplifies to $\displaystyle\lim_{r\to 0} 3r^2\cos^2\theta\sin\theta = 0$.
- Check that $\displaystyle\lim_{\substack{(x,y)\to(0,0)\\ x\neq -y}} \frac{x+y-\sin(x+y)}{(x+y)^3} = \frac{1}{6}$, by noticing that this function is constant on $x + y = k$. This suggests using the substitution $u = x+y$ and applying the single-variable version of l'Hospital's Rule. This is also a good limit to first investigate numerically, plugging in small values of x and y.
- Go over Exercises 33 and 34 (page 766) using polar coordinates.
- Discuss the squeeze principle by looking at the three graphs below, and computing $\displaystyle\lim_{(x,y)\to(0,0)} f(x, y)$ by squeezing f between g and h.



$f(x, y) = \dfrac{3x^2|y|}{x^2+y^2}$



$g(x, y) = 3|y|$



$h(x, y) = 0$

Group Work 1: Even Mathematicians Have Their Limits

When a group is finished doing the problems as stated (plugging in different values of x and y) have them attempt to prove their results mathematically, establishing path-independence, either by changing to polar coordinates, or by using a substitution.

Group Work 2: Limits in $\mathbb{R}^3$

Group Work 3: There Is No One True Path

This is a difficult project, but might be of interest for strong students.

Have the students use a calculator or CAS to do an analysis of $\lim_{(x,y)\to(0,0)} \frac{x^3y^2}{x^4+y^8}$. Guide them through the following steps:

1. Setting $x = 0$ and letting $y \to 0$ gives a limit of 0.
2. More generally, for any m, approaching the origin along the path $y = mx$ gives a limit of 0.
3. Following the path $y = x^2$ gives a limit of 0 as well.
4. However, the actual limit is not zero! For example, following the path $y = \sqrt{x}$ gives a limit of 1.

Homework Problems

Core Exercises: 1, 2, 4, 5, 8, 11, 25, 30

Sample Assignment: 1, 2, 4, 5, 8, 11, 13, 19, 25, 28, 30, 33, 35, 36

Note: Exercise 19 requires a CAS.

Exercise	C	A	N	G	V
1	×				
2	×				
4			×		
5–18		×			
19				×	×
25–32		×			
35		×			
36		×		×	

Group Work 1, Section 11.2
Even Mathematicians Have Their Limits

Try to estimate the following limits by plugging small values of x and y into the appropriate function. Remember that path independence is important: Try some values where $x = y$ and some where $x \neq y$.

1. $\displaystyle\lim_{(x,y)\to(0,0)} 5$

2. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{e^{xy}}{x^2+y^2}$

3. $\displaystyle\lim_{(x,y)\to(0,0)} \ln\left(x^2+y^2\right)$

4. $\displaystyle\lim_{\substack{(x,y)\to(0,0)\\ x\neq -y}} \frac{\sin\left(2\left(x+y\right)\right)}{x+y}$

Group Work 2, Section 11.2
Limits in $\mathbb{R}^3$

Determine whether or not the following limits exist. If a limit exists, compute its value.

1. $$\lim_{(x,y,z)\to(0,0,0)} \frac{x+y+z}{x^2+y^2+z^2}$$

2. $$\lim_{(x,y,z)\to(0,0,0)} \frac{\sin\left(x^2+y^2+z^2\right)}{x^2+y^2+z^2}$$

3. $$\lim_{(x,y,z)\to(0,0,0)} \frac{(x+y+z)^2}{x^2+y^2+z^2}$$

4. $$\lim_{(x,y,z)\to(0,0,0)} \frac{x+y+z}{e^{x^2+y^2+z^2}}$$

11.3 Partial Derivatives

Suggested Time and Emphasis

$\frac{3}{4}$–1 class   Essential material

Transparencies Available

- Transparency 43 (Figures 2–5, page 770)
- Transparency 44 (Exercise 7, page 777)

Points to Stress

1. The meaning of f_x and f_y, both analytically and geometrically.
2. The various notations for f_x and f_y. [Be sure to point out that in the notation $f_x(x, y)$, x is playing two different roles: $f_x(x, y)$ can be written as $f_1(x, y)$, where the 1 indicates that the derivative is taken with respect to the first variable.]
3. Higher-order partial derivatives.

Text Discussion

- In computing $f_x(1, 1)$ for $f(x, y) = 4 - x^2 - 2y^2$, what function of x are we differentiating at $x = 1$?
- Describe the line which has slope $f_y(1, 1)$ for $f(x, y) = 4 - x^2 - 2y^2$.
- Find a function $f(x, y)$ for which $\frac{\partial f}{\partial x} = x + y$ and $\frac{\partial f}{\partial y} = x$.

Materials for Lecture

- Provide an alternate geometric interpretation for the partial derivative in terms of vector functions. The graph C_1 of the function $g(x) = f(x, b)$ is the curve traced out by the *vector* function $\mathbf{g}(x) = \langle x, b, f(x, b)\rangle$ whose *vector* derivative $\mathbf{g}'(a) = \langle 1, 0, f_x(a, b)\rangle$ is determined by $f_x(a, b)$. [That is, its "slope" is $f_x(a, b)$.] Similarly, the graph C_2 of $h(y) = f(a, y)$ is the curve traced out by the vector function $\mathbf{h}(y) = \langle a, y, f(a, x)\rangle$ whose vector derivative $\mathbf{h}'(b) = \langle 0, 1, f_y(a, b)\rangle$ is determined by $f_y(a, b)$.
- Compute $\frac{\partial f}{\partial x}(0, 0)$ and $\frac{\partial f}{\partial y}(0, 0)$ for $f(x, y) = \sin(\pi x e^{xy})$.
- Let $f(x, y, z) = xy^4z^3$ or some other easy-to-differentiate function. Verify that $f_{xyz} = f_{xzy} = \cdots = f_{zyx}$. Perhaps then show that $f_{xzz} = f_{zxz}$.
- The idea of partial derivatives being continuous is going to be very important in this chapter, so make sure to stress both the hypothesis and the conclusion of Clairaut's Theorem.

- Define the function $f(x,y) = \begin{cases} x^{4/3}\sin(y/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ and compute $\frac{\partial f}{\partial x}(0,1)$ and $\frac{\partial f}{\partial y}(0,1)$ using the definitions of f_x and f_y.

- Demonstrate that the functions $f(x,y) = 5xy$, $f(x,y) = e^x \sin y$, and $f(x,y) = \arctan(y/x)$ all solve the Laplace equation $f_{xx} + f_{yy} = 0$.

▲ Workshop/Discussion

We believe that it is crucial that the students do not leave their workshop/discussion session without knowing how to compute partial derivatives. There should be some opportunity for the students to practice in class, even by just trying one or two easy problems, so they can get instant feedback.

- Compute f_x and f_y for $f(x,y) = (x+y^2)^3 + \sin(x+y) + e^{x^2 y}$ and g_x and g_y for $g(x,y) = \frac{xy}{x^2+y^2}$.

- Compute some second- or even third-order partial derivatives for $f(x,y,z) = \ln(xy^2 + z^3)$.

- Expand on the text discussion of Clairaut's Theorem by showing that $f_{xyy} = f_{yxy} = f_{yyx}$ (provided that all the first-, second-, and third-order partial derivatives are continuous).

- Set up the tangent plane determined by the partials f_x and f_y by using the vector functions $\mathbf{g}(x) = \langle x, b, f(x,b)\rangle$, $\mathbf{h}(y) = \langle a, y, f(a,y)\rangle$ and the vector derivatives $\mathbf{a} = \langle 1, 0, f_x(a,b)\rangle$, $\mathbf{b} = \langle 0, 1, f_y(a,b)\rangle$ (as discussed in Materials for Lecture above). Form the plane through $(a, b, f(a,b))$ with normal vector $\mathbf{N} = \mathbf{a} \times \mathbf{b}$. Find the plane tangent to $f(x,y) = e^{xy}$ at the point $(1, 2, e^2)$.

- Have the students do Exercise 61 or Exercise 63 (page 778), either as a class or in groups.

▲ Group Work 1: Partial Derivatives on the Sphere

▲ Group Work 2: Partial Derivatives on Hyperboloids

▲ Group Work 3: Clarifying Clairaut's Theorem

Problem 3 foreshadows the process of solving exact differential equations by finding $f(x,y)$ given that $f_{xy} = f_{yx}$. Students should be led carefully through this component.

▲ Group Work 4: Back to the Park

Note that Problem 5 foreshadows the technique of linear approximation.

▲ Group Work 5: The Geometry of Partial Derivatives

This group work gives a concrete example to illustrate the geometry behind the usual way of computing partial derivatives. It might help the students if a graph of $z = xy$ is given to them when they get to Problem 3.

Homework Problems

Core Exercises: 1, 4, 21, 30, 36, 46, 53, 60, 63

Sample Assignment: 1, 2, 4, 8, 18, 21, 24, 30, 33, 36, 42, 46, 53, 60, 61, 62(b), 62(c), 63, 65, 66, 76, 79

Note:
- Exercise 76 is a particularly good modeling problem.
- Exercises 76 and 79 require a CAS.

Exercise	C	A	N	G	V
1	×				
2	×		×		
4	×		×		
8			×	×	×
13–34		×			
36		×			
42		×			
46		×			
53		×			

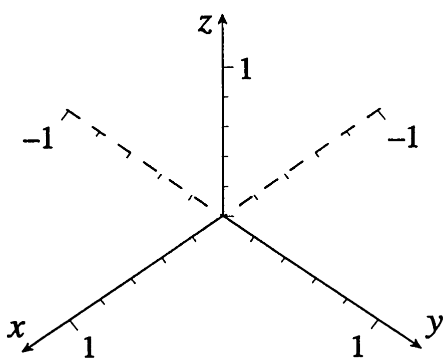
Exercise	C	A	N	G	V
60				×	×
61		×			
62		×			
63		×			
65		×			
66		×			
76	×	×		×	
79	×	×	×	×	×

Group Work 1, Section 11.3
Partial Derivatives on the Sphere

Consider the surface formed by the top half of the unit sphere $x^2 + y^2 + z^2 = 1$, $z \geq 0$.

1. Write the equation for the top half of the sphere in the form $z = f(x, y)$.

2. Draw the surface.



3. Compute $\dfrac{\partial z}{\partial x}(0,0)$ and $\dfrac{\partial z}{\partial y}(0,0)$, and justify your answer by looking at your drawing.

4. Compute $\dfrac{\partial z}{\partial x}\left(\frac{1}{\sqrt{2}},0\right)$ and $\dfrac{\partial z}{\partial y}\left(\frac{1}{\sqrt{2}},0\right)$ and similarly justify your answer.

Group Work 2, Section 11.3
Partial Derivatives on Hyperboloids

1. Consider the hyperboloid of one sheet $x^2 + y^2 - z^2 = 1$.

(a) For what values of (x, y) is $z = f(x, y)$, $z \geq 0$ defined?

(b) Sketch a graph of the surface.

(c) Compute $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, both in general and at the point $(2, 2)$.

2. Next consider the hyperboloid of two sheets $x^2 + y^2 - z^2 = -1$.

(a) For what values of (x, y) is $z = g(x, y)$, $z \geq 0$ defined?

(b) Sketch a graph of the surface.

(c) Compute $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, both in general and at the point $(2, 2)$.

3. Compare your answers for part (c) of Problems 1 and 2. Can you give a geometric explanation why the values of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ in Problem 1 are larger than the corresponding values in Problem 2?

Group Work 3, Section 11.3
Clarifying Clairaut's Theorem

Consider $f(x, y) = x^2 \cos(y^3 + 2y)$.

1. Why do we know that $f_{yxyyxx} = 0$ without doing any computation?

2. Do we also know, without doing any computation, that $f_{yxyyxy} = 0$? Why or why not?

3. Suppose that $f_x = 3x + ay^2$, $f_y = bxy + 2y$, $f_y(1, 1) = 3$, and f has continuous mixed second partial derivatives f_{xy} and f_{yx}.

(a) Find values for a and b and thus equations for f_x and f_y. *Hint:* What does Clairaut's Theorem say about the mixed partial derivatives of a function? When does the theorem apply?

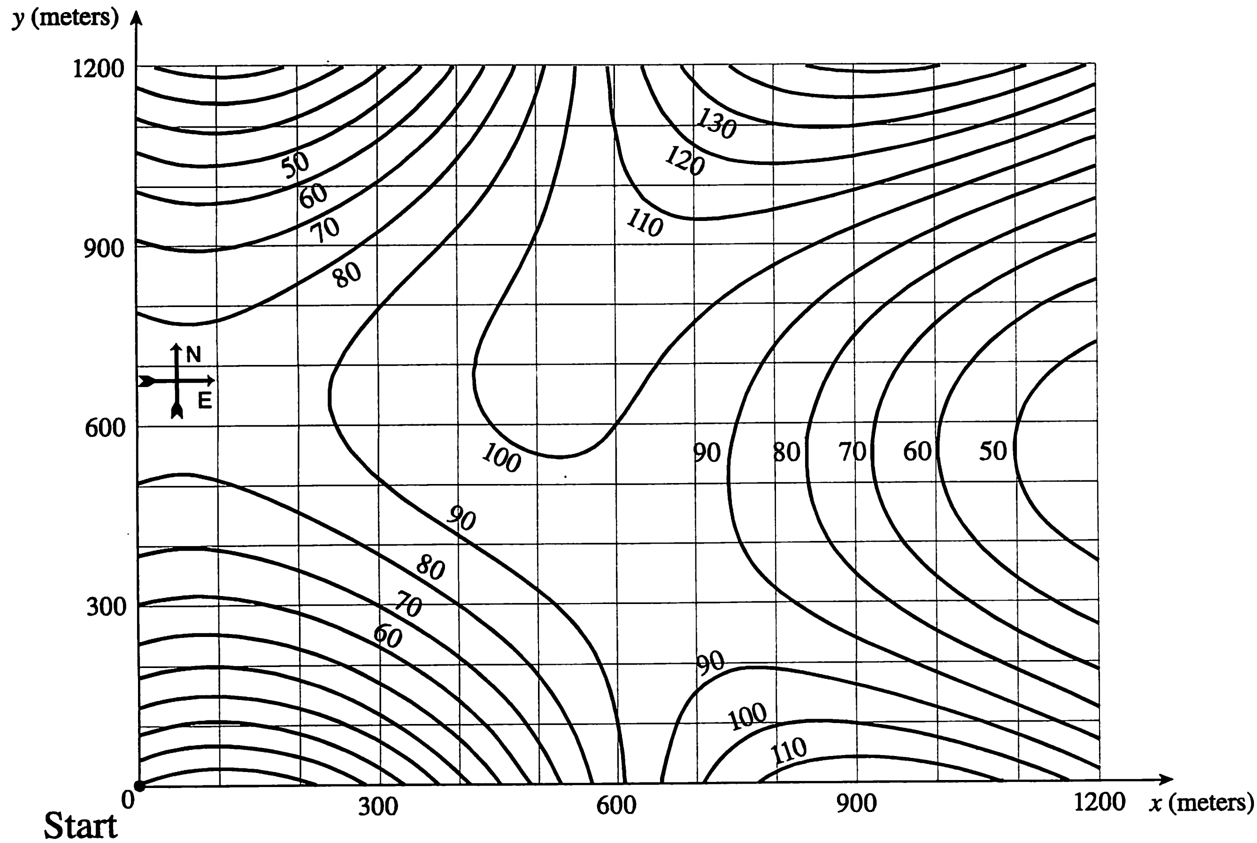
(b) Can you find a function $F(x, y)$ such that $\dfrac{\partial F}{\partial x} = f_x$ in part (a)?

(c) Can you find a function $G(x, y) = F(x, y) + k(y)$ such that $\dfrac{\partial G}{\partial y} = f_y$ in part (a)? What is $k(y)$?

(d) What is $\dfrac{\partial G}{\partial x}$? Can you now find $f(x, y)$?

Group Work 4, Section 11.3
Back to the Park

The following is a map with curves of the same elevation of a region in Orangerock National Park:



We define the altitude function, $A(x, y)$, as the altitude at a point x meters east and y meters north of the origin ("Start").

1. Estimate $A(300, 300)$ and $A(500, 500)$.

2. Estimate $A_x(300, 300)$ and $A_y(300, 300)$.

3. What do A_x and A_y represent in physical terms?

4. In which direction does the altitude increase most rapidly at the point $(300, 300)$?

5. Use your estimates of $A_x\,(300, 300)$ and $A_y\,(300, 300)$ to approximate the altitude at $(320, 310)$.

Group Work 5, Section 11.3
The Geometry of Partial Derivatives

Consider the function $f(x, y) = xy$.

1. Write an equation for the points in the intersection of the graph $z = xy$ with the plane $x = 1.5$. Since $x = 1.5$ is constant, we can consider this curve to be the graph of a function $g_{1.5}(y) = f(1.5, y)$. Compute the slope of the curve $g_{1.5}(y)$ and explain why $g'_{1.5}(y) = \frac{\partial f}{\partial y}(1.5, y)$.

2. Do the same thing for the intersection of $z = xy$ and the plane $y = 2$. As before, we can think of this curve as the graph of a function $h_2(x) = f(x, 2)$. Compute the slope of the curve $h_2(x)$ and explain why $h'_2(x) = \frac{\partial f}{\partial x}(x, 2)$.

3. Graph the function $z = xy$. Why are the curves on the graph that are parallel to either the xz-plane or the yz-plane always straight lines? What variables are being held constant in these cases? Imagine yourself walking on the surface. If you are at the point $(1.5, 2, 3)$ on the surface, the slope in the x-direction is 2 and the slope in the y-direction is 1.5. Verify this by using the results of parts (a) and (b) and by looking at the graph.

11.4 Tangent Planes and Linear Approximations

Suggested Time and Emphasis

1–$1\frac{1}{4}$ classes Recommended material

Transparencies Available

- Transparency 45 (Figure 2, page 781)
- Transparency 46 (Figure 7, page 785)

Points to Stress

1. The tangent plane and its analogy with the tangent line.

2. Approximation along the tangent plane and its analogy with approximation along the tangent line.

3. The meaning of differentiability in $\mathbb{R}^2$ and $\mathbb{R}^3$.

4. The difference between f being differentiable and the existence of f_x and f_y.

Text Discussion

- Is it possible for a function f to be differentiable at (a, b) even though f_x and f_y do not exist at (a, b)?
- Is it possible for a function f to be *not* differentiable at (a, b) even though f_x and f_y exist at (a, b)?

Materials for Lecture

- Discuss tangent planes using both the development on pages 779–80, and the geometric and vector approaches. This latter approach uses the "vector" derivative with $\mathbf{a} = \langle 1, 0, f_x(a, b)\rangle$, $\mathbf{b} = \langle 0, 1, f_y(a, b)\rangle$, and normal $\mathbf{N} = \mathbf{a} \times \mathbf{b}$ to define the tangent plane at $(a, b, f(a, b))$.
- Find an equation for the tangent plane to the top half of the unit sphere $x^2 + y^2 + z^2 = 1$ at the point $(0, 0, 1)$ and then at $\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$ using both algebraic and geometric reasoning.
- Discuss differentiability using both the definition on page 782, and this alternate definition involving the tangent plane approximation:
  f is differentiable at (a, b) if both $f_x(a, b)$ and $f_y(a, b)$ exist and no matter how we choose (x, y) sufficiently close to (a, b), the linearization of f at (a, b) closely approximates $f(x, y)$.
- One good example of a function that is not differentiable at the origin is $f(x, y) = \begin{cases} \dfrac{(x-y)^2}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 1 & \text{if } (x, y) = (0, 0) \end{cases}$ Here $f(x, 0) \equiv 1$ and $f(0, y) \equiv 1$, so f_x and f_y are both zero, but the tangent plane $z = 1$ fails to be a good approximation, no matter how close to the origin we look. For example, $f(x, x) \equiv 0$ for all $x \neq 0$.
- Note that if the partial derivatives exist and are continuous, then the tangent plane exists.

- Present examples of functions which are not differentiable, such as $f(x,y) = \begin{cases} \dfrac{x-y}{x+y} & \text{if } x+y \neq 0 \\ 0 & \text{if } x+y=0 \end{cases}$ Include an example, such as $g(x,y) = \sqrt{x^2+y^2}$, where the function is continuous.

Workshop/Discussion

- Visit or revisit examples of functions such as $f(x,y) = \begin{cases} \dfrac{x-y}{x+y} & \text{if } x+y \neq 0 \\ 0 & \text{if } x+y=0 \end{cases}$ and $g(x,y) = \sqrt{x^2+y^2}$, providing details where necessary.
- Compute some approximations to values of differentiable functions. For example, if $f(x,y) = \sin\left(\pi\left(x^2+xy\right)\right)$, then $f\left(\frac{1}{2},0\right) = \frac{1}{\sqrt{2}}$. Show the students how to use this fact and the partial derivatives of f to estimate $f(0.55,-0.01)$.
- Go over Example 5 in detail. Point out that although the errors are not small numbers in absolute terms, they are small numbers relative to the total volume (approximately 2618 cm^3) at the specified values.
- Use the approach on pages 779–80 to find the plane tangent to a surface described by a function. For example, given $f(x,y) = x^2y^3$, find the equation for the tangent plane at $(1,1,1)$, using the formula $z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$ (Answer: $z = 2x+3y-4$). Then have the students find the tangent plane at the point $(3,1,9)$ (Answer: $z = 6x+27y-36$).
  This example can be extended by asking how well the tangent plane approximates the function at the given point, perhaps by comparing $f(1.1,1.1) = 1.61$ to the approximating function $2x+3y-4$ at $(1.1,1.1)$, which has the value 1.5, and then comparing $f(1.01,1.01) \approx 1.051$ to $2x+3y-4$ at $(1.01,1,01)$, which has the value 1.05. Point out that $f(2,2) = 32$, while the value of $2x+3y-4$ at $(2,2)$ is 6.
- Review the geometric and vector approaches to planes using a vector normal to the plane. Consider the graph of f and the curves C_1 and C_2 as described in Section 11.3 on pages 769–70. Given that the vectors $\langle 1,0,f_x\rangle$ and $\langle 0,1,f_y\rangle$ lie in the tangent plane, taking the cross product gives a normal vector $\mathbf{N} = \langle -f_x,-f_y,1\rangle$. Now we can compute the equation of the tangent plane to be $z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$, exactly as derived by a different approach in the text on pages 779–80.
- Describe the tangent plane for a surface using either geometric reasoning or formal computations. A good example is the ellipsoid $\frac{x^2}{4} + y^2 + \frac{z^2}{4} = 1$ and its tangent planes at $(0,0,4)$, $\left(\sqrt{2},0,\sqrt{2}\right)$, $\left(\sqrt{2},0,-\sqrt{2}\right)$, and $(0,0,-4)$.

Group Work 1: Trying it All Out

Problem 1 of this exercise requires the student to recognize where a complicated function is continuous. Problem 2 can be done without a picture, but students should be encouraged to draw a picture to verify their conclusions.

Group Work 2: Voluminous Approximations

Group Work 3: Self-Intersection of Surfaces

This one is a challenge for the students. For Problem 2, it will probably be necessary to give them a hint. If $\mathbf{r}(u_1, v_1) = \mathbf{r}(u_2, v_2)$, then we know that $u_1 = \pm u_2$ and $v_1 = \pm v_2$ in order that the first two coordinates be the same. Given this fact, we have to find the relationship between the pairs to make $u_1 + 2v_1 = u_2 + 2v_2$. There are three nontrivial cases to check:

$$u_1 = u_2, v_1 = -v_2 \qquad u_1 = -u_2, v_1 = v_2 \qquad u_1 = u_2, v_1 = -v_2$$

When all is said and done, the only nontrivial result occurs when $u_1 = -u_2$ and $v_1 = -v_2$, and that result is $u = -2v$. So the self-intersecting set occurs when $u = -2v$, and the self-intersecting points are of the form $(4v^2, v^2, 0)$, that is, the half-line $x = 4y$, $y \geq 0$.

Homework Problems

Core Exercises: 1, 4, 10, 18, 20, 26, 33

Sample Assignment: 1, 4, 5, 8, 10, 11, 13, 18, 20, 25, 26, 31, 33, 36, 40

Note: Exercises 5, 8, 33, and 36 require a CAS.

Exercise	C	A	N	G	V
1		×			
4		×			
5		×		×	
8		×		×	
10		×			
11		×			
13		×	×		
18			×		

Exercise	C	A	N	G	V
20		×			
25		×	×		
26		×	×		
31		×	×		
33		×			
36		×		×	
40		×			

Group Work 1, Section 11.4
Trying it All Out

1. Determine for what points the following functions are differentiable, and describe (qualitatively) why your answer is correct.

(a) $f(x, y) = e^{xy} \cos(\pi(xy + 1))$

(b) $g(x, y) = \dfrac{x^4 - y^4}{x + y}$

(c) $h(x, y) = x - 2y \ln|x + y|$

2. Consider the surface $\dfrac{x^2}{4} + \dfrac{y^2}{9} + \dfrac{z^2}{16} = 1.$

(a) Find the equation of the tangent plane to this surface at the point $(0, 3, 0)$.

(b) Can you find a point at which the tangent plane to this surface is horizontal? Is there any other such point?

(c) Can you find a point at which the tangent plane to this surface is vertical? Is there any other such point?

3. Consider the surface $\dfrac{x^2}{4} + \dfrac{y^2}{9} - \dfrac{z^2}{16} = 1.$

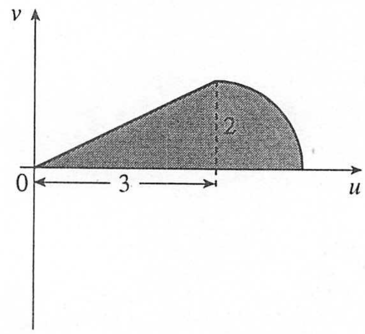
(b) Find the equation of the tangent plane to this surface at the point $(0, 3, 0)$.

(b) Can you find a point at which the tangent plane to this surface is horizontal? Is there any other s point?

(c) Can you find a point at which the tangent plane to this surface is vertical? Is there any c point?

Group Work 2, Section 11.4
Voluminous Approximations

Consider the solid obtained when rotating the following region about the u-axis.



Note that the region is composed of a right triangle and a quarter-circle.

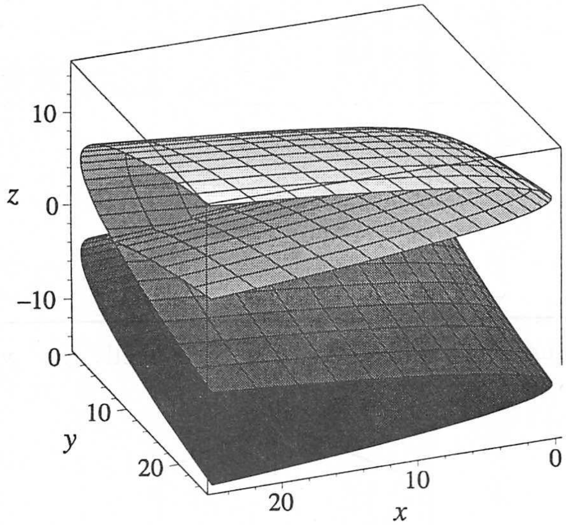
1. Compute the volume of this solid.

2. Find the volume $V(x, y)$ of a similar solid created by rotating a region with horizontal dimension x and vertical dimension y instead of 3 and 2.

3. Oh yeah — we forgot to tell you that in Problem 1, the "2" and the "3" were really just rounded-off numbers. The actual quantities can be off by up to 0.5 in either direction. Use linear approximation to estimate the maximum possible error in your answer to part (a).

Group Work 3, Section 11.4
Self-Intersection of Surfaces

Consider the parametric surface given by $\mathbf{r}(u, v) = (u^2, v^2, u + 2v)$, shown below.



1. There is a set of points at which this surface intersects itself. If we know that $\mathbf{r}(u_1, v_1) = \mathbf{r}(u_2, v_2)$, what conditions does this place on u_1 and u_2, and on v_1 and v_2?

Hint: Look at each of the three coordinates separately.

2. Using your answer to Problem 1, find a relationship between u and v which describes the set of self-intersecting points.

3. Give a geometric description of the set of points on the surface that are self-intersecting points.

11.5 The Chain Rule

Suggested Time and Emphasis

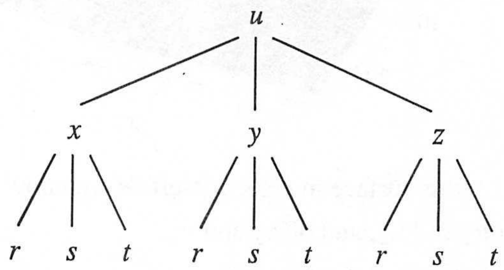
1 class   Essential material

Points to Stress

1. The extension of the Chain Rule for functions of several variables.
2. Tree diagrams.
3. Implicit differentiation.

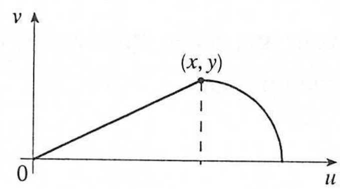
Text Discussion

- What was the following figure illustrating in the text? Specifically, how was it used?



Materials for Lecture

- Review the single-variable Chain Rule $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$ using the same language and symbols that will be used in presenting the multivariable Chain Rule.
- Develop the formulas and derivatives for chain rules involving two and three independent variables.
- Find the volume $V(x, y)$ generated by rotating the region at right about the u-axis. (Note that the region is composed of a right triangle and a quarter-circle.) Now suppose that x and y vary with time: $x = t + \sin t$, $y = 2^t - t\cos t$. Compute $\frac{dV}{dt}$.

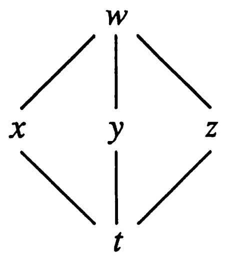


- Carefully state the Implicit Function Theorem. Illustrate it for the "fat circle" $x^4 + y^4 = 1$, and show why it fails when $\frac{\partial F}{\partial y} = 0$ [that is, at the points $(-1, 0)$ and $(1, 0)$.]
- Give an example of implicit differentiation on the ellipsoid $x^2 + \frac{y^2}{2} + \frac{z^2}{3} = 1$. Compute $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, both in general and at the point $\left(0, 1, \sqrt{\frac{3}{2}}\right)$.
- Consider a cylindrical can of radius r and height h. Let V and S be the volume and surface area of the can. Find $\frac{\partial V}{\partial r}$ and $\frac{\partial S}{\partial h}$ and discuss what these quantities mean in practical terms. Then find $\frac{\partial V}{\partial h}$ when $r = 5$.

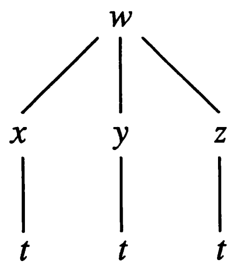
Workshop/Discussion

- Set up tree diagrams in two ways for the set of functions

$$w = f(x,y,z) \qquad x = g(t) \qquad y = h(t) \qquad z = k(t)$$



or



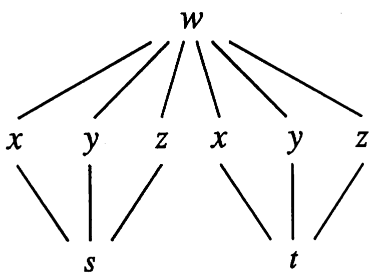
Then write out the Chain Rule for this case:

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

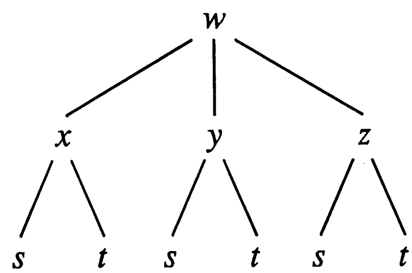
Extend the demonstration by considering

$$w = f(x,y,z) \qquad x = g(s,t) \qquad y = h(s,t) \qquad z = k(s,t)$$

Set out the relevant tree diagrams as shown below and write out the Chain Rule for $\frac{\partial w}{\partial s}$ and $\frac{\partial w}{\partial t}$.



or



- Consider the function $f(x,y) = x^2+y^2$, where $x = 3t$ and $y = e^{2t}$. Compute $\frac{df}{dt}$, first by using the Chain Rule, and then by actually performing the substitution to get $f(t) = 9t^2 + e^{4t}$ and taking the derivative. Show how this process of first substituting can become more complicated when using functions of two variables by discussing the function $g(x,y) = (x^2 + xy + y^2)^2$ with $x = 3(t+s)$ and $y = e^{2st}$, and computing $\frac{\partial g}{\partial t}$.

Group Work 1: Chain Rule Examples

This exercise contains three problems from very different disciplines. It would probably be overkill to have every group do every problem. Assign the one that most closely matches the interest of the class, and then groups that finish early can work on a different one.

Group Work 2: The Elephant Podium

Notice that Problems 1 and 2 can be done with calculus, by rotating the relevant line segment around the y-axis, or without calculus, by subtracting the volume of one cone from the volume of another cone. There may be even more methods, but they are surely irrelephant.

Group Work 3: Partial to Polar

Consider a surface $z = f(x, y)$ in polar coordinates and compute $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial \theta}$. Do an explicit example, such as $z^2 = x^2 - y^2$, and evaluate $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial \theta}$ when $r = 1$ and $\theta = \frac{\pi}{2}$.

Homework Problems

Core Exercises: 2, 7, 11, 16, 22, 28, 34, 36

Sample Assignment: 1, 2, 7, 10, 11, 16, 22, 25, 28, 29, 34, 35, 36, 40

Note: Many students remember the Chain Rule by telling themselves that $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$ "because the dx's cancel". In single variable calculus, it is hard to persuade them that this is not what is really happening. Problem 45 shows the students an example of where that kind of thinking can get them into trouble.

Exercise	C	A	N	G	V
1		×			
2		×			
7		×			
10		×			
11		×			
16		×			
22		×			

Exercise	C	A	N	G	V
25		×			
28	×	×	×		
29		×	×	×	
34		×			
35		×			
36		×			
40		×			

Group Work 1, Section 11.5
Chain Rule Examples

Example A: Chemistry 101

Given n moles of gas, the relationship between pressure P, temperature T, and volume V can be approximated by the formula

$$PV = nRT$$

where P is in atmospheres, V is in Liters, T is in degrees Kelvin (degrees Kelvin = degrees Celsius +273.15), and R is the ideal gas constant [0.08206 L · atm/ (mol · K)]

Assume we have 10 moles of gas in a balloon-type bladder. Initially we have a volume of 1 liter at "STP" ($T = 273.15$, $P = 1$). As time goes on, the gas is heated. The following expresses the temperature T of the gas as a function of the time elapsed t since the beginning of the experiment:

$$T = 323.15 - \frac{50}{t+1}$$

The bladder begins to expand over time as a function also of the strength s of its material, with the following formula describing how the volume V of the gas which can occupy the bladder changes as a function of the number of minutes t and the material strength s (where s is measured in millikents.)

$$V = 2\left(2 - e^{-3ts}\right)$$

1. Describe how the pressure of the gas in the bladder changes as a function of time.

2. Describe how the pressure of the gas in the bladder changes as a function of the strength of the bladder.

3. If the experiment takes place in a one-millikent bladder, what is the pressure of the gas in the box after 4 minutes?

4. If the experiment is allowed to run for a very long time, what value will P approach? Wh $\frac{dP}{dt}$ approach?

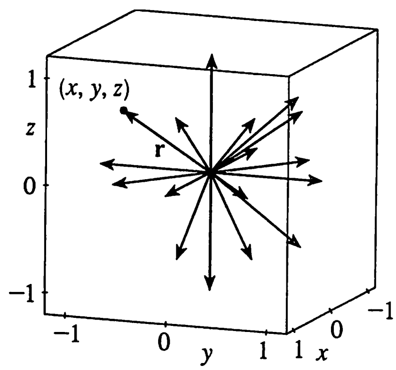
Group Work 1, Section 11.5
Chain Rule Examples

Example B: Electric Fields

Here we introduce the idea of a vector field or function defined for position vectors in space. In a vacuum, the electric field r units from a charge q at the origin is given by the vector field

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_0 r^2}\mathbf{r} = \langle E_1, E_2, E_3 \rangle$$

where ε_0 is a constant, q is the charge on an electron, and $\mathbf{r}$ is a unit vector that points radially outward from the charge as shown in the figure.



1. What is the field at $\left(\frac{1}{2}, 0, 0\right)$? At $(0, 1, 0)$? At $(0, 0, 2)$?

2. Assume that we are concerned with the electric field at a point $p = (x, y, z)$. Why can we write $\mathbf{r} = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}} \langle x, y, z \rangle$?

3. Find the total x-change $\dfrac{\partial \mathbf{E}}{\partial x} = \left\langle \dfrac{\partial E_1}{\partial x}, \dfrac{\partial E_2}{\partial x}, \dfrac{\partial E_3}{\partial x} \right\rangle$ of the electric field if we start at $(5, 5, 5)$ and move in a direction parallel to the x-axis. Is this derivative a vector or a scalar?

4. Find the rate of change $\dfrac{d\mathbf{E}}{dt}$ of the electric field as we move along the line $z = 0$, $y = 2x$, starting at the point $(1, 2, 0)$, with parametrization $\mathbf{L}(t) = \langle t + 1, 2(t + 1), 0 \rangle$, $t \geq 0$. Is this derivative a vector or a scalar?

Group Work 1, Section 11.5
Chain Rule Examples

Example C: The Mutual Fund

One of the hottest investments on Wall Street today is the Share-All Mutual Fund. The Share-All Fund has issued 500,000 shares for eager investors to buy. Each share, therefore, represents $\frac{1}{500,000}$ of the fund's total net asset value. The fund owns 100,000 shares of stock in four companies, as described below, on a given day.

Company Name	Current price/share	Number of Shares Owned by Share-All	Total $ value
Allied Oil	30	30,000	900,000
Beck Keyboard Manufacturing	15	20,000	300,000
Jasmine Tea	23	28,000	644,000
Lapland Importing-Exporting	22	22,000	484,000
Total asset value			2,328,000

So, on this day, the total asset value is $2,328,000, and the price of one share of Share-All is
$\frac{2,328,000}{500,000} = \$4.656$.

Many factors affect the price of a stock. For example, the worldwide exchange rate* w affects the Lapland Importing-Exporting company much more than it does the primarily domestic Beck Keyboard Manufacturing company. Similarly, the United States prime lending rate p affects the highly indebted Allied Oil company more than it affects the relatively debt-free Jasmine Tea company. Thus, we can develop models for the price of these stocks as functions of the world-wide exchange rate, the prime lending rate, and other economic factors q, r, s, and v, which are independent of these variables.

Let w be the world-wide exchange rate, and p be the U.S. prime lending rate. Let $a(w,p,q)$, $b(w,p,r)$, $j(w,p,s)$, and $l(w,p,v)$ be the current price per share of Allied, Beck, Jasmine and Lapland respectively. We have

$$\begin{aligned} a(w,p,q) &= 0.1w^2 - 100\sqrt{p} + q \\ b(w,p,r) &= 0.01w - p + r \\ j(w,p,s) &= 2w + s \\ l(w,p,v) &= -(1+w)^3 - p + v \end{aligned}$$

where q, r, s and v are composite variables representing the myriad other factors that affect the price of these stocks, and are independent of w and p. (*Note:* If we knew q, r, s, and v exactly, then we would be able to predict the price of the stock with unrealistic accuracy.)

* The worldwide exchange rate measures the value of the dollar measured versus a weighted average of other relevant currencies. In other words, it is a statistic that we made up, but which could probably fool some people.

Chain Rule Examples

1. If the current worldwide exchange rate is 0.85, and the current U.S. prime lending rate is 0.06, what are the current values of q, r, s, and v?

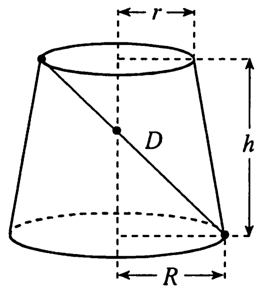
2. Assume that the values of q, r, s, and v are fixed at the quantities computed in Problem 1. Let S be the price of a share of Share-All. Write S as a function of w and p.

3. If S is the price of a share of Share-All, what is $\frac{\partial S}{\partial w}$?

4. What is $\frac{\partial S}{\partial p}$?

Group Work 2, Section 11.5
The Elephant Podium

The solid shown below has top radius r, bottom radius R, and height h.



1. If $r = 2$, $R = 4$, and $h = 8$, then what is the volume V of the solid?

2. Find a general expression for V in terms of r, R, and h.

3. Compute $\dfrac{\partial V}{\partial R}$.

4. Compute $\dfrac{\partial V}{\partial h}$.

5. If $r = 2$, $R = 4$, and $h = 8$, then what is the length of the diagonal D of the solid?

6. Find a general expression for D in terms of r, R, and h

7. Compute $\dfrac{\partial D}{\partial R}$.

8. Compute $\dfrac{\partial D}{\partial h}$.

9. Compute $\dfrac{\partial R}{\partial r}$ if we know that $V = 10\pi$. What does this quantity represent in practical terms?

11.6 Directional Derivatives and the Gradient Vector

Suggested Time and Emphasis

$1\text{–}1\frac{1}{4}$ classes   Essential Material

Transparencies Available

- Transparency 47 (Figure 3, page 799 and Figure 5, page 801)

Points to Stress

1. The geometric meaning of a directional derivative.
2. The geometric meanings of the gradient vector, as defining the direction of greatest change of the directional derivative, and as a normal vector to a surface.
3. The relationships between directional derivatives, gradient vectors, and tangent planes.

Text Discussion

- The text shows that $D_{\mathbf{u}}f(x,y) = \nabla f(x,y) \cdot \mathbf{u}$, where $\mathbf{u}$ is a unit vector Why does this show that the directional derivative in the direction of $\mathbf{u}$ is the scalar projection of the gradient vector onto $\mathbf{u}$?

Materials for Lecture

- Give geometric interpretations of the gradient, particularly of how the gradient gives the direction of maximal increase of a function $\left[\text{using the unit vector } \mathbf{u} = \dfrac{\nabla f(x,y)}{|\nabla f(x,y)|}\right]$ and its negative the direction of maximal decrease.
- Consider the surface $f(x,y) = xy$ at the point $(0,0)$. Note that although the maximum rate of change is zero at that point, it is not the case that the function is identically zero near the origin. Thus, if $\nabla f(a,b) = \mathbf{0}$, we cannot talk about the direction of maximal change at (a,b).
- Take any function $z = f(x,y)$ and write out its linearization at a point (a,b):

$$z = g(x,y) = f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b)$$

  Show that the two functions f and g have the same gradient at the point (a,b), namely

$$\nabla f = \nabla g = \langle f_x(a,b), f_y(a,b) \rangle$$

  Hence their directional derivatives are the same at (a,b). However, the graph of the function g is a plane, so it is visually obvious that there is a unique direction of maximal increase, *unless* the plane is horizontal. Note that the plane is horizontal precisely when

$$f_x(a,b) = f_y(a,b) = 0$$

  that is, when (a,b) is a critical point of f. This reasoning gives an informal explanation as to why f has only one direction of maximal increase.
- Let $f(x,y) = x^2y^2$. Compute $D_{\mathbf{u}}f(x,y)$ for unit vectors $\mathbf{u}$ making angles of 0, $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$, and $\frac{\pi}{2}$ with the positive x-axis, and fill in the following table. Point out that the coefficient of $2xy^2$ decreases from

1 to 0 while the coefficient of $2x^2y$ increases from 0 to 1. Have the students reason intuitively why this should be the case, just using the concept of "directional derivative". Have them figure out (intuitively) $D_{\mathbf{u}}f(x,y)$ for angles π and $\frac{3\pi}{2}$.

Angle	$D_uf(x,y)$
0	$(1)\,2xy^2 + (0)\,2x^2y$
$\frac{\pi}{6}$	$(0.866\ldots)\,2xy^2 + (0.5)\,2x^2y$
$\frac{\pi}{4}$	$(0.707\ldots)\,2xy^2 + (0.707\ldots)\,2x^2y$
$\frac{\pi}{3}$	$(0.5)\,2xy^2 + (0.866...)\,2x^2y$
$\frac{\pi}{2}$	$(0)\,2xy^2 + (1)\,2x^2y$

Note: At a given point (x,y), one cannot maximize D_u merely by looking at the chart.

- Review that the direction of any vector $\mathbf{v} \neq \mathbf{0}$ is determined by the unit vector $\mathbf{u} = \dfrac{\mathbf{v}}{|\mathbf{v}|} = \langle\cos\theta, \sin\theta\rangle$ where θ is the angle that $\mathbf{v}$ makes with the positive x-axis. Using this interpretation, the directional derivative formula can be rewritten as

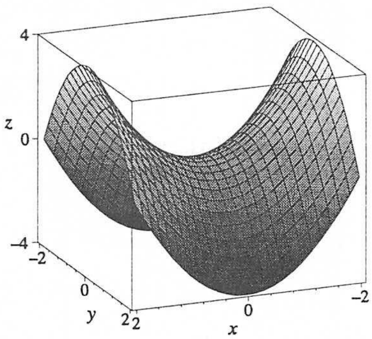
$$D_uf = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta$$

- Define S to be a level surface of the function $f(x,y,z)$. Explain why $\nabla f(x_0,y_0,z_0)$ is orthogonal to the surface at any point $P(x_0,y_0,z_0)$ on S. Define the tangent plane to S at P to be the plane with normal vector $\nabla f(x_0,y_0,z_0)$.
- Show how Equation 2 in Section 11.4 (page 780) is really just a special case of Equation 19 in this section (page 806).

Workshop/Discussion

- Have the students practice finding the directional derivatives of $f = x^2y$ and $f = e^{xy}$ in the directions $-\mathbf{i}$, $\mathbf{i}+\mathbf{j}$, $-\mathbf{i}-\mathbf{j}$, and $\mathbf{i}-\mathbf{j}$. Also have them find the directional derivatives of $f(x,y,z) = z^2e^{xy^2}$ in the directions of $\langle -1,-1,-1\rangle$ and $\langle 0,0,-1\rangle$.
- Show that the gradient vector $\nabla f(x_0,y_0)$ is normal to the line tangent to the level curve $k = f(x,y)$ at the point (x_0,y_0). Look at the example $f(x,y) = 5x^4 + 4xy + 3y^2$ and show that $\nabla f(-1,-1) = \langle -24,-10\rangle$. Conclude that we now know that f is decreasing in both the x- and y-directions and that the direction of maximal increase is $\langle -24,-10\rangle$. Ask the students to resolve these seemingly contradictory observations: That the gradient is supposed to point in the direction of maximal *increase*, yet the components $f_x(-1,-1) = -24$ and $f_y(-1,-1) = -10$ of the gradient are pointing in the direction of decreasing x and y. (Although no real paradox exists, students are often confused by this type of situation.)
- Note that $-\nabla f$ points in the direction of maximal decrease of f, and that the rate of change is $-|\nabla f|$.
- Do Exercise 32 (page 810), adding some additional points on the outer contour lines and drawing the curves of steepest ascent, and then choosing a point on the innermost contour line and drawing the curve of steepest descent.

- Solve the problem of finding the direction $\mathbf{u}$ for which the rate of increase of $f(x, y) = x^2y^3 + x + y$ at $(1, 1)$ is 2. Pose the same problem to the students for a rate of increase of 4 and for a rate of decrease of 5.
- Analyze Figure 13. Ask why the gradients near the y-axis point toward the vicinity of the origin and downhill (have negative z-coordinate) while those near the x-axis are pointing uphill, as the text claims. Show how the shape of $z = x^2 - y^2$ reflects this behavior.



$z = x^2 - y^2$

Group Work 1: Two Ways

Group Work 2: Computation Practice

It is a good idea to give the students a chance for guided practice using the types of computations that will be required on the homework. We recommend having them do either Problem 1 or Problem 2 in groups, and then handing the remaining problem out as a worksheet.

Group Work 3: Bowling Balls and Russian Weebles

This exercise may seem trivial, but it is a good setup for discussions of Lagrange multipliers. If the students do this exercise, ask them to remember the result, and make sure to remind them of the bowling balls and Russian weebles when discussing Lagrange multipliers.

Homework Problems

Core Exercises: 1, 5, 8, 14, 19, 23, 30, 32, 41

Sample Assignment: 1, 5, 8, 10, 12, 14, 19, 20, 23, 28, 30, 31, 32, 34, 35, 39, 41, 44, 53

Note: Exercise 39 requires a CAS.

Exercise	C	A	N	G	V
1			×	×	×
5		×			
8		×			
10		×			
12		×			
14		×			
19		×			
20		×			
23		×			
28		×			

Exercise	C	A	N	G	V
30		×			
31		×			
32				×	
34	×			×	
35		×			
39		×		×	
41		×		×	
44		×			
53	×	×			

Group Work 1, Section 11.6
Two Ways

Consider the function $f(x,y) = x^2 + 2xy^2$.

1. What is $f(1,1)$?

2. What is the directional derivative $D_u f(1,1)$ if $\mathbf{u} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$?

3. What is the directional derivative $D_u f(1,1)$ if $\mathbf{u}$ is the unit vector that makes an angle θ with the positive x-axis?

4. In Problem 3, you expressed $D_u f(1,1)$ as a function of the angle θ. Let's say we want to find the maximum value of the directional derivative. This is now a single-variable calculus problem! Use your single-variable calculus techniques, coupled with your answer to Problem 3, to find the find the maximum value of the directional derivative.

5. What is the angle θ for which f increases the fastest? (You should be able to use your computations for Problem 4 to answer this one quickly.)

6. What is the unit vector that makes that angle θ with the positive x-axis?

7. Now compute $\dfrac{\nabla f(1,1)}{|\nabla f(1,1)|}$, but before you do so, discuss with your group members what the answer should be. You should be able to anticipate the correct answer.

Group Work 2, Section 11.6
Computation Practice

1. Find the directional derivative of the function at the given point in the direction of the given vector $\mathbf{v}$.

(a) $f(x,y) = e^{xy} - x^2$, $(1,1)$, $\mathbf{v} = \langle 1,0 \rangle$

(b) $f(x,y) = e^{xy} - x^2$, $(1,1)$, $\mathbf{v} = \langle 0,1 \rangle$

(c) $f(x,y) = e^{xy} - x^2$, $(1,1)$, $\mathbf{v} = \langle 1,1 \rangle$

(d) $f(x,y,z) = z \ln(x^2 + y^2)$, $(-1,1,0)$, $\mathbf{v} = \langle 1,1,-1 \rangle$

(e) $f(x,y,z) = z \ln(x^2 + y^2)$, $(-1,1,0)$, $\mathbf{v} = \langle 2,1,1 \rangle$

2. Find the maximum and minimum rates of change of f at the given point and the directions in which they occur.

(a) $f(x,y) = e^{xy} - x^2$, $(1,1)$

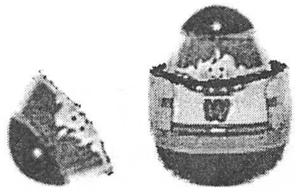
(b) $f(x,y,z) = z\ln\left(x^2+y^2\right)$, $(-1,1,0)$

Group Work 3, Section 11.6
Bowling Balls and Russian Weebles

1. Assume that there are two bowling balls in a ball-return machine. They are touching each other. At the point at which they touch, what can you say about their respective normal vectors? What about their tangent planes? What would happen if the bowling balls were of different sizes?

2. A *weeble* is a doll that is roughly egg-shaped. It is an ideal toy for little children, because weebles wobble but they don't fall down.

A Russian Weeble is a hollow weeble, with one or more weebles inside it. Picture two nested hollow eggs as shown.



At the point at which two nested Russian weebles touch each other, what can you say about their respective normal vectors? What about their tangent planes?

3. Now picture your two favorite differentiable surfaces that touch at exactly one point. What can you say about their normal vectors at the point where they touch? What about their tangent planes?

11.7 Maximum and Minimum Values

Suggested Time and Emphasis

1 class Essential Material

Transparencies Available

- Transparency 48 (Figures 7–9, page 815)

Points to Stress

1. The contrast between optimization problems in single-variable calculus (relatively few cases) and in multivariable calculus (many possible solutions).
2. Critical points and local maxima and minima.
3. The Second Derivative Test.
4. Absolute maxima and minima.

Text Discussion

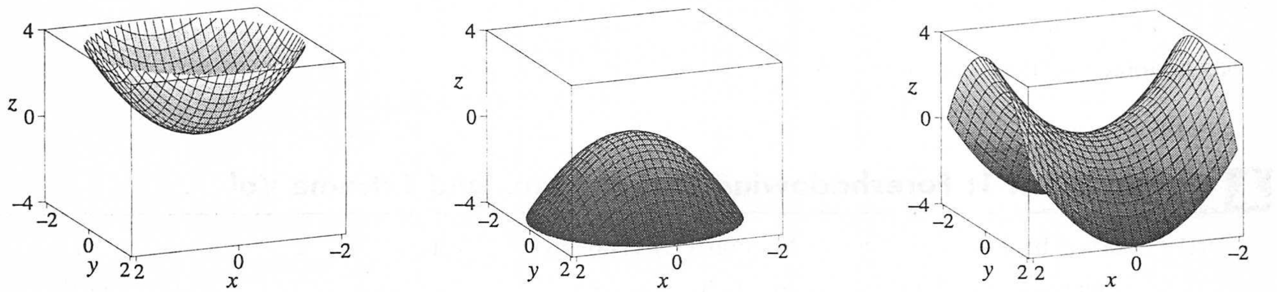
- Can a differentiable function f have a local maximum at a point (a, b) with $f_x(a, b) = 3$?
- Can you give an example of a function f with the property that $f_x(a, b) = 0$, $f_y(a, b) = 0$, and f does *not* have a local maximum or minimum at (a, b)?
- Can you always apply the Second Derivative Test at any critical point? If you can, does it always give information?

Materials for Lecture

Note: One way to introduce this topic is to have the students initially do Group Work 1: Foreshadowing Critical Points and Extreme Values.

- Stress geometric interpretations: If f is differentiable at a local maximum or minimum, then the tangent plane must be horizontal.
- Note that there are critical points at which there is no local maximum or minimum. For example, examine the saddle points at the origin for $f(x, y) = xy$ and $g(x, y) = x^2 - y^2$.

- Illustrate the idea behind the Second Derivative Test, using $f(x,y) = \frac{1}{2}(ax^2 + by^2)$. Note that $D = ab$ means that
  - $D > 0$, $a > 0$ gives $b > 0$ and hence $f(x,y)$ has a local minimum at $(0,0)$ [See Picture (a)]
  - $D > 0$, $a < 0$ gives $b < 0$ and hence $f(x,y)$ has a local maximum at $(0,0)$ [See Picture (b)]
  - $D < 0$ means that a and b have opposite signs, and hence $f(x,y)$ has a saddle point at $(0,0)$ [See Picture (c)]



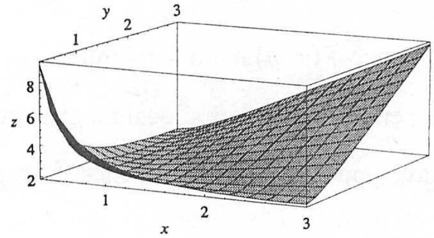
(a) Minimum (b) Maximum (c) Saddle Point ($a > 0$, $b < 0$)

- Describe some of the ways saddle points can occur for functions of two variables. (For examples, see Example 3 and Figure 4 on page 813, and Example 4 and Figures 7 and 8 on pages 813–15.) Contrast with the single-variable case, where there are fewer possibilities.
- Use $f(x,y) = x^4 + y^4$, $g(x,y) = x^4 - y^4$, $h(x,y) = -(x^4 + y^4)$ to show that no information is given about local extrema when $D = 0$.
- Review Example 5 (pages 815–16), which finds the distance between a point and a plane. Contrast this approach to the method used in Example 8 in Section 9.5 (page 682).
- Briefly discuss absolute maximum and minimum values of continuous functions on closed bounded sets. Point out that these values can occur either at boundaries or "inside" the region. One good example is $f(x,y) = x^2 + y^2$ on the rectangle $D = \{(x,y) \mid -1 \le x \le 1, -1 \le y \le 2\}$, where the absolute minimum value is $f(0,0) = 0$ and the absolute maximum value is $f(\pm 1, 2) = 5$.

Workshop/Discussion

- Pose the problem of finding the absolute maximum of $f(x,y) = ax + by + c$ on the set of points $x^2 + y^2 \le 4$. Note that the gradient of f is never zero, so the maximum and minimum values must occur on the boundary. One way to find these maximum and minimum values is by parametrizing the boundary $x^2 + y^2 = 4$ by $\mathbf{r}(\theta) = \langle 2\cos\theta, 2\sin\theta \rangle$, where θ is the angle made by the position vector with the x-axis, and then optimizing the function $g(\theta) = f(\mathbf{r}(\theta))$.
- Expand on Example 5 (pages 815–16) by indicating that it is sometimes easier to optimize $f^n(x,y)$ instead of $f(x,y)$ for a function f that is always positive. Point out that f^n has the same maxima as f for any n. One good example to use is $f(x,y) = \left[(x-1)^2 + (y+1)^2 + 1\right]^{1/3}$. Another example is the problem of finding the points on the surface $z^2 = xy + x^2 + 1$ that are closest to the origin. [Answer: $\left(\frac{1}{7}, -\frac{4}{7}, \frac{3\sqrt{2}}{7}\right)$]

- Discuss local and absolute maxima and minima for $f(x,y) = xy + 1/(xy)$, $x > 0$, $y > 0$.



- Consider $f(x,y) = x^2 - y^2 + 2xy$. Show that f has no absolute maximum or minimum and also that f has no local maximum or minimum.

Group Work 1: Foreshadowing Critical Points and Extreme Values

This group work is best done just before Section 11.7 is covered. First present the single-variable definitions of local and global maximum and minimum. (This was done in single-variable calculus, but the students have probably forgotten the technical definitions by this point.) Then put the students into groups and ask them to come up with good multivariable definitions of the same concepts. They should present their definitions and discuss them. At the end of the activity, look up the definition presented in the text, and compare it with the student definitions.

If there is time, do a similar activity for the various types of critical points. Graph $y = x^2$, $y = -x^2$, $y = x^3$, $y = -x^3$, $y = |x|$, and $y = -|x|$ on the board to show different types of critical values at $x = 0$. Then have the students try to come up with the variety of types that can occur for functions of two variables.

Group Work 2: The Squares Conjecture

If a group finishes this problem early, have them try to solve it without using any calculus at all.

Group Work 3: Strange Critical Points

In this case, f_x and f_y do not exist at the critical point $(1, -1)$ and so the students cannot use the Second Derivative Test. Acceptable answers include graphing the surface or recognizing that it is an elliptic cone.

Extended Lab Project: The Genetic Algorithm

The use of "genetic algorithms" for finding maxima and minima for functions of several variables has become popular in recent years. Usually this technique is used to optimize functions of hundreds of variables, but we'll look at the simpler case of functions of two variables.

Although we don't intend to give a complete description of how genetic algorithms work, an outline is as follows:

Suppose you want to maximize a function of several variables. Start by selecting several arbitrary points (at random or otherwise) from your domain. Select two points among these which give the two largest values of your function. Now choose several more arbitrary points close to these selected points. Continue to repeat this process until you have what seems to be a maximum value.

We will study this process for the complicated function $100e^{-(|x|+1)(|y+1|+1)}\dfrac{\sin(y\sin x)}{1+x^2y^2}$. Let D be the square $[-3,3]\times[-3,3]$.

(i) Use your computer program to select 5 points at random in this square and then evaluate the function at these 5 points.

(ii) Select the points which give the two largest values for $f(x,y)$ and then select 4 points at random close to each of these points. Again, selecting the points at random near these points isn't so trivial. Evaluate the function at the 10 points you now have. Select the two points among these which give the largest value for $f(x,y)$. Repeat (b) until it appears that you have a maximum.

(iii) Is the value you found in (ii) likely to be an absolute maximum?

Homework Problems

Core Exercises: 2, 3, 7, 8, 12, 16, 23, 25

Sample Assignment: 1, 2, 3, 4, 7, 8, 12, 13, 15, 16, 23, 25, 29, 32, 35, 40, 45

Note:
- Exercise 29 requires a CAS.
- Problem 8 in Focus on Problem Solving (page 837) makes a good group project, particularly because it incorporates material from several sections of the text.

Exercise	C	A	N	G	V
1					
2	×				
3		×		×	×
4		×		×	×
5–14		×		×	
15		×		×	
16		×		×	

Exercise	C	A	N	G	V
23		×			
25		×			
29		×		×	
32		×			
35		×			
40		×			
45		×			

Group Work 2, Section 11.7
The Squares Conjecture

You are given a government grant to prove or disprove the Squares Conjecture:

> There exist three positive numbers, r, s, t whose product is 100, yet have the property that the sum of their squares is less than 65.

Either find three such numbers, or show that none exist.

Group Work 3, Section 11.7
Strange Critical Points

Let $f(x, y) = 2 + \sqrt{3(x-1)^2 + 4(y+1)^2}$.

1. Find the critical points of f.

2. Find the local and absolute minimum values of f. Where do these values occur?

Applied Project: Designing a Dumpster

This project requires the students to solve an extended real-world problem that involves them going out and measuring a nearby dumpster. They will have to make approximations, and figure out how best to get an accurate answer. A good sample answer is given in the *Complete Solutions Manual.*

Discovery Project: Quadratic Approximations and Critical Points

Problems 1–3 serve as a good introduction to Taylor's Theorem for two variables, and to quadratic polynomial approximation in two variables. Problems 4 and 5 justify the Second Derivative Test, the proof of which is given in Appendix E.

11.8 Lagrange Multipliers

Suggested Time and Emphasis

$\frac{3}{4}$–1 class Essential Material: one-constraint problems.
Optional Material (if time permits): two-constraint problems

Transparencies Available

- Transparency 49 (Figures 2 and 3, pages 824–25)

Points to Stress

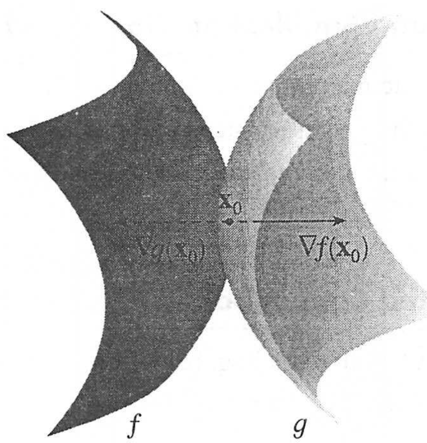
1. The geometric justification for the method of Lagrange multipliers
2. How to apply the method of Lagrange multipliers, including the extension of the method for two-constraint problems

Text Discussion

- How does the equation $\nabla f(x,y) = \lambda \nabla g(x,y)$, subject to the constraint $g(x,y) = k$, lead to three equations with three unknowns? What are the unknowns?

Materials for Lecture

- Make sure that students understand the actual "nuts and bolts" of the one-constraint method.
- Draw a picture like the one below illustrating that if two surfaces are tangent, they have parallel normals at the point of tangency.



- Give an example to show that, with functions of two variables, there are often alternate methods other than Lagrange multipliers to solve max/min problems. Perhaps redo Example 2, substituting $x^2 = 1 - y^2$ into $f(x,y) = x^2 + 2y^2$ to get the single-variable problem $g(y) = 1 + y^2$, minimize to get $y = 0$ (with $x = \pm 1$), and then get $h(x) = 2 - x^2$ with maximum at $x = 0$, $y = \pm 1$.
- Note that, for the two-variable case, $\nabla f = \lambda \nabla g$ implies that $\nabla f \times \nabla g = \mathbf{0}$. This condition can sometimes be used to replace Lagrange multipliers.
- Present some additional explanation of the use of Lagrange multipliers for functions of three variables.

- Present a complementary problem similar to Examples 2 and 3 (pages 824–25), such as finding extreme values for $f(x,y) = \left(x^2 + y^2\right)^{3/2}$ on the ellipse $\frac{1}{4}x^2 + \frac{1}{9}y^2 = 1$.

- Go through the analytic solution to Example 4 in detail.

Workshop/Discussion

- Find the volume of the largest rectangular solid that can be inscribed in a sphere, that is, maximize $V(x,y,z) = (2x)(2y)(2z)$ given that $x^2 + y^2 + z^2 = a^2$.

- Discuss the geometric solution to Example 4. What basic geometric principle is being used?

Group Work 1: The Inscribed Rectangle Race

Divide the class in half. Write the following problem on the board: "What is the area of the largest rectangle that can be inscribed in a circle of radius 4?" Have one half of the class try to solve this problem using Lagrange multipliers, and the other half try to use single-variable calculus. See which side finishes first, and which side found the problem more difficult. At the end, the students should see both methods presented.

If a group finishes early, or after all groups have presented, have the students further practice the two techniques by maximizing xy^2 on the ellipse $\frac{1}{5}x^2 + \frac{1}{7}y^2 = 1$.

Group Work 2: Biggest and Smallest on Closed and Bounded Sets

This activity involves finding the absolute maximum and minimum values of a function of several variables on a closed and bounded set. Review the necessary steps outlined on page 817 in Section 11.7. Note that Problem 1 may be easier to solve by plugging in the appropriate value for x or y along a boundary curve and using single-variable methods.

Group Work 3: The Heated Cannonball

This problem appears to be quite difficult at first reading, but letting x, y, and z be the angles (in radians) and using Lagrange multipliers leads to a very easy solution.

Group Work 4: The Sum of the Sines

Group Work 5: Find the Error

This group work presents a problem where the absolute maximum value occurs on the boundary and the absolute minimum is inside the region. The idea is to reinforce the point that finding critical points is not sufficient to locate absolute extrema for functions of two variables.

Homework Problems

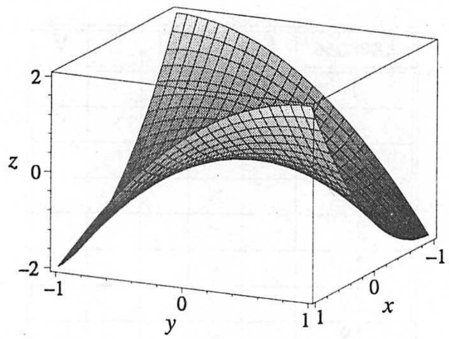
Core Exercises: 1, 2, 5, 13, 19, 23

Sample Assignment: 1, 2, 5, 8, 13, 19, 23, 26, 29, 38

Exercise	C	A	N	G	V
1	×			×	×
2		×		×	×
3–17		×			
19		×			
23		×			
26		×			
29		×			
38		×			

Group Work 2, Section 11.8
Biggest and Smallest on Closed and Bounded Sets

Let $f(x, y) = x^2 - y^2 + 2xy$.



1. What are the absolute maximum and absolute minimum values of this function on the unit square $\{0 \le x \le 1, 0 \le y \le 1\}$?

2. What are the absolute maximum and absolute minimum values of this function on the disk $x^2 + y^2 \le 1$?

3. What are the absolute maximum and absolute minimum values of this function on the elliptical disk $x^2 + \frac{y^2}{4} \le 1$?

Group Work 3, Section 11.8
The Heated Cannonball

One of the wonderful things about the British army in the eighteenth century was that they were very polite. For example, during the Revolutionary War, during the battle of Valley Forge, it was standard practice for them to gently warm their cannonballs before firing them at the colonists. Suppose that a particular cannonball with radius 1 foot has a temperature distribution $T(x, y, z) = 48 + 16(y^2 + z^2 - x^2)$ (where the center of the cannonball is at the origin).

1. What are the maximum and minimum temperatures in the cannonball, and where do they occur?

2. What is the shape of the wire frame used to apply the heat to the surface of the cannonball?

Group Work 4, Section 11.8
The Sum of the Sines

1. What is the description of the triangle for which the sum of the sines of its angles is a maximum?

2. Find the maximal value of the sums of the sines of the angles of a triangle.

3. Repeat Problems 1 and 2 if we now assume that the triangle is

(a) a right triangle.

(b) a triangle one of whose angles is 30°.

Group Work 5, Section 11.8
Find the Error

Consider the ellipsoid $\frac{1}{9}x^2 + \frac{1}{4}y^2 + z^2 = 1$. You want to locate the point(s) on this ellipse which are furthest from the origin.

1. Is it sufficient to maximize the square D of the distance instead of the distance itself?

2. Writing the ellipsoid as $z^2 = 1 - \frac{1}{9}x^2 - \frac{1}{4}y^2$, find an equation for the square D of the distance in terms of x and y.

3. Show that $(0,0)$ is a critical point of D.

4. The points corresponding to $(x,y) = (0,0)$ are $(0,0,\pm 1)$ and $D = 1$ at these points. But at the point $\left(0, 1, \frac{\sqrt{3}}{2}\right)$, $D = 1 + \frac{3}{4} = \frac{7}{4}$. So the points $(0,0,\pm 1)$ are not furthest from the origin. So calculus doesn't locate this point. What is wrong? How can you find the point(s) on the surface furthest from the origin?

Applied Project: Rocket Science

This is an excellent example of Lagrange multipliers presented in a realistic setting. If not assigned as a project, it can be given as a supplementary reading. The computations required for this problem are extensive. A CAS might help, but is not required.

Applied Project: Hydro-Turbine Optimization

The Great Northern Paper Company is a real company that has hired engineers to solve the same problem that the students are faced with. If this project is assigned, the students should be informed that they have the opportunity to solve a real engineering problem.

11 Sample Exam

Problems marked with an asterisk (*) are particularly challenging and should be given careful consideration.

1. (a) Consider the function $f(x,y) = \dfrac{1}{x^2+y^2+1}$. Find equations for the following level surfaces for f, and sketch them.

(i) $f(x,y) = \frac{1}{5}$

(ii) $f(x,y) = \frac{1}{10}$

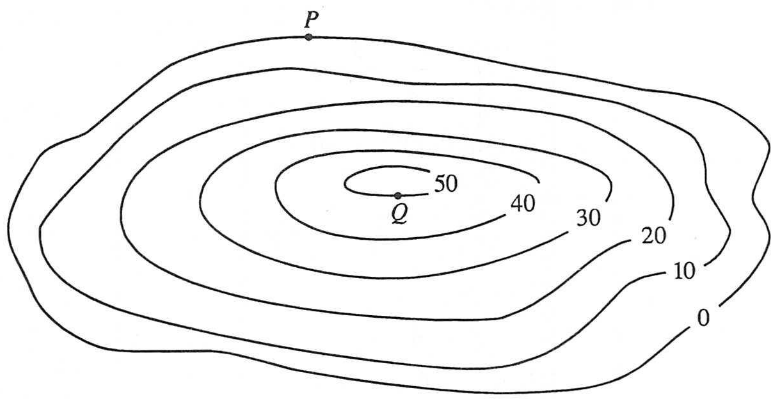
(b) Find k such that the level surface $f(x,y) = k$ consists of a single point.

(c) Why is k the global maximum of $f(x,y)$?

2. Is the function $f(x,y) = \sin^2(xy^2)$ a solution to the partial differential equation
$\dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y} = (2x+y)(2y)\cos(xy^2)\sqrt{f}$ when $\sin(xy^2) \geq 0$?

3. Is it possible to find a function for which it is true that, for all $x > 0$ and $y > 0$, $f_x > 0$ and $f_y < 0$, and $f(x,y) > 0$? If so, give an example. If not, why not?

4.



The above is a topographical map of a hill.

(a) Starting at P, sketch the path of steepest ascent to the peak elevation of 50 yards.

(b) Suppose it rains, and water runs down the hill starting at Q. At what point would you expect the water to reach the bottom? Justify your answer.

5. Find the absolute maximum and minimum of $f(x,y) = x^2+xy+y^2$ on the disk $\{(x,y) \mid x^2+y^2 \leq 9\}$.

6. Consider the ellipsoid $\dfrac{x^2}{4} + 2z^2 + \dfrac{y^2}{4} = 1$. Using geometric reasoning or otherwise, find the equation of the tangent plane at

(a) $(\sqrt{2}, \sqrt{2}, 0)$.

(b) $\left(0, 0, \frac{1}{\sqrt{2}}\right)$.

7. Describe the level surfaces $f(x,y,z) = k$ for the function $f(x,y,z) = 1 - x^2 - \dfrac{y^2}{2} - \dfrac{z^2}{3}$ and the values $k = -1$, $k = 1$, and $k = 2$.

8. Suppose that the amount of energy $F(x,y,z)$ emanating from a source at $(0,0,0)$ is inversely proportional to one more than the square of the distance from the origin measured only in the the xy-plane, and is directly proportional to the height above the xy-plane. Assume that all of the constants of proportionality are equal to 1.

(a) What is an equation for the energy as a function of x, y, and z?

(b) Where is there no energy at all?

(c) Sketch the level surface $F(x,y,z) = 1$.

9. Consider the function

$$f(x,y) = \frac{x+y}{|x|+|y|}$$

(a) Evaluate the following

(i) $f(1,1)$

(ii) $f(1,-1)$

(iii) $f(-1,1)$

(iv) $f(-1,-1)$

(b) Does this function have a limit at $(0,0)$?

10. Consider the function

$$f(x,y) = \begin{cases} \dfrac{2x^2+3y^2}{x-y} & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

(a) Compute $f_x(0,0)$ directly from the limit definition of a partial derivative

$$f_x(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0+h,y_0) - f(x_0,y_0)}{h}$$

(b) Compute $f_y(0,0)$.

11. If $f(0,0) = 0$, $f_x(0,0) = 0$, $f_y(0,0) = 0$, and $f(x,y)$ is differentiable at $(0,0)$, does this imply that $f(x,y) = 0$ for some point $(x,y) \neq (0,0)$? Justify your result, or give a counterexample.

12. Consider the sphere $x^2+y^2+z^2 = 9$. Find the equation of the plane tangent to this sphere at

(a) $(3,0,0)$.

(b) $(2,2,1)$.

13. Suppose that $f(x,y) = e^{x-y}$ and $f(\ln 2, \ln 2) = 1$. Use the technique of linear approximation to estimate $f(\ln 2 + 0.1, \ln 2 + 0.04)$.

14. Let $g(u)$ be a differentiable function and let $f(x,y) = g(x^2+y^2)$.

(a) Show that $yf_x = xf_y$.

(b) Find the direction of maximal increase of f at $(1,1)$ in terms of g'.

15. Let f be a function of two variables with the following properties:

- $\dfrac{\partial f}{\partial x}$ is defined near $(0,0)$, continuous at $(0,0)$ and $\dfrac{\partial f}{\partial x}(0,0) = 0$
- $\dfrac{\partial f}{\partial y}$ is defined near $(0,0)$, continuous at $(0,0)$ and $\dfrac{\partial f}{\partial y}(0,0) = 0$

- $\dfrac{\partial^2 f}{\partial x \partial y}(0,0) = 1$

- $\dfrac{\partial^2 f}{\partial y \partial x}(0,0) = -1$

Answer true or false to the following, and give reasons for your answers.

(a) f is differentiable at $(0,0)$.

(b) There is a horizontal plane that is tangent to the graph of f at $(0,0)$.

(c) The functions $\dfrac{\partial^2 f}{\partial x \partial y}$ and $\dfrac{\partial^2 f}{\partial y \partial x}$ are both continuous at $(0,0)$.

(d) The linear approximation to $f(x,y)$ at $(0,0)$ is $L(x,y) = x - y$.

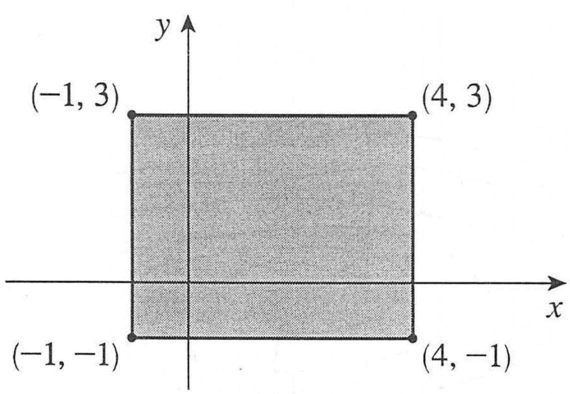
16. Suppose $\mathbf{u} = \langle 1, 0 \rangle$, $\mathbf{v} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$, $D_{\mathbf{u}}(f(a,b)) = 3$ and $D_{\mathbf{v}}(f(a,b)) = \sqrt{2}$.

(a) Find $\nabla f(a,b)$.

(b) What is the maximum possible value of $D_{\mathbf{w}}(f(a,b))$ for any $\mathbf{w}$?

(c) Find a unit vector $\mathbf{w} = \langle w_1, w_2 \rangle$ such that $D_{\mathbf{w}}(f(a,b)) = 0$.

17. Let $f(x,y) = e^{-(x^2+y^2)}$. Find the maximum and minimum values of f on the rectangle shown below. Justify your answer.



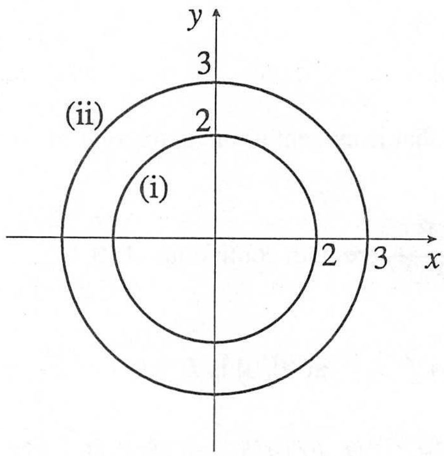
18. Which point on the surface $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = 1$, $x, y, z > 0$ is closest to the origin?

11 Sample Exam Solutions

1. $f(x,y) = \dfrac{1}{x^2+y^2+1}$

(a) (i) $f(x,y) = \frac{1}{5} \Rightarrow 5 = x^2+y^2+1$ (ii) $f(x,y) = \frac{1}{10} \Rightarrow x^2+y^2 = 9$

$\Rightarrow x^2+y^2 = 4$



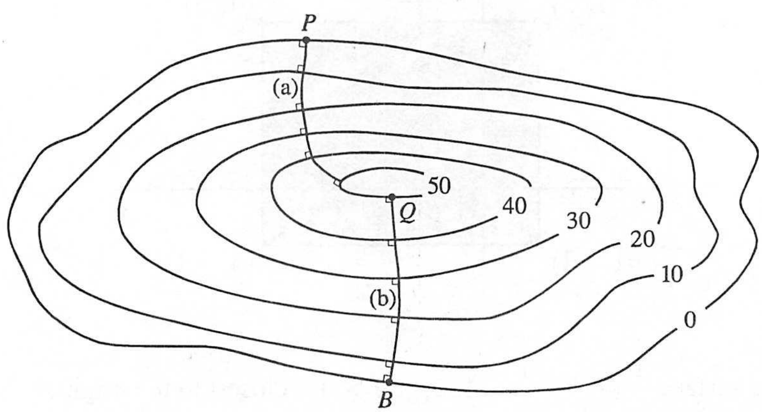
(b) $f(x,y) = 1$ consists of a single point $(0,0)$. Otherwise, $k < 1$ always gives the circle $x^2+y^2 = 1 - 1/k$.

(c) $\dfrac{1}{x^2+y^2+1} \leq 1$ for any point (x,y), since $x^2+y^2+1 \geq 1$.

2. Yes. On the left-hand side we get $(2x+y)\,2y\cos(xy^2)\sin(xy^2)$ and on the right-hand side we get $(2x+y)\,2y\cos(xy^2)\left|\sin xy^2\right|$, so these are equal for $\sin(xy^2) \geq 0$.

3. Yes. There are many examples of such functions. One which works for all x and y is $f(x,y) = e^x + e^{-y}$, which has $f_x = e^x$ and $f_y = -e^{-y}$. A good strategy is to write $f(x,y) = g(x) + h(y)$, where $g'(x) > 0$, $h'(y) < 0$.

4.



5. $f(x,y) = x^2 + xy + y^2$ on the disk $\{(x,y) \mid x^2 + y^2 \le 9\}$.
$\nabla f(x,y) = \langle 2x + y, 2y + x \rangle = \langle 0,0 \rangle \Leftrightarrow y = -2x$ and $x = -2y \Leftrightarrow (x,y) = (0,0)$. So the minimum value on the interior of the disk is $f(0,0) = 0$.
Using Lagrange multipliers for the boundary, we solve $\nabla f = \lambda \nabla g$ where $g(x,y) = x^2 + y^2 = 9$. So $2x + y = \lambda 2x \Rightarrow \lambda = 1 + y/(2x)$ and $2y + x = \lambda 2y \Rightarrow \lambda = 1 + x/2y \Rightarrow x^2 = y^2$. But $x^2 + y^2 = 9$, so $2x^2 = 9 \Rightarrow x = \pm\frac{3}{\sqrt{2}}$ and $y = \pm\frac{3}{\sqrt{2}}$. Thus the maximum value on the boundary is $f\left(\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right) = f\left(-\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right) = \frac{27}{2}$ and the minimum value on the boundary is $f\left(\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right) = f\left(-\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right) = \frac{9}{2}$.
The absolute minimum value is $f(0,0) = 0$ and the absolute maximum value is $f\left(\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right) = f\left(-\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right) = \frac{27}{2}$.

6. (a) Let $g(x,y,z) = \dfrac{x^2}{4} + \dfrac{y^2}{4} + 2z^2$, so $\nabla g = \left\langle \frac{x}{2}, \frac{y}{2}, 4z \right\rangle$ and $\nabla g\left(\sqrt{2}, \sqrt{2}, 0\right) = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right\rangle$ which is normal to the surface. So the tangent plane satisfies $\dfrac{x}{\sqrt{2}} + \dfrac{y}{\sqrt{2}} = k$ and goes through $\left(\sqrt{2}, \sqrt{2}, 0\right)$. Thus $k = 1$ and the tangent plane is $\dfrac{x}{\sqrt{2}} + \dfrac{y}{\sqrt{2}} = 1$.

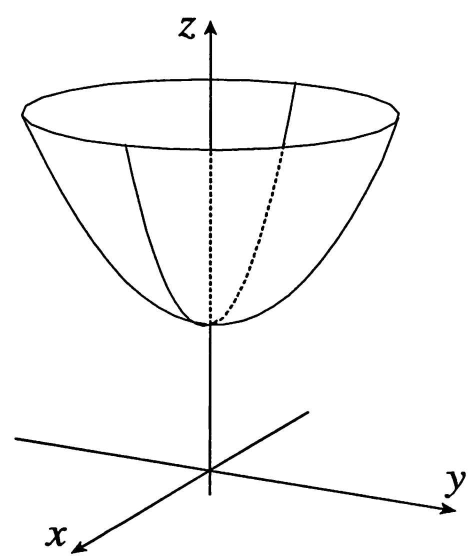
(b) Since this is a maximum value of z, the tangent plane is horizontal, that is, $z = \frac{1}{\sqrt{2}}$. Analytically, $\nabla g\left(0, 0, \frac{1}{\sqrt{2}}\right) = \langle 0, 0, 2\sqrt{2} \rangle$, so the tangent plane is $2\sqrt{2}z = 2$ or $z = \frac{1}{\sqrt{2}}$.

7. Ellipsoid for $k = -1$, single point $(0,0,0)$ for $k = 1$, no surface for $k = 2$.

8. (a) $F(x,y,z) = \dfrac{z}{1 + x^2 + y^2}$

(b) $z = 0$ is the only place where $F(x,y,z) = 0$. So there is no energy on the xy-plane.

(c) $F(x,y,z) = 1$ gives $1 = \dfrac{z}{1 + x^2 + y^2}$ or $z = 1 + x^2 + y^2$, a circular paraboloid.



9. $f(x,y) = \dfrac{x+y}{|x|+|y|}$

(a) (i) $f(1,1) = 1$

(ii) $f(1,-1) = 0$

(iii) $f(-1,1) = 0$

(iv) $f(-1,-1) = -1$

(b) No, the function does not have a limit at $(0,0)$, since if $y = -x$, then $f(x,-x) = 0$ and if $y = x$, $f(x,x) = \dfrac{x}{|x|} = \pm 1$.

10. $f(x,y) = \begin{cases} \dfrac{2x^2+3y^2}{x-y} & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$

(a) $f_x(0,0) = \lim\limits_{h\to 0} \dfrac{f(h,0) - f(0,0)}{h} = \lim\limits_{h\to 0} \dfrac{\frac{2h^2}{h} - 0}{h} = \lim\limits_{h\to 0} \dfrac{2h^2}{h^2} = \lim\limits_{h\to 0} 2 = 2$

(b) $f(0,y) = \dfrac{3y^2}{-y} = 3y = g(y)$. Then $f_y(0,0) = g'(0) = -3$.

11. A counterexample is $f(x,y) = x^2 + y^2$. For this function $f_x(0,0) = f_y(0,0) = 0$; $f(0,0) = 0$ and $f(x,y) \neq 0$ for $(x,y) \neq (0,0)$.

12. $x^2 + y^2 + z^2 = 9$

(a) The tangent plane at $(3,0,0)$ is $x = 3$.

(b) Let $g(x,y,z) = x^2 + y^2 + z^2$. Then $\nabla g = \langle 2x, 2y, 2z \rangle$ and $\nabla g(2,2,1) = \langle 4,4,2 \rangle$, which is normal to the surface. So the tangent plane is $4x + 4y + 2z = k$ and goes through $(2,2,1)$, so $k = 18$, and the tangent plane is $2x + 2y + z = 9$.

13. $f(x,y) = e^{x-y}$, $f_x(x,y) = e^{x-y}$, $f_y(x,y) = -e^{x-y}$.
$L(x,y) = f(\ln 2, \ln 2) + f_x(\ln 2, \ln 2)(x - \ln 2) + f_y(\ln 2, \ln 2)(y - \ln 2)$. So the linear approximation is $f(\ln 2 + 0.1, \ln 2 + 0.04) \approx L(\ln 2 + 0.1, \ln 2 + 0.04) = 1 + 1(0.1) - 1(0.04) = 1.06$.

14. (a) $yf_x = y\left[g'(x^2+y^2)\,2x\right] = 2xyg'(x^2+y^2)$, $xf_y = x\left[g'(x^2+y^2)\,2y\right] = 2xyg'(x^2+y^2)$.

(b) The maximal increase is in the direction of $\mathbf{u} = \langle 2g'(2), 2g'(2) \rangle$, which is the same as that of $\mathbf{w} = \langle 1,1 \rangle$.

15. (a) True; the partials are continuous.

(b) True (in fact the plane is $z = 0$).

(c) False; if they were continuous, then we would have $f_{xy}(0,0) = f_{yx}(0,0)$.

(d) False; the linear approximation is $L(x,y) = 0$.

16. $\mathbf{u} = \langle 1, 0 \rangle$, $\mathbf{v} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$, $D_{\mathbf{u}}(f(a,b)) = 3$ and $D_{\mathbf{v}}(f(a,b)) = \sqrt{2}$

(a) $\nabla f(a,b) = \langle f_1, f_2 \rangle$ and $\langle f_1, f_2 \rangle \cdot \mathbf{u} = 3 \;\Rightarrow\; f_1 = 3$. $\langle f_1, f_2 \rangle \cdot \mathbf{v} = \sqrt{2} \;\Rightarrow\; \dfrac{f_1}{\sqrt{2}} + \dfrac{f_2}{\sqrt{2}} = \sqrt{2}$

$\Rightarrow\; \dfrac{3}{\sqrt{2}} + \dfrac{f_2}{\sqrt{2}} = \sqrt{2} \;\Rightarrow\; 3 + f_2 = 2 \;\Rightarrow\; f_2 = -1$. So $\nabla f(a,b) = \langle 3, -1 \rangle$.

(b) $D_{\mathbf{w}}(f(a,b))$ is maximized when $\mathbf{w}$ is in the direction of $\langle 3, -1 \rangle$. So $\mathbf{w} = \left\langle \frac{3}{\sqrt{10}}, -\frac{1}{\sqrt{10}} \right\rangle$ and since $\mathbf{w} = \frac{4}{\sqrt{10}}\mathbf{u} - \frac{1}{\sqrt{5}}\mathbf{v}$, $D_{\mathbf{w}}(f(a,b)) = \frac{4}{\sqrt{10}} D_{\mathbf{u}}(f(a,b)) - \frac{1}{\sqrt{5}} D_{\mathbf{v}}(f(a,b)) = \frac{4}{\sqrt{10}} \cdot 3 - \frac{1}{\sqrt{5}} \cdot \sqrt{2} = \sqrt{10}$

(c) $D_{\mathbf{w}}(f(a,b)) = 0$ if $\mathbf{w} \cdot \langle 3, -1 \rangle = 0$, so $3w_1 - w_2 = 0$ and $w_1^2 + w_2^2 = 1$ gives $w_1^2 + 9w_1^2 = 1$, $w_1 = \frac{1}{\sqrt{10}}$ and $w_2 = \frac{3}{\sqrt{10}}$, so $\mathbf{w} = \left\langle \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \right\rangle$.

17. Since f is a function which is constant on circles $x^2 + y^2 = R$ and since f is decreasing as the radius of the circle increases, then the maximum is $f(0,0) = 1$ and the minimum is $f(4,3) = e^{-25}$.

18. Let $d^2 = x^2 + y^2 + z^2$ and minimize d^2 subject to the constraint $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = 1$, $x, y, z > 0$. The method of Lagrange multipliers gives the point $(3, 3, 3)$.

12 Multiple Integrals

Note: This material is very difficult for students who do not remember their single variable integration principles and techniques. One possible strategy is to give the students a review session or gateway exam on integration techniques before starting Chapter 12.

12.1 Double Integrals over Rectangles

Suggested Time and Emphasis

$\frac{1}{2}$–$\frac{3}{4}$ class   Essential Material

Transparencies Available

- Transparency 50 (Figures 4 and 5, page 841)
- Transparency 51 (Figures 7 and 8, pages 842–43)
- Transparency 52 (Figure 11, page 845)
- Transparency 53 (Figures 12 and 13, pages 845–46)

Points to Stress

1. The definition of the double integral.
2. The analogy between single and double integration.
3. Volume interpretations of double integrals.
4. Average value in two dimensions.

Text Discussion

- Compute $\sum_{i=1}^{2}\sum_{j=1}^{3} 2^i 3^j$.
- If we partition $[a, b]$ into m subintervals of equal length and $[c, d]$ into n subintervals of equal length, what is the value of ΔA for any subrectangle R_{ij}?
- For a positive function $f(x, y)$, what is a physical interpretation of the average value of f over a region R?

Materials for Lecture

- Briefly review a few basic properties of double sums, such as $\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij} = \sum_{i=1}^{m}\left(\sum_{j=1}^{n} a_{ij}\right)$ and $\sum_{i=1}^{m}\sum_{j=1}^{n} a_i b_j = \left(\sum_{i=1}^{m} a_i\right)\left(\sum_{j=1}^{n} b_j\right)$, if needed.

- Show how double integration extends single-variable integration, including Riemann sums.
- Do a problem on numerical estimation, such as estimating $\iint (x^2 + 2y^2)\,dA$ over $0 \le x \le 2$, $0 \le y \le 2$, using both the lower left corners and midpoints as sample points. Also approximate f_{ave} over R.
- Use a geometric argument to directly compute $\iint_R (3 + 4x)\,dA$ over $R = [0,1] \times [0,1]$.
- Discuss what happens when $f(x,y)$ takes negative values over the region of integration. Start with an odd function such as $x^3 + y^5$ being integrated over $R = [-1,1] \times [-1,1]$, and then discuss what happens when the function goes negative, but is not odd (for example $z = 2 - 3x$ on $R = [0,1] \times [0,1]$). Illustrate numerically. (This can also be done using the group work "An Odd Function".)
- Briefly discuss average value. Point out that if f is continuous, then for some point (x_0, y_0) in the region of integration R, $f_{\text{ave}} = f(x_0, y_0)$, as in the single-variable case. So $\iint_R f(x,y)\,dA = A(R) \cdot f(x_0, y_0)$, as shown in Figure 11 (page 845).

Workshop/Discussion

- The following example can be used to help solidify the idea of approximating an area. Consider a square pyramid with vertices at $(1,1,0)$, $(1,-1,0)$, $(-1,1,0)$, $(-1,-1,0)$ and $(0,0,1)$. Derive an equation for the surface of the pyramid, using functions such as $z = 1 - \max(|x|, |y|)$, and then the equations of the planes containing each of the five faces ($z = 0$, $y + z = 1$, and so on). Approximate the volume using the Midpoint Rule for the following equal subdivisions. Note that m is the number of equal subdivisions in the x-direction and n is the number of equal subdivisions in the y-direction.

  1. $m = n = 2$ (Approximation is 2)

  2. $m = n = 3$ (Approximation is $\frac{44}{27}$)

  3. $m = n = 4$ (Approximation is $\frac{3}{2}$)

  4. $m = n = 5$ (Approximation is $\frac{36}{25}$)

  Compare your answers with the actual volume $\frac{4}{3}$ computed using the formula $V = \frac{1}{3}Bh$.
- Consider $\iint \sqrt{4 - y^2}\,dA$, with $R = [0,3] \times [-2,2]$. Use a geometric argument to compute the actual volume after approximating as above with $m = n = 3$ and $m = n = 4$. Show that the average value is $\frac{6\pi}{12} = \frac{\pi}{2}$, and that the point $\left(0, \sqrt{4 - \frac{1}{4}\pi^2}\right) \approx (0, 1.24)$ in R satisfies $f\left(0, \sqrt{4 - \frac{1}{4}\pi^2}\right) = \frac{\pi}{2}$.

Group Work 1: Back to the Park

This group work is similar to Example 4 (page 845) and uses the Midpoint Rule.

Group Work 2: An Odd Function

The goal of the exercise is to estimate $\iint_{[-2,2] \times [-2,2]} (x^3 + y^5)\,dA$ numerically. There are three different problem sheets, each one suggesting a different strategy to obtain sample points for the estimation. After all the students are finished, have them compare their results. The exact value of the integral is zero, by symmetry.

Group Work 3: Justifying Properties of Double Integrals

Put the students into groups. Have them read page 847 of the text carefully, and then have some groups try to justify Equation 7, some Equation 8, and some Equation 9 for nonnegative functions f and g. They don't have to do a formal proof, but they should be able to justify these equations convincingly, either using sums or geometrical reasoning.

Group Work 4: Several Ways to Compute Double Integrals

Homework Problems

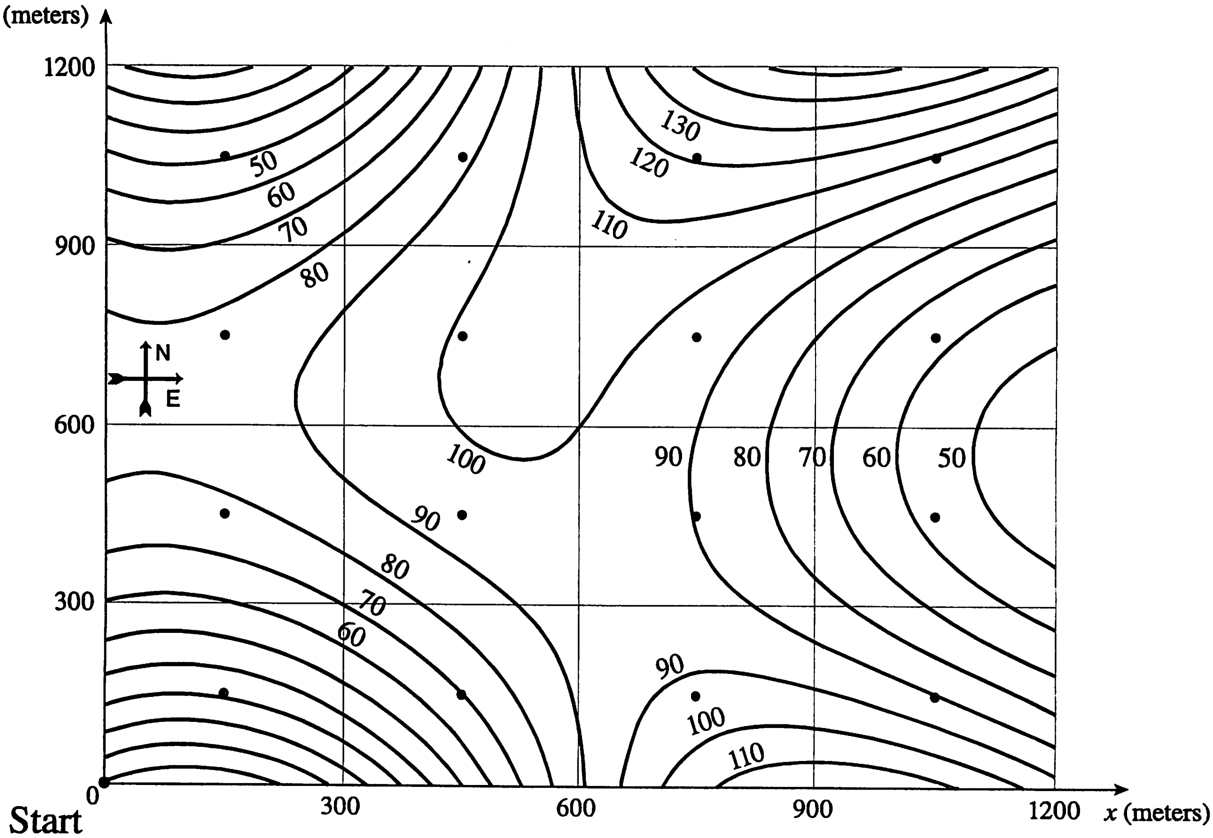
Core Exercises: 1, 3, 5, 6, 7, 10

Sample Assignment: 1, 3, 5, 6, 7, 10, 12, 17

Exercise	C	A	N	G	V
1			×		
3			×		
5			×		
6			×		
7			×		
10			×	×	
12			×		×
17		×			

Group Work 1, Section 12.1
Back to the Park

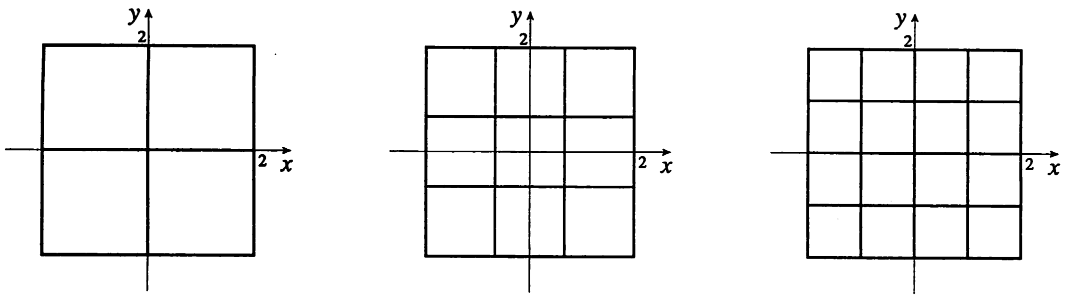
The following is a map with curves of the same elevation of a region in Orangerock National Park:



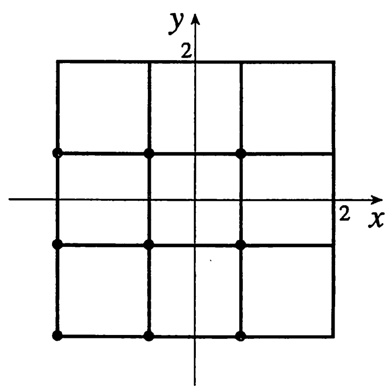
Estimate (numerically) the average elevation over this region using the Midpoint Rule.

Group Work 2, Section 12.1
An Odd Function (Version 1)

In this exercise, we are going to try to approximate the double integral $\iint_{[-2,2]\times[-2,2]} (x^3 + y^5)\, dA$. We start by partitioning the region $[-2,2]\times[-2,2]$ into four smaller regions, then nine, then sixteen, like this:



Now we approximate the double integral as discussed in the text, picking one point in every smaller region. To make things simple, just choose the lower left corner of every small region, like so:



Approximation for four regions: ________________

Approximation for nine regions: ________________

Approximation for sixteen regions: ________________

When you are finished, try again using a point of your choice in each region.

Using Arbitrarily Chosen Points

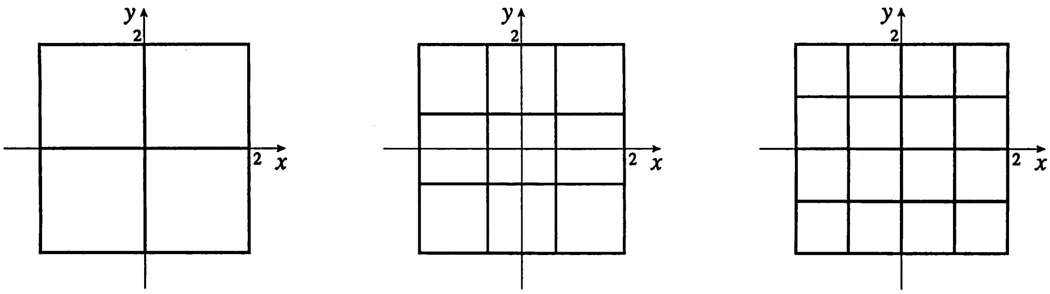
Approximation for four regions: ________________

Approximation for nine regions: ________________

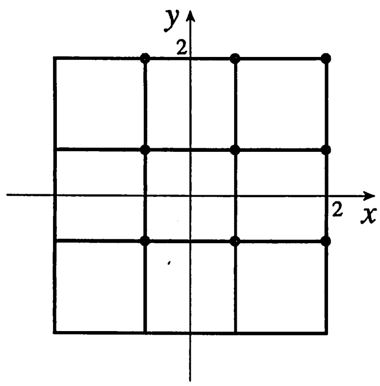
Approximation for sixteen regions: ________________

Group Work 2, Section 12.1
An Odd Function (Version 2)

In this exercise, we are going to try to approximate the double integral $\iint_{[-2,2]\times[-2,2]} \left(x^3 + y^5\right) dA$. We start by partitioning the region $[-2,2] \times [-2,2]$ into four smaller regions, then nine, then sixteen, like this:



Now we approximate the double integral as discussed in the text, picking one point in every smaller region. To make things simple, just choose the upper right corner of every small region, like so:



Approximation for four regions: ______________________

Approximation for nine regions: ______________________

Approximation for sixteen regions: ______________________

When you are finished, try again using a point of your choice in each region.

Using Arbitrarily Chosen Points

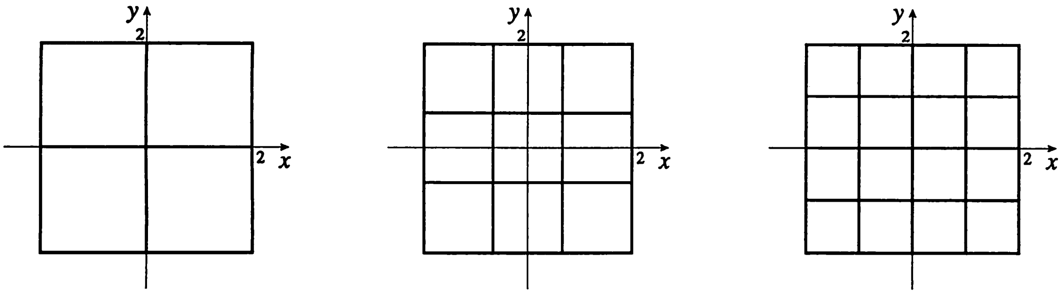
Approximation for four regions: ______________________

Approximation for nine regions: ______________________

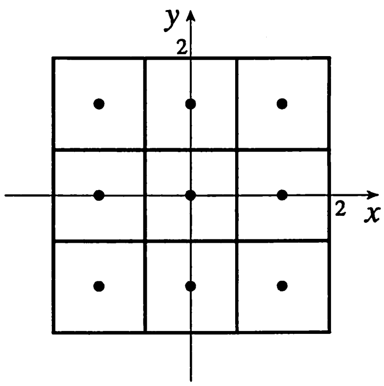
Approximation for sixteen regions: ______________________

Group Work 2, Section 12.1
An Odd Function (Version 3)

In this exercise, we are going to try to approximate the double integral $\iint_{[-2,2]\times[-2,2]} (x^3 + y^5)\, dA$. We start by partitioning the region $[-2,2] \times [-2,2]$ into four smaller regions, then nine, then sixteen, like this:



Now we approximate the double integral as discussed in the text, picking one point in every smaller region. To make things simple, just choose the midpoint of every small region, like so:



Approximation for four regions: ______________

Approximation for nine regions: ______________

Approximation for sixteen regions: ______________

When you are finished, try again using a point of your choice in each region.

Using Arbitrarily Chosen Points

Approximation for four regions: ______________

Approximation for nine regions: ______________

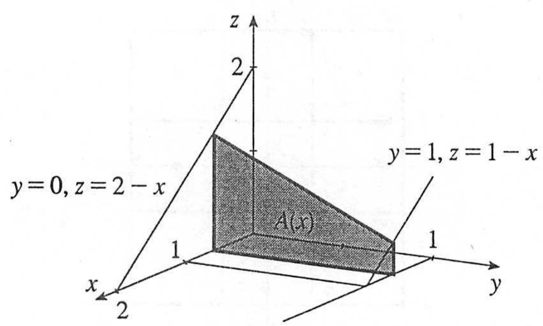
Approximation for sixteen regions: ______________

Group Work 4, Section 12.1
Several Ways to Compute Double Integrals

Consider the double integral $\iint (2 - x - y)\, dA$, where $R = [0, 1] \times [0, 1]$.

1. Estimate the value of the double integral, first using two equal subdivisions in each direction, then three.

2. Fix x such that $0 \le x \le 1$. What is the area $A(x)$ of the slice shown below?



3. Find the exact volume of the solid with cross-sectional area $A(x)$ using single variable calculus.

12.2 Iterated Integrals

Suggested Time and Emphasis

$\frac{3}{4}$–1 class   Essential Material

Points to Stress

1. The meaning of $\int_a^b \int_c^d f(x,y)\,dy\,dx$ for a positive function $f(x,y)$ over a rectangle $[a,b] \times [c,d]$.

2. The geometric meaning of Fubini's Theorem: slicing the area in two different ways.

3. The statement of Fubini's Theorem and how it makes computations easier.

Text Discussion

- Consider Figures 1 and 2 in the text. Why is $\int_a^b A(x)\,dx = \int_c^d A(y)\,dy$?

Materials for Lecture

- Use an alternate approach to give an intuitive idea of why Fubini's Theorem is true. Using equal intervals of length Δx and Δy in each direction and choosing the lower left corner in each rectangle, we can write the double sum in Definition 12.1.5 (page 841) as the iterated sum

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f\left(x_{ij}^*, y_{ij}^*\right)\Delta x\,\Delta y = \sum_{i=1}^{m}\left(\sum_{j=1}^{n} f\left(x_{ij}^*, y_{ij}^*\right)\Delta x\right)\Delta y$$

  which in the limit gives an iterated integral. Go through some examples such as $\iint_{[-1,3]\times[-1,3]} xy\,dA$ and $\iint_{[0,1]\times[0,1]} (2-x-y)\,dA$ to demonstrate this approach.

- Revisit the example $\iint_{[-2,2]\times[-2,2]} \left(x^3+y^5\right)dA$ using integrated integrals, to illustrate the power of the technique of iteration.

- Illustrate that for rectangles, $\iint_R f(x)\,g(y)\,dA = \int_a^b f(x)\,dx \int_c^d g(y)\,dy$ using $\dfrac{x}{y}e^{x^2}$ over $[0,1]\times[1,2]$. Stress that this does *not* mean that $\iint_R f(x,y)\,g(x,y)\,dA = \iint_R f(x,y)\,dA \iint_R g(x,y)\,dA$.

Workshop/Discussion

- Remind the students how, in single-variable calculus, volumes were found by adding up cross-sectional areas. Take the half-cylinder $f(x,y) = \sqrt{1-y^2}$, $0 \le x \le 2$, and find its volume, first by the "old" method, then by expressing it as a double integral. Show how the two techniques are, in essence, the same.

- Have the students work several examples, such as $\iint_{[0,1]\times[0,1]} y\sqrt{1-x^2}\sin\left(2\pi y^2\right)dA$, which can be computed as $\int_0^1 f(x)\,dx \int_0^1 g(y)\,dy$.

- Find the volume of the solids described by $\iint_{[-\sqrt{2},\sqrt{2}]\times[-2,3]} \left(2-x^2\right)dA$ and $\iint_{[-1,1]\times[-1,1]} \left(1+x^2+y^2\right)dA$.

Group Work 1: Regional Differences

If the students get stuck on this one, give them the hint that Problem 1(b) can be done by finding the double integral over the square, and then using the symmetry of the function to compute the area over R. The regions in the remaining problems can be broken into rectangles.

Group Work 2: Practice with Double Integrals

It is a good idea to give the students some practice with straightforward computations of the type included in Problem 1. It is advised to put the students in pairs or have them work individually, as opposed to putting them in larger groups. The students are not to actually compute the integral in Problem 2; rather, they should recognize that each slice integrates to zero. Think about what happens when integrating with respect to x first.

Group Work 3: The Shape of the Solid

In Problem 3, the students could first change the order of integration to more easily recognize the "pup tent" shape of the resulting solid.

Homework Problems

Core Exercises: 2, 5, 10, 16, 17, 25

Sample Assignment: 2, 5, 7, 10, 15, 16, 17, 20, 25, 31

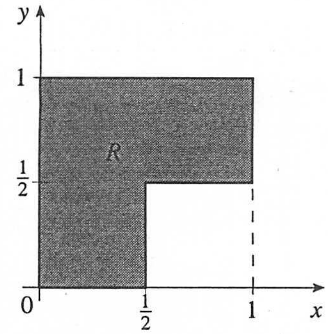
Note:
- Exercise 31 requires a CAS.
- Problem 5 from Focus on Problem Solving (page 914) can be assigned as an optional project.

Exercise	C	A	N	G	V
2		×			
3–10		×			
15		×			
16		×			
17					×
20		×			×
25		×			×
31	×				

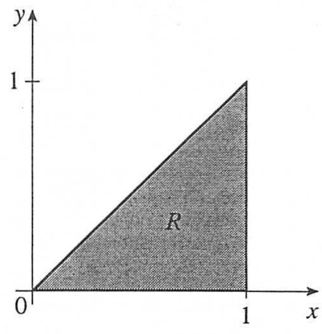
Group Work 1, Section 12.2
Regional Differences

1. Calculate the double integral $\iint_R (x + y)\, dA$ for the following regions R:

(a)

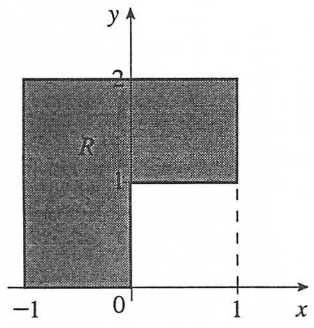


(b)

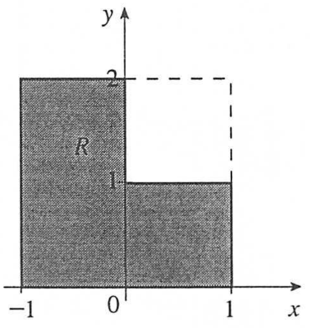


2. Calculate the double integral $\iint_R (xy - y^3)\, dA$ for the following regions R.

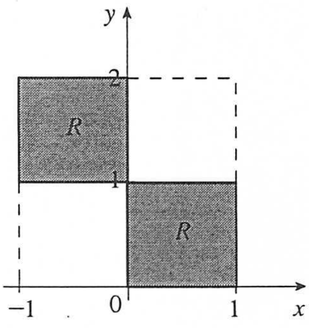
(a)



(b)



(c)



Group Work 2, Section 12.2
Practice with Double Integrals

Compute the following double integrals:

1. (a) $\displaystyle\iint_{[1,2]\times[0,1]} x\sqrt{1+y+x^2}\,dA$

(b) $\displaystyle\iint_{[0,1]\times[1,2]} \frac{x}{x+y}\,dx\,dy$

(c) $\displaystyle\iint_{[0,1]\times[1,2]} ye^{xy}\,dx\,dy$

2. Is the statement $\iint_{[0,1]\times[0,1]} \cos\left(2\pi\left(y^2+x\right)\right)dA = 0$ true or false?

Group Work 3, Section 12.2
The Shape of the Solid

For each of the following integrals, describe the shape of the solid whose volume is given by the integral, then compute the volume.

1. $\iint_{[0,1]\times[0,1]} (3 - 2x - y)\,dA$

2. $\int_{-3}^{3}\int_{-2}^{2} \sqrt{4 - y^2}\,dy\,dx$

3. $\int_{-1}^{1}\int_{-2}^{3} (1 - |x|)\,dy\,dx$

12.3 Double Integrals over General Regions

Suggested Time and Emphasis

1 class   Essential Material

Points to Stress

1. The geometric interpretation of $\int_a^b \int_{f(y)}^{g(y)} dx\, dy$ and $\int_c^d \int_{h(x)}^{k(x)} dy\, dx$.

2. Setting up the limits of double integrals, given a region over which to integrate.

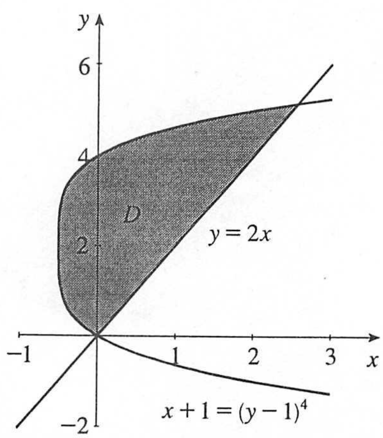
3. Changing the order of integration.

Text Discussion

- Why is the type I region illustrated in Figure 8 (page 856) not considered type II?
- Sketch a region that is type II and not type I, and then sketch one that is both type II and type I.
- Is it true that $\int_0^1 \int_x^1 f(x,y)\, dy\, dx = \int_0^1 \int_y^1 f(x,y)\, dx\, dy$?

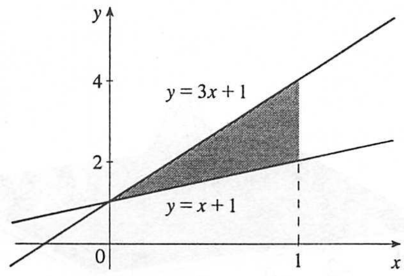
Materials for Lecture

- Clarify why we bother with the fuss of defining $F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in D \\ 0 & \text{if } (x,y) \notin D \end{cases}$ instead of "just integrating over D".
- Show how the area under a curve in the xy-plane can be thought of as the double integral $\int_a^b \int_0^{f(x)} dy\, dx$.
- Let D is the region shown below. Set up $\iint_D f(x,y)\, dA$ both as a type I integral $\left[\iint_D f(x,y)\, dx\, dy\right]$ and a type II integral $\left[\iint_D f(x,y)\, dy\, dx\right]$.



- Discuss Example 4, emphasizing that $z = 2 - x - 2y$ is the *height* of the solid being described. Show how to integrate with respect to x first, noting that the answer is still $\frac{1}{3}$.

- Show that the order of integration matters when computing $\iint e^{x^2}\,dA$, where D is the region shown below.



- Show how to change order of integration in $\int_0^1 \int_{x^4}^{x^{1/3}} f(x,y)\,dy\,dx$.

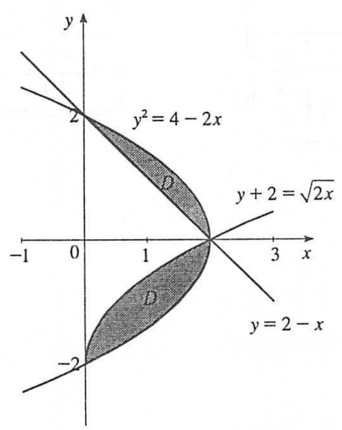
Workshop/Discussion

- Point out that $\iint_D 1\,dA$ gives the area of D, and that $\int_a^b \int_{f(x)}^{g(x)} 1\,dy\,dx$ gives the usual formula for the area between curves.

- Evaluate $\iint_D yx^2\,dA$, where D is the unit circle, both as a type I integral and as a type II integral.

- Evaluate $\iint_D h\,dA$, where D is a circle of radius r and h is constant. Show how this gives the general formula for the volume of a cylinder. Ask the students to evaluate $\iint_D h\,dA$, where D is the parallelogram with vertices $(1,1)$, $(2,3)$, $(5,1)$, and $(6,3)$. Have them interpret their answer in terms of volume.

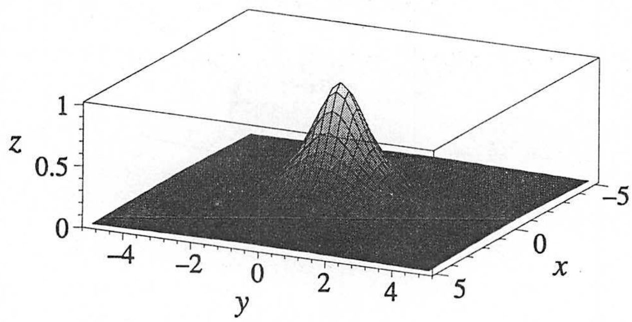
- Evaluate $\iint_D x\,dA$, where D is the region shown below.



- Change order of integration for $\int_0^1 \int_{(2/\pi)\arcsin y}^{y^{1/3}} f(x,y)\,dx\,dy$.

- Find a good upper bound for $\displaystyle\iint_D \frac{1}{x^2+y^2+1}\,dA$ where D is $[-5,5]\times[-5,5]$. Perhaps show that $\frac{100}{51}$

is a lower bound, since $\dfrac{1}{x^2+y^2+1} \geq \dfrac{1}{51}$ on D. The graph of $\dfrac{1}{x^2+y^2+1}$ is given below.



Group Work 1: Type I or Type II?

Before handing out this activity, remind the students of the definitions of type I and type II regions given in the text.

Group Work 2: Fun with Double Integrals

Group Work 3: Writing a Quarter-Annulus as One or More Iterated Integrals

Problem 2 can be done using Problem 1 and symmetry.

Group Work 4: Bounding on a Disk

Homework Problems

Core Exercises: 2, 8, 19, 22, 30, 35

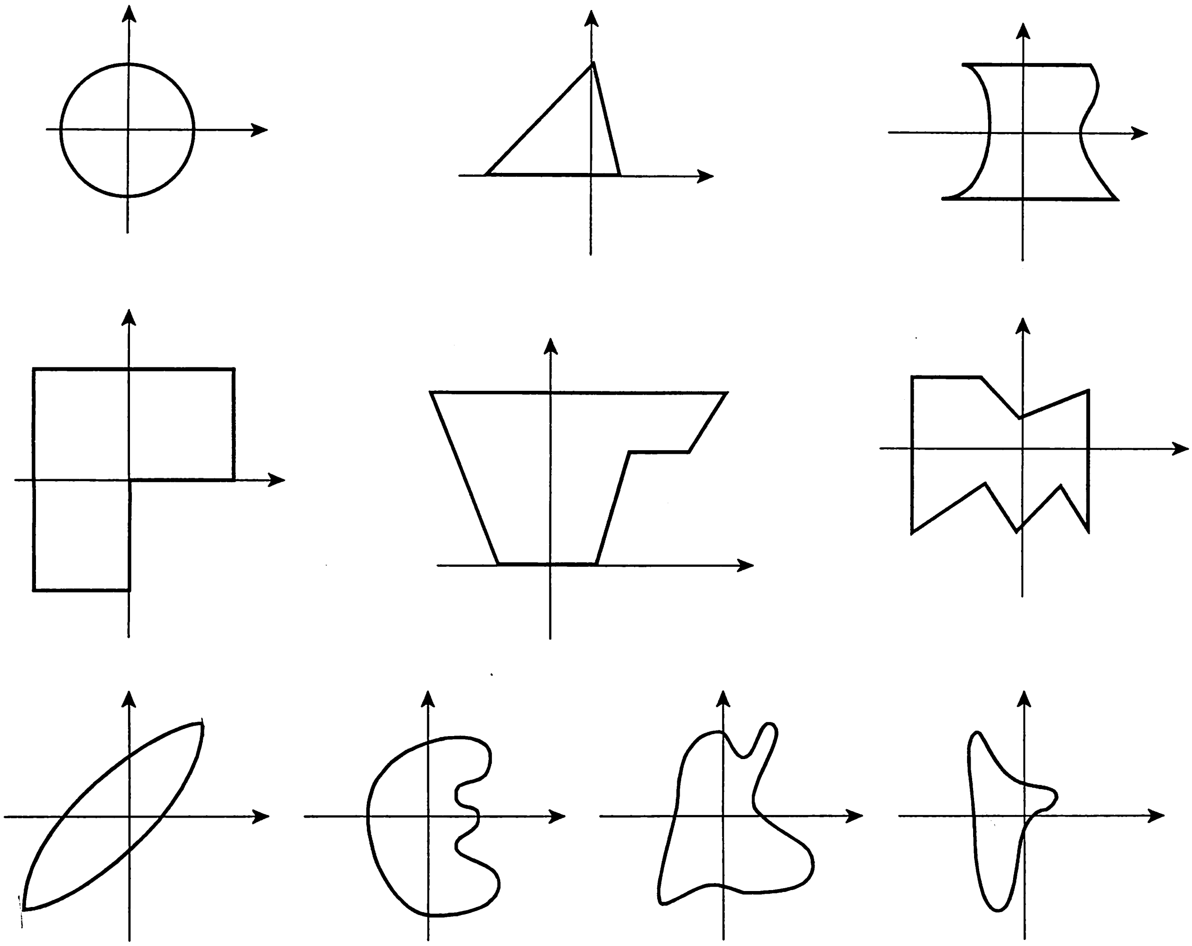
Sample Assignment: 2, 5, 8, 11, 15, 19, 22, 24, 30, 33, 35, 38, 44, 46, 49

Exercise	C	A	N	G	V
2		×			
5		×			
7–16		×			
17–24		×			×
30		×			×
33		×			×

Exercise	C	A	N	G	V
35		×			
38		×			
44			×		
46		×			×
49		×			×

Group Work 1, Section 12.3
Type I or Type II?

Classify each of the following regions as type I, type II, both, or neither.

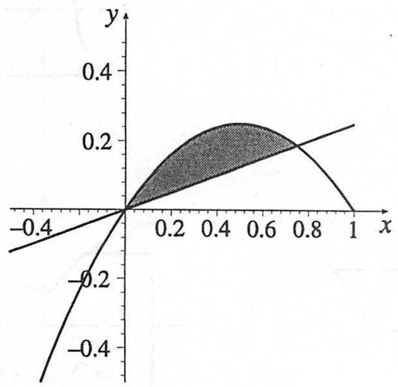


Group Work 2, Section 12.3

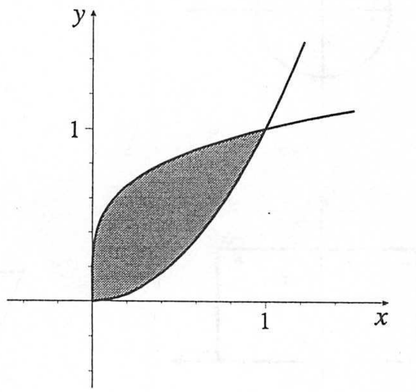
Fun with Double Integration

1. Write double integrals that represent the following areas.

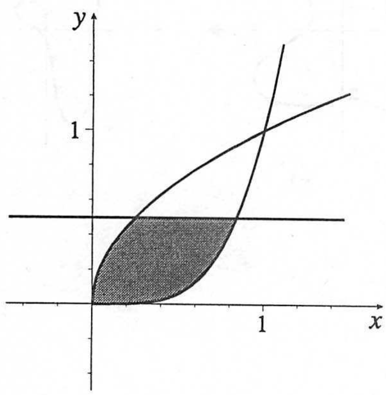
(a) The area enclosed by the curve $y = x - x^2$ and the line $y = \frac{x}{4}$



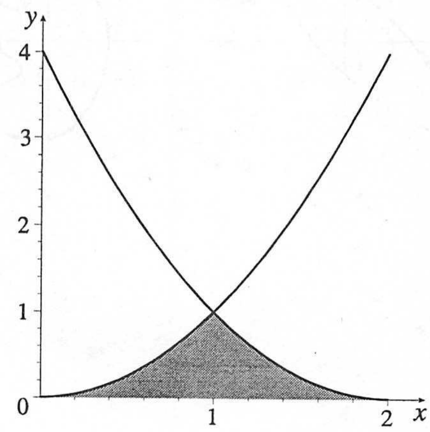
(b) The area enclosed by the curves $y = \sqrt[4]{x}$ and $\sqrt{y} = x$



(c) The area enclosed by the curves $y = \sqrt{x}$ and $\sqrt[4]{y} = x$, and the line $y = \frac{1}{2}$



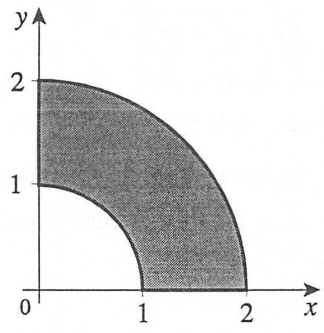
(d) The area enclosed by the curves $y = x^2$ and $y = (x-2)^2$, and the line $y = 0$



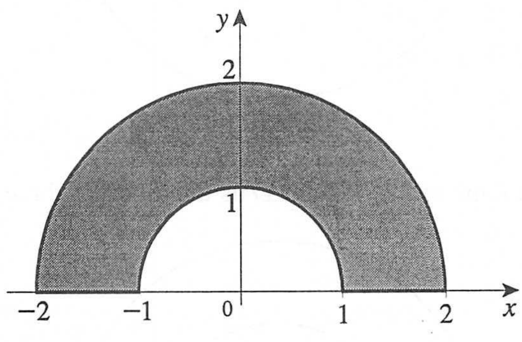
2. What solid region of $\mathbb{R}^3$ do you think is represented by $\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1 - x^2 - y^2}\, dy\, dx$?

Group Work 3, Section 12.3
Writing a Quarter-Annulus as One or More Iterated Integrals

1. Express $\iint_D \sqrt{x^2+y^2}\, dA$ using iterated integrals, where D is given by the region sketched below.



2. Express $\iint_D \sqrt{x^2+y^2}\, dA$ using iterated integrals, where D is the region sketched below.



Group Work 4, Section 12.3
Bounding on a Disk

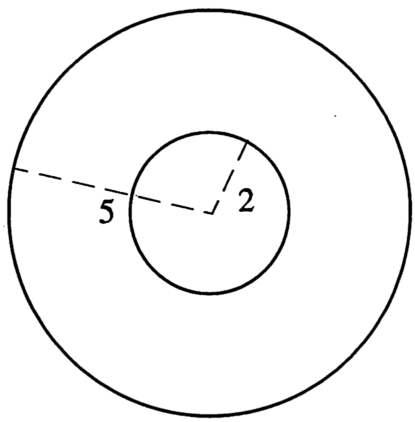
One integral that is very important in Nentebular science is the Brak integral:

$$\iint_D \frac{1}{x^2+y^2+1}\,dA$$

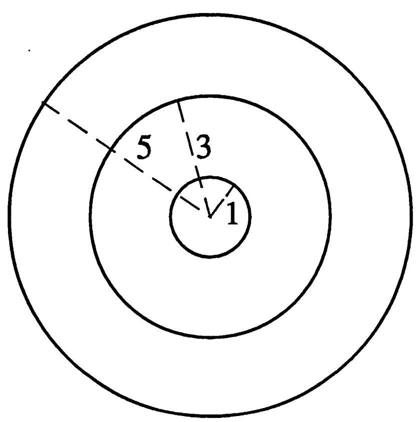
For these sorts of integrals the domain D is usually a disk. In this exercise, we are going to find upper and lower bounds for this integral, where D is the disk $0 \le x^2+y^2 \le 25$.

1. It is possible to get some crude upper and lower bounds for this integral over D without any significant calculations? Find upper and lower bounds for this integral (perhaps crude ones) and explain how you know for sure that they are true bounds.

2. One can get a better estimate by splitting up the domain as shown in the graph below, and bounding the integral over the inside disk and then over the outside ring. Using this method, what are the best bounds you can come up with?



3. Now refine your bounds by looking at the domain D as the union of three domains as shown.



12.4 Double Integrals in Polar Coordinates

Suggested Time and Emphasis

1 class   Essential Material

Note: If polar coordinates have not yet been covered, the students should read Appendix H.1, pages A58–A65.

Points to Stress

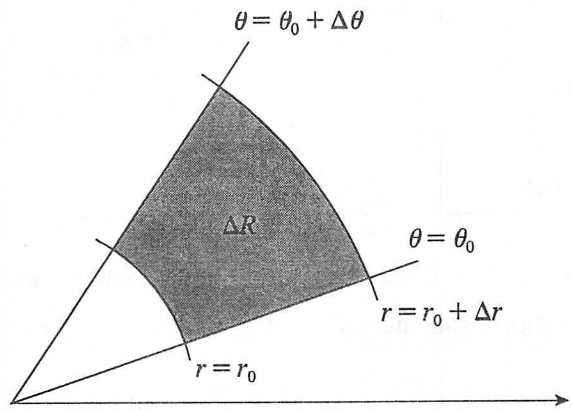
1. The definition of a polar rectangle: what it looks like, and its differential area $r\,dr\,d\theta$
2. The idea that some integrals are simpler to compute in polar coordinates
3. Integration over general polar regions

Text Discussion

- Is the area of a polar rectangle $R = \{(r,\theta) \mid a \le r \le b, \alpha \le \theta \le \beta\}$ equal to $(b-a)(\beta-\alpha)$?
- In polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$. So in the formula $\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$, where does that extra r come from?

Materials for Lecture

- Start by reminding the students of polar coordinates, and ask them what they think the polar area formula will be: what will replace $dx\,dy$?
- Draw a large picture of a polar rectangle, and emphasize that it is the region between the gridlines $r = r_0$, $r = r_0 + \Delta r$, $\theta = \theta_0$, and $\theta = \theta_0 + \Delta\theta$.



The following method can be used to show that Area$(\Delta R) \approx r\,\Delta r\,\Delta\theta$ if Δr and $\Delta\theta$ are small:

1. Using the formula for the area of a circular sector, we obtain

$$\begin{aligned}\text{Area}(\Delta R) &= \tfrac{1}{2}(r+\Delta r)^2\,\Delta\theta - \tfrac{1}{2}(r)^2\,\Delta\theta \\ &= r\,\Delta r\,\Delta\theta + \tfrac{1}{2}(\Delta r)^2\,\Delta\theta\end{aligned}$$

2. We now take the limit as $\Delta r, \Delta\theta \to 0$, of $\dfrac{\text{Area}(\Delta R)}{r\,\Delta r\,\Delta\theta}$ and show that this limit is equal to 1.

$$\lim_{\Delta r\to 0,\,\Delta\theta\to 0} \frac{\text{Area}(\Delta R)}{r\,\Delta r\,\Delta\theta} = \lim_{\Delta r\to 0,\,\Delta\theta\to 0} \frac{r\,\Delta r\,\Delta\theta + \frac{1}{2}(\Delta r)^2\,\Delta\theta}{r\,\Delta r\,\Delta\theta}$$

$$= \lim_{\Delta r\to 0,\,\Delta\theta\to 0} 1 + \frac{\Delta r}{2r} = 1$$

3. The result follows, since when Δr and $\Delta\theta$ are small,

$$\frac{\text{Area}(\Delta R)}{r\,\Delta r\,\Delta\theta} \approx 1$$

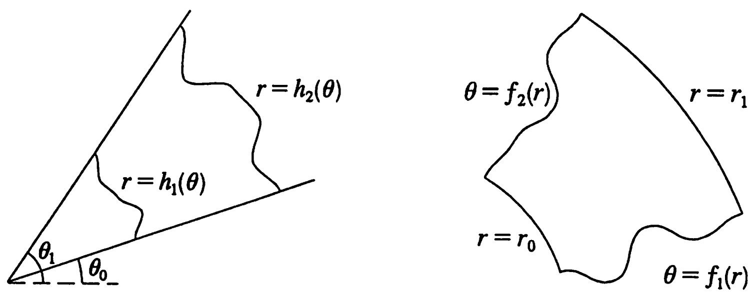
or

$$\text{Area}(\Delta R) \approx r\,\Delta r\,\Delta\theta$$

- If the quarter-annulus group activity was done in the previous section, point out that the integration problem posed can now be solved using one simple integral with the methods from this section.

- Indicate to the students that polar coordinates are most useful when one has an obvious center of symmetry for the region R in the xy-plane.

- Show how to set up the two general types of polar regions. (The second type occurs less frequently, and may be omitted.)



Workshop/Discussion

- Point out that polar areas can be found by setting up (double) polar integrals of the function $f(r\cos\theta, r\sin\theta) = 1$ and compare this method to using the formula $A = \int_a^b \frac{1}{2}r^2\,d\theta$. Calculate

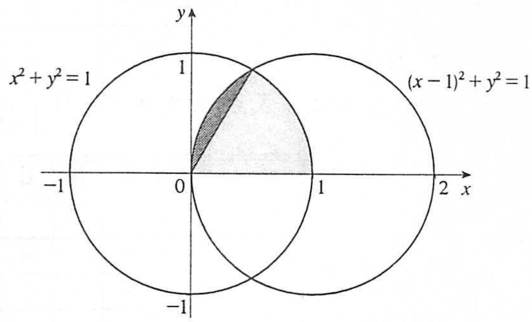
  1. The area between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 2$ in the second quadrant.

  2. The area inside the spiral $r = \theta$ where $0 \le \theta \le \pi$.

  3. $A = \left\{(x, y) \mid -1 \le x \le 1, 0 \le y \le \sqrt{1 - x^2}\right\}$.

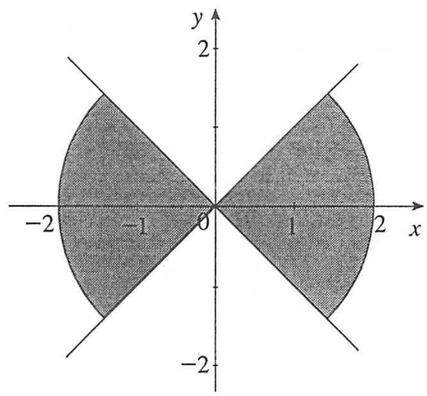
  4. The area inside the first loop of the curve $r = 2\sin\theta\cos\theta$.

- A more difficult problem is to find the area inside the circles $x^2+y^2 = 1$ and $(x-1)^2+y^2 = 1$ (or $r = 2\cos\theta$). This involves finding the point of intersection [which is $\left(1, \frac{\pi}{3}\right)$ in polar coordinates] and then breaking the double integral for the portion in the first quadrant into 2 pieces, one of which is just the area of the sector of angle $\frac{\pi}{3}$ of the unit disk.



- Give some different uses of polar integrals:

1. Compute $\iint_R \left(x^2+y^2\right)^2 \, dA$, where R is the region enclosed by $x^2+y^2 = 4$ between the lines $y = x$ and $y = -x$.



2. Compute the volume between the cone $z^2 = x^2+y^2$ and the paraboloid $z = 4 - x^2 - y^2$.

Group Work 1: The Polar Area Formula

Group Work 2: Fun with Polar Area

Many students will write $\int_0^{3\pi} \int_0^{\theta} r \, dr \, d\theta$ for Problem 1(c). Try to get them to see for themselves that they must now subtract the area of the region that is counted twice in this expression. Note that in Problem 2 they are computing the volume of a hemisphere.

Group Work 3: Fun with Polar Volume

If the students get stuck on Problem 2, point out that the intersection can be shown to be $x^2+y^2 = \frac{3}{4}$, $z = \frac{1}{2}$, and hence the integral is over the region $x^2+y^2 \leq \frac{3}{4}$. Problem 3 is challenging.

Homework Problems

Core Exercises: 3, 4, 8, 17, 23, 29

Sample Assignment: 1, 3, 4, 6, 8, 9, 17, 21, 23, 29, 30, 32

Exercise	C	A	N	G	V
1–6		×			×
8		×		×	
9		×			
17		×			
21		×			

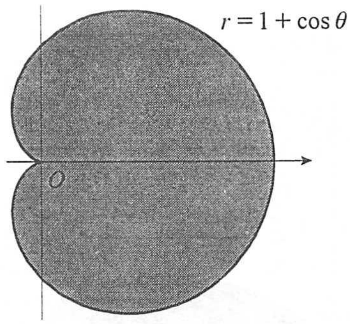
Exercise	C	A	N	G	V
23		×			
29		×			
30		×			
32		×			

Group Work 1, Section 12.4
The Polar Area Formula

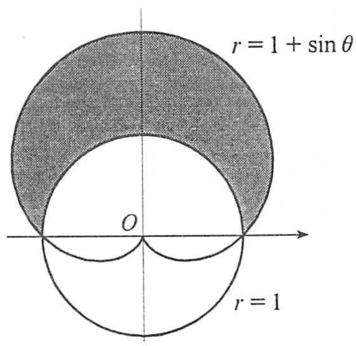
The following formula, used for finding the area of a polar region described by the polar curve $r = f(\theta)$, $a \le \theta \le b$, can be found in Appendix H.2, page A69:

$$A = \int_a^b \frac{1}{2}[f(\theta)]^2 \, d\theta$$

1. Use this formula to compute the area inside the cardioid $r = 1 + \cos\theta$.



2. Use the formula to find the area inside the cardioid $r = 1 + \sin\theta$ and outside the unit circle $r = 1$.

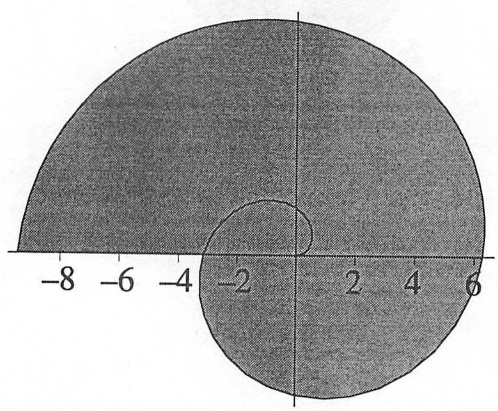


Group Work 2, Section 12.4
Fun with Polar Area

1. Sketch the following polar regions, and find their area:

(a) The region inside $r = 3\cos\theta$ and outside $\left(x - \frac{3}{2}\right)^2 + y^2 = \frac{9}{4}$

(b) The region inside the curve $r = \theta, 0 \leq \theta \leq 3\pi$



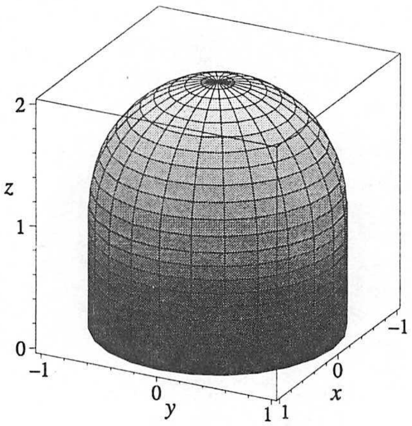
(c) The region between $\theta = \sqrt{2\pi r}$ and $\theta = r^2$ with $0 \leq r \leq \sqrt{2\pi}$

2. Compute $\iint_R f(x, y)\, dA$ if f is the positive-valued function given implicitly by $x^2 + y^2 + z^2 = 4$ and R is the region inside the circle $x^2 + y^2 = 4$.

3. Rewrite $\int_{-2}^{2} \int_0^{\sqrt{4-x^2}} e^{-x^2-y^2}\, dy\, dx$ as a polar integral and evaluate it.

Group Work 3, Section 12.4
Fun with Polar Volume

1. Find the volume of the region bounded above by the upper hemisphere of the sphere $x^2+y^2+(z-1)^2=1$ and bounded below by the xy-plane.



2. Find the volume of the region bounded above by the sphere $x^2+y^2+z^2=1$ and below by the sphere $x^2+y^2+(z-1)^2=1$.

3. Find the volume of the ellipsoid $\dfrac{x^2}{9}+\dfrac{y^2}{9}+z^2=1$.

12.5 Applications of Double Integrals

Suggested Time and Emphasis

$\frac{3}{4}$–1 class   Optional material

Points to Stress

We recommend stressing only one of the following topics:

1. Density, mass, and centers of mass (for an engineering- or physics-oriented course)

2. Probability and expected values (for a course oriented toward biology or the social sciences)

Text Discussion

- If a lamina has a uniform density, and an axis of symmetry, what information do we then have about the location of the center of mass?

- What is a logical reason that the total area under a joint density curve should be equal to one?

Materials for Lecture

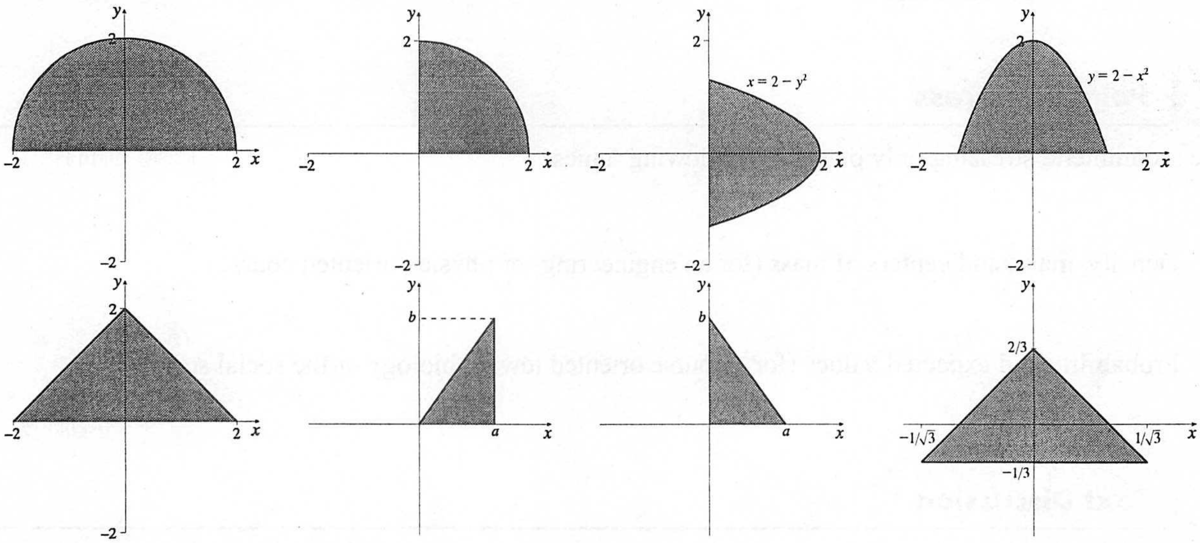
- Describe the general ideas behind continuous density functions, computations of mass, and centers of mass.

- Do one interesting mass problem. A good exercise is the mass over the unit disk if $\rho(x, y) = |x| + |y|$. This reduces to $4\int_0^1 \int_0^{\sqrt{1-x^2}} (x + y)\,dy\,dx$ which, surprisingly, equals $\frac{8}{3}$.

- Describe the general idea of the joint density function of two variables. Similarly, describe the concept of expected value. If time permits, show that $f(x, y) = \dfrac{1}{2\pi}\dfrac{1}{(1 + x^2 + y^2)^{3/2}}$ describes a joint density function.

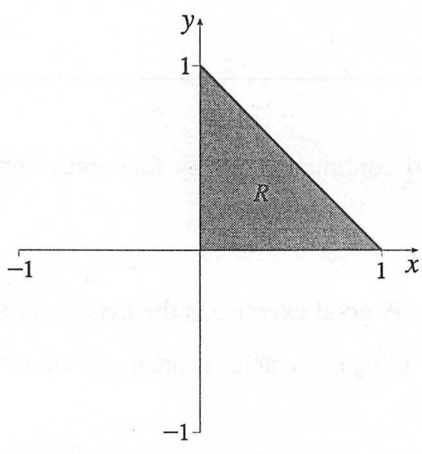
Workshop/Discussion

- Define the centroid $(\overline{x}, \overline{y})$ of a plane region R as the center of gravity, obtained by using a density of 1 for the entire region. So $\overline{x} = \dfrac{1}{A(R)} \iiint x\,dx\,dy$ and $\overline{y} = \dfrac{1}{A(R)} \iiint y\,dx\,dy$. Show the students how to find the centroid for two or three figures like the following:



Show the students that if $x = 0$ is an axis of symmetry for a region R, then $\overline{x} = 0$, and more generally, that $(\overline{x}, \overline{y})$ is on the axis of symmetry. Point out that if there are two axes of symmetry, then the centroid $(\overline{x}, \overline{y})$ is at their intersection.

- Consider the triangular region R shown below, and assume that the density of an object with shape R is proportional to the square of the distance to the origin. Set up and evaluate the mass integral for such an object, and then compute the center of mass $(\overline{x}, \overline{y})$.



Group Work 1: Fun with Centroids

Have the students find the centroids for some of the eight regions described earlier in Workshop/Discussion.

Group Work 2: Generating the Bivariate Normal Distribution

Go over, in detail, Exercise 32 from Section 12.4 (page 868), which involves computing the integral $\int_{-\infty}^{\infty} e^{-x^2}\,dx$ using its double integral counterpart. Since these functions are related to the normal distribution and the bivariate normal distribution, the students can then actually show that $\displaystyle\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-(x/\mu)^2/(2\sigma)} = 1$, as it should be. Show that this is also true for $\displaystyle\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-((x-a)/\mu)^2/(2\sigma)}$ by noting that replacing x by $x - a$ just corresponds to a horizontal shift of the integrand.

Group Work 3: A Slick Model

This activity requires a CAS and is based on the results of Group Work 2.

Homework Problems

Core Exercises: 1, 5, 10

Sample Assignment: 1, 5, 6, 9, 10, 14, 20, 24

Exercise	C	A	N	G	V
1		×			
5		×			
6		×			
9		×			

Exercise	C	A	N	G	V
10		×			
14		×			
20		×			
24	×	×			

Group Work 3, Section 12.5
A Slick Model

An oil tanker has leaked its entire cargo of oil into the middle of the Pacific Ocean, far from any island or continent. The oil has spread out in all directions in a thin layer on the surface of the ocean. The slick can be modeled by the two-dimensional density function $K \exp\left(-2\frac{x^2+y^2}{w^2}\right)$, where w is a fixed constant and the origin of the xy-plane represents the location of the tanker. Assuming that none of the oil evaporates, the density function must account for all of the oil and hence can be interpreted as a probability distribution.

1. Suppose $w = 2$. Find the value of K which ensures that $K \exp\left(-2\frac{x^2+y^2}{w^2}\right)$ is a probability distribution.

2. Find the expected values μ_x and μ_y of x and y in the probability distribution from Problem 1. Interpret your answer geometrically.

3. Find the radius of the circle centered at the origin which contains exactly 99% of the oil in the slick.

12.6 Surface Area

Suggested Time and Emphasis

$\frac{3}{4}$–1 class Recommended material, particularly if Stokes' Theorem and the Divergence Theorem are to be covered. However, it can be deferred to just before Section 13.6.

Points to Stress

1. Review of parametric surfaces and their "partial derivatives" $\frac{\partial \mathbf{r}}{\partial u}$ and $\frac{\partial \mathbf{r}}{\partial v}$ along grid curves $u = u_0$, $v = v_0$ (Sections 10.5 and 11.4).

2. Computation of the area element $\left|\frac{\partial r}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right|$ of a parametric rectangle.

Text Discussion

- Why does the area of the parallelogram determined by vectors $\Delta u\, \mathbf{r}_u^*$ and $\Delta v\, \mathbf{r}_v^*$ involve the cross product $\mathbf{r}_u^* \times \mathbf{r}_v^*$?

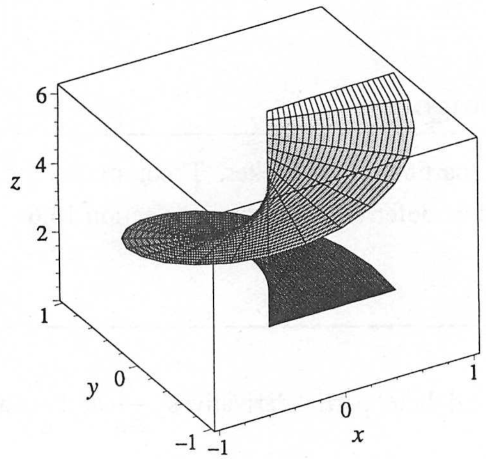
Materials for Lecture

- Give an intuitive presentation developing the area element $\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dx\, dy$ for the surface area of $z = f(x, y)$ as follows: If we have a plane $z = ax + by + c$, then the vectors $\mathbf{v}_1 = \langle -a, 0, 1\rangle$ and $\mathbf{v}_2 = \langle 0, -b, 1\rangle$ are in the plane, and $(\Delta x)\, \mathbf{v}_1$ and $(\Delta y)\, \mathbf{v}_2$ generate a small rectangle in the plane with area $|\mathbf{v}_1 \times \mathbf{v}_2|\, \Delta x\, \Delta y = \sqrt{1 + a^2 + b^2}\, \Delta x\, \Delta y = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, \Delta x\, \Delta y$. If we have a surface $z = f(x, y)$, then approximating a small part of the surface near a point by a small rectangle in the tangent plane at the point gives area $\approx \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, \Delta x\, \Delta y = \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, \Delta x\, \Delta y$. So the surface area above a domain D is $\iint_D \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dx\, dy$.

- Give an intuitive presentation of the area element $|\mathbf{r}_u \times \mathbf{r}_v|\, dA$ for general parametric surfaces.

- Compute the surface area of the surface given in cylindrical coordinates by $z = \theta$ above the unit disk

$0 \le x^2 + y^2 \le 1$ and below $z = 2\pi$. You may use the polar coordinates r and θ as parameters.



Workshop/Discussion

- Set up an integral to compute the surface area of the surface S obtained by rotating $y = x^2$, $0 \le x \le 2$ about the x-axis. Point out that this integral is hard to compute by hand, but a CAS can do it very easily.
- Find the area of the portion of the surface $z = 16 - x^2 - y^2$ that lies above the xy-plane.
- Set up an integral to compute the surface area of the portion of the cone $z = r$ lying above the region enclosed by the polar curve $r = \sqrt{\cos 2\theta}$, $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$.
- Show how to compute the surface area of the portion of the surface $z = xy$ inside the cylinder $x^2+y^2 = a^2$. Point out that some of the surface lies above the plane $z = 0$ (above points in the first and third quadrants) and the remainder lies below $z = 0$. Then show how the full surface integration as a single integral requires using cylindrical coordinates.

Group Work 1: Setting up Surface Integrals

The first two problems are straightforward. After the students have started the third problem, hand each group "a hint sheet". Unbeknownst to them, different groups will get different hint sheets. After they are finished, they can present four different ways of solving the problem. If a group finishes early, give them the bonus problem. Depending on the students' facility with computation, and/or their access to a CAS, the instructor may want them to compute all the integrals, or just set them all up.

Group Work 2: Surfaces of Revolution

This activity leads the students through computations of the surface areas of solids of revolution. If Section 10.5 has not been covered, students may need more of an introduction. Note that Problem 3 leads to an improper integral.

Group Work 3: Time to Blow your Geographical Minds

The surface area of Wyoming is 96,988 square miles, and that of Colorado is 103,598 square miles. In this activity, the students compute these surface areas and obtain numbers that are much less. The reason for the discrepancy between their model and reality is that the model does not take mountains into account. If a hint is needed, perhaps show the students the states on a relief map or globe. Note that you may need to remind students how latitude and longitude are measured.

Homework Problems

Core Exercises: 1, 6, 9, 18

Sample Assignment: 1, 2, 6, 9, 12, 13, 18, 22, 24

Note: Exercises 13(b), 18(c), and 18(d) require a CAS.

Exercise	C	A	N	G	V
1		×			×
2		×			×
6		×			×
9		×			×
12		×			
13		×	×		
18		×	×	×	
22		×			×
24		×			

Group Work 1, Section 12.6
Setting Up Surface Integrals

1. Set up the surface integral for the parametric surface $\mathbf{r}(u,v) = \langle 3u\sin v, 4u\cos v, u^3 \rangle$ over the domain $\{(u,v) \mid 0 \le u \le 5, 0 \le v \le \pi\}$

2. Compute the surface area of the part of the surface $z = y^2$ above the triangle with vertices $(0,1,0)$, $(1,0,0)$, and $(1,1,0)$.

3. Compute the surface area of a cone with height 4 and radius 3.

Group Work 1, Section 12.6
Setting Up Surface Integrals (Hint Sheet)

To compute the surface area of a cone with height 4 and radius 3, set up the shape as $z = f(x, y)$ and compute a surface integral.

Group Work 1, Section 12.6
Setting Up Surface Integrals (Hint Sheet)

To compute the surface area of a cone with height 4 and radius 3, set up the shape as $z = f(r, \theta)$ and compute a surface integral.

Group Work 1, Section 12.6
Setting Up Surface Integrals (Hint Sheet)

To compute the surface area of a cone with height 4 and radius 3, set up the shape in spherical coordinates (ϕ is constant) and compute a surface integral.

Group Work 1, Section 12.6
Setting Up Surface Integrals (Hint Sheet)

To compute the surface area of a cone with height 4 and radius 3, set up the shape as generated by a line segment rotated about the z-axis and compute a surface integral.

Group Work 1, Section 12.6
Setting Up Surface Integrals (Bonus Problem)

Set up the surface integral of the piece of the unit sphere above the polar circle $r = \sin\theta$. Compute the surface area. How would the answer have differed had we used $r = \cos\theta$?

Group Work 2, Section 12.6
Surfaces of Revolution

For certain surfaces there is a formula for the surface area which can be written as a single integral instead of as a double integral. These surfaces are the so-called surfaces of revolution which we get by rotating the graph of a function about one of the axes. The formula for a surface formed by rotating the graph of $y = f(x)$ about the x-axis is $A = 2\pi \int_a^b f(x) \sqrt{1 + [f'(x)]^2}\, dx$.

1. Derive this formula using Formula 3 from Section 10.5 (page 740).

2. There is a similar formula for the area of a surface formed when the graph of $x = g(y)$ is rotated about the y-axis. Derive this formula.

3. Use the formula from Problem 1 to find the surface area of a hemisphere of the unit sphere.
Hint: First find the formula for $0 \le x \le a$, and then take the limit as $a \to 1^-$.

4. Use the formula found in Problem 2 to set up an integral for the surface area of the ellipsoid $x^2 + \frac{1}{4}y^2 + z^2 = 1$ as a surface of revolution about the y-axis.

Group Work 3, Section 12.6
Time to Blow your Geographic Minds

The boundaries of Wyoming are given by the lines 41° North latitude, 45° North latitude, 104° West longitude, and 111° West longitude. The boundaries of Colorado are given by the lines 37° North latitude, 41° North latitude, 102° West longitude, and 109° West longitude.

1. Which state takes up more surface area on a globe?

2. Assume that the radius of the Earth is 3986 miles. What are the approximate areas of Wyoming and Colorado, in square miles?

3. Ask your instructor what the actual areas are. What do you think accounts for the discrepancy between these figures and your answers in Part 2?

12.7 Triple Integrals

Suggested Time and Emphasis

$\frac{3}{4}$–1 class   Essential Material

Points to Stress

1. The basic definition of a triple integral.
2. The various types of volume domain, and how to set up the volume integral based on a given domain.
3. Changing the order of integration in triple integrals.

Text Discussion

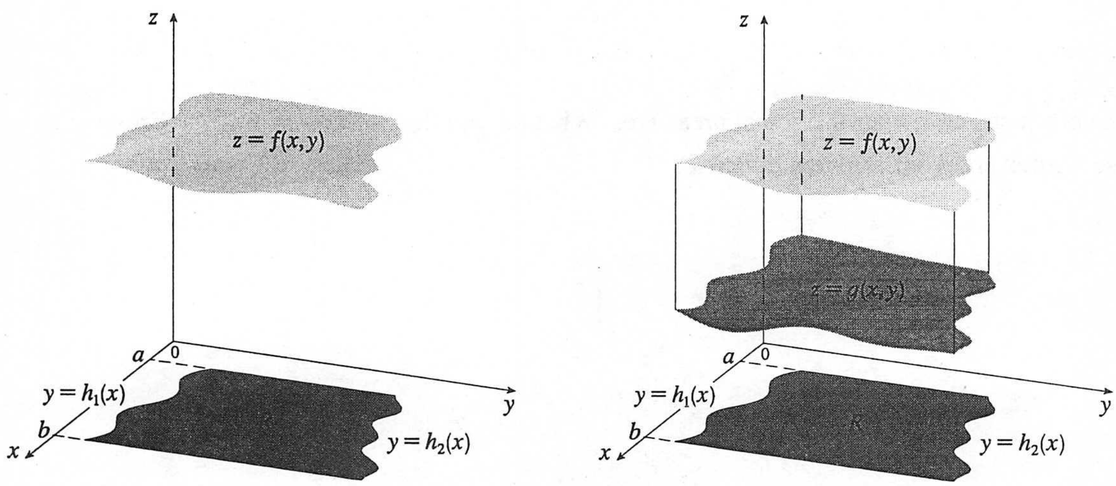
- Which variable ranges between two functions of the other two variables in a type 1 region? In a type 3 region?
- Give an example of a region that is both type 1 and type 2.

Materials for Lecture

- One way to introduce volume integrals is by revisiting the concept of area, pointing out that area integrals can be viewed as double integrals (for example $\int_0^{10} f(x)\,dx = \int_0^{10}\int_0^{f(x)} dy\,dx$) and then showing how some volume integrals work by an analogous process. Set up a typical volume integral of a solid S using double integrals and similarly transform it into a triple integral:

$$V = \iint_R f(x,y)\,dA = \int_a^b \int_{h_1(x)}^{h_2(x)} f(x,y)\,dy\,dx = \int_a^b \int_{h_1(x)}^{h_2(x)} \int_0^{f(x,y)} dz\,dy\,dx$$

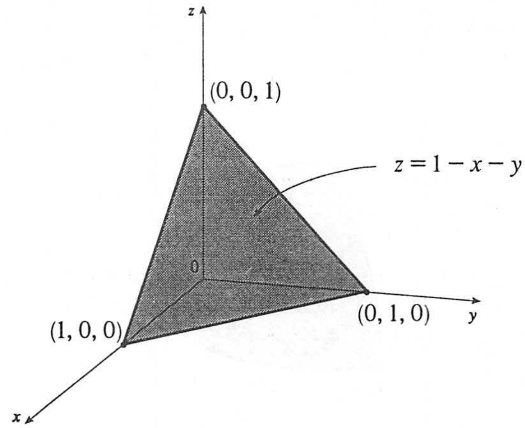
Then "move" the bottom surface of S up to $z = g(x,y)$, so S has volume $V = \int_a^b \int_{h_1(x)}^{h_2(x)} \int_{g(x,y)}^{f(x,y)} dz\,dy\,dx$.



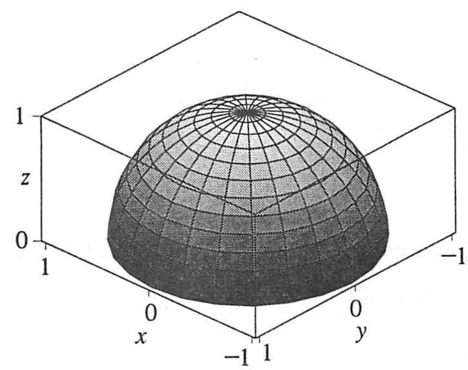
If we now have a function $k(x,y,z)$ defined on S, then the triple integral of k over S is

$$\iiint_S k(x,y,z)\,dV = \int_a^b \int_{h_1(x)}^{h_2(x)} \int_{g(x,y)}^{f(x,y)} k(x,y,z)\,dz\,dy\,dx$$

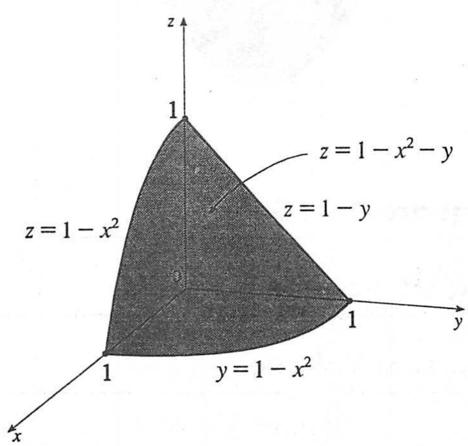
- Show the students why the region shown below is type 1, type 2, and type 3, and describe it in all three ways.



- Show how to identify the region of integration E shown below for the volume integral $\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{0}^{\sqrt{1-x^2-y^2}} f(x, y, z)\, dz\, dy\, dx$, and then rewrite the integral as an equivalent iterated integral of the form $\iiint_E f(x, y, z)\, dx\, dz\, dy$.

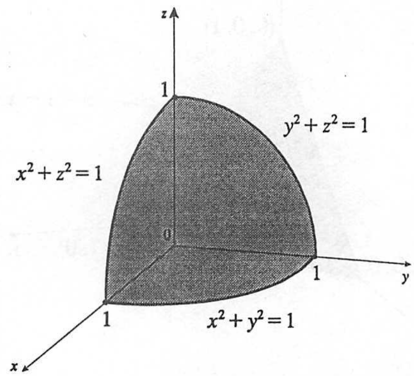


- Do a sample computation, such as the triple integral of $f(x, y, z) = z + xy^2$ over the volume V bounded by the surface sketched below, in the first octant.

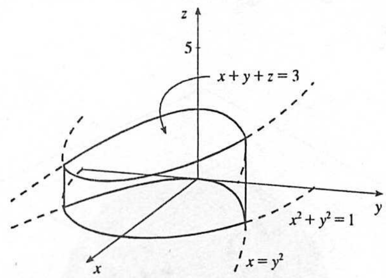


Workshop/Discussion

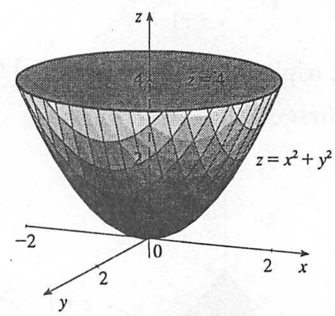
- Set up a triple integral for the volume V of the piece of the sphere of radius 1 in the first octant, with different orders of integration.



- Sketch the solid whose volume is given by the triple integral $\int_{-\sqrt{(\sqrt{5}+1)/2}}^{\sqrt{(\sqrt{5}+1)/2}} \int_{y^2}^{\sqrt{1-y^2}} \int_0^{3-x-y} dz\, dy\, dx$. Such a sketch is shown below.



- Compute $\iiint_E \left(x^2+y^2\right)^{1/2}\, dV$, where E is the solid pictured below, by first integrating with respect to z and then using polar coordinates in place of $dx\, dy$.



Group Work 1: The Square-Root Solid

Group Work 2: Setting Up Volume Integrals

Group Work 3: An Unusual Volume

This is a challenging group work for more advanced students. The idea is to show that just because a solid looks simple, the computation of its volume may be difficult. The line generated by P_1 and P_2 has equation

$z = \dfrac{L_1 - L_2}{2R}x + L_2$, and hence this equation, interpreted in three dimensions, is also the equation of the plane S. The integral $V(E) = \int_{-R}^{R} \int_{-\sqrt{R^2-(x-R)^2}}^{\sqrt{R^2-(x-R)^2}} \left(\dfrac{L_1 - L_2}{2R}x + L_2 \right) dy\, dx$ requires polar coordinates to solve by hand (since the bounding circle has equation $r = 2R\cos\theta$, $0 \le \theta \le \pi$) and also requires the students to remember how to integrate $\cos^2\theta$ and $\cos^4\theta$. Note that the problem can be simplified by moving the solid so that the z-axis runs through the center of D. Point out that a simple geometric solution can be obtained by replacing S by the horizontal plane $z = \dfrac{L_1 + L_2}{2}$, thus giving a standard cylinder.

Homework Problems

Core Exercises: 2, 5, 10, 26, 29

Sample Assignment: 2, 5, 8, 10, 13, 16, 20, 21, 26, 29, 37, 45

Note: Problem 7 from Focus on Problem Solving (page 914) can be assigned here as a project for an advanced student or group of students.

Exercise	C	A	N	G	V
2		×			
5		×			
8		×			
10		×			×
13		×			×
16		×			×

Exercise	C	A	N	G	V
20			×		
21		×			
26		×			
29		×			×
37		×			
45		×			

Group Work 1, Section 12.7
The Square-Root Solid

Consider the volume integral over the solid S given by $V = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{1} dz\, dy\, dx$.

1. Identify the solid S by drawing a picture.

2. Rewrite the volume integral as $V = \iiint_S dx\, dy\, dz$.

3. Rewrite the volume integral as $V = \iiint_S dz\, dx\, dy$.

4. Starting with the original iterated integral, compute the volume by any means at your disposal.

Group Work 2, Section 12.7
Setting Up Volume Integrals

Set up volume integrals for the following solids:

1. S is the solid formed by hollowing out a square cylinder of side length 1 centered along the z-axis through the solid sphere $x^2 + y^2 + z^2 = 4$.

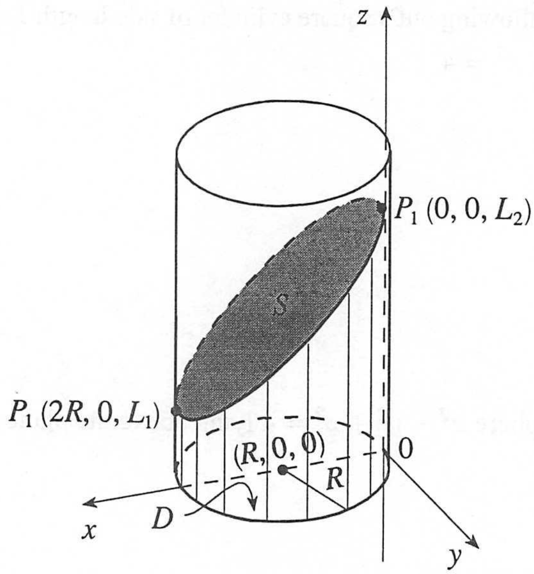
2. S is the part of the solid sphere $x^2 + y^2 + z^2 = 4$ lying above the plane $z = 1$.

3. S is the solid formed by drilling a cylindrical hole of radius 1 centered along the z-axis through the solid sphere $x^2 + y^2 + z^2 = 4$.

4. S is the solid formed by hollowing out a square cylinder of side length 1 centered along the y-axis through the solid sphere $x^2 + y^2 + z^2 = 4$.

Group Work 3, Section 12.7
An Unusual Volume

Consider the solid E shown below.



1. Find the equation of the plane S, parallel to the y-axis, which forms the top cap of E.

2. Set up a volume integral for E of the form $V(E) = \iiint_E dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} dz\right] dA$.

3. Compute $V(E)$ by any means at your disposal. (**Hint:** Try polar coordinates.)

Discovery Project: Volumes of Hyperspheres

Problems 1 and 2 review computations that the students may already know. Problem 4 is optional for this project, but it is highly recommended. To extend this project, students can be asked to find a book or article that discusses hyperspheres, and add some geometric discussion of these objects to their reports.

12.8 Triple Integrals in Cylindrical and Spherical Coordinates

Suggested Time and Emphasis

$\frac{3}{4}$–1 class Essential Material

Transparencies Available

- Transparency 54 (Figure 7, page 895)
- Transparency 55 (Discovery Project: The Intersection of Three Cylinders, page 901)

Points to Stress

1. The basic shapes of cylindrical and spherical rectangular solids
2. Volume integrals in cylindrical and spherical coordinates

Text Discussion

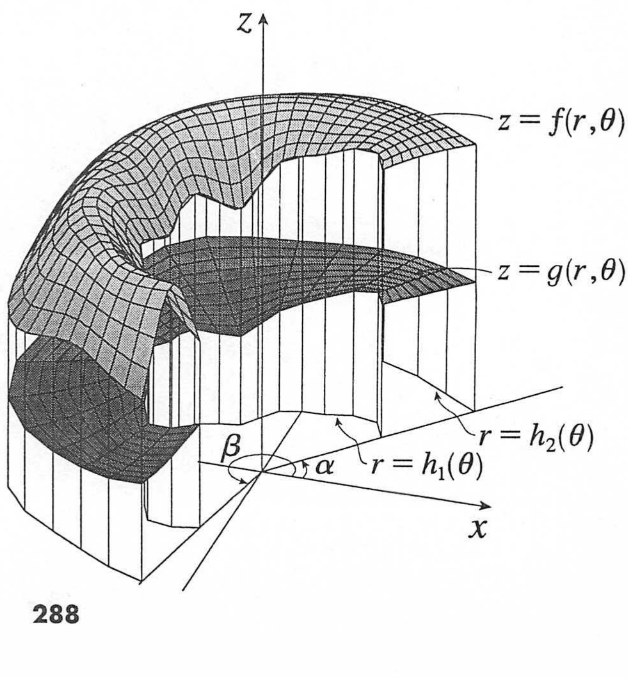
- Does the region of Example 2 have an axis of symmetry? If so, what does it say about the choice of using cylindrical coordinates?
- Why should we choose to use spherical coordinates in Example 4?

Materials for Lecture

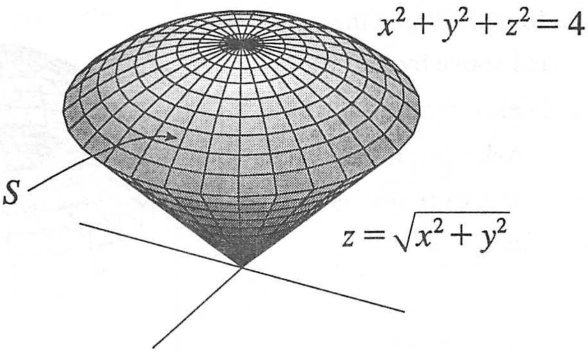
- Convert a typical cylindrical volume integral of a solid S computed using double integrals into a triple integral:

$$V = \iint_R f(r,\theta)\, r\, dr\, d\theta = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r,\theta)\, r\, dr\, d\theta = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_0^{f(r,\theta)} r\, dz\, dr\, d\theta$$

Move the bottom surface of S up to $z = g(r,\theta)$, as pictured at right. The volume now becomes $\int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{g(r,\theta)}^{f(r,\theta)} r\, dz\, dr\, d\theta$. The basic volume element is given in Figure 3 of the text. Conclude with the situation where we have $h(r,\theta,z)$ defined on S. Then the triple integral of h on S is $\int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{g(r,\theta)}^{f(r,\theta)} h(r,\theta,z)\, r\, dz\, dr\, d\theta$.



- Develop a straightforward example such as the region depicted below:

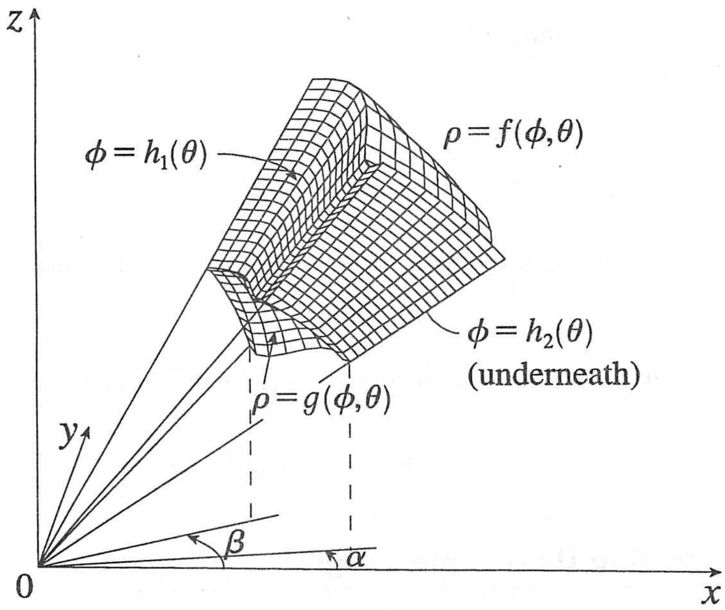


The Capped Cone

Set this volume up as a triple integral in cylindrical coordinates, and then find the volume. (The computation of this volume integral is not that hard, and can be assigned to the students.) Conclude by setting up the volume integral of $h(r,\theta,z) = rz$ over this region.

- Draw a basic spherical rectangular solid S and compute that its volume is approximately $\rho^2 \sin\phi\,\Delta\rho\,\Delta\phi\,\Delta\theta$ if $\Delta\rho$, $\Delta\phi$, and $\Delta\theta$ are small. Calculate the volume of the solid pictured below to be

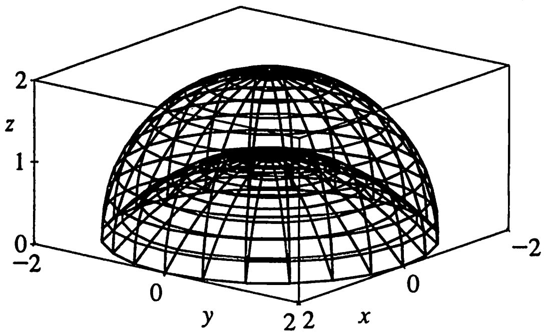
$$V = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{g(\phi,\theta)}^{f(\phi,\theta)} \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta$$



As usual, if there is a function $l(\rho,\phi,\theta)$ on S, then the triple integral is $\int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{g(\phi,\theta)}^{f(\phi,\theta)} l(\rho,\phi,\theta)\,\rho^2 \sin\phi\,d\rho\,d\phi\,d\theta$.

- Redo "The Capped Cone" example, this time in spherical coordinates. (This is similar to Example 4 in the text.)

- Indicate to the students why using spherical coordinates is a good choice for calculating the volume of E, the solid bounded above by the sphere $x^2 + y^2 + z^2 = 4$ and below by the paraboloid $4z = 4 - x^2 - y^2$. Ask them if they think cylindrical coodinates would work just as well. Then compute $V(E)$ using either method.



Workshop/Discussion

- Give a geometric description of the solid S whose volume is given in spherical coordinates by $V = \int_0^{\pi} \int_{\pi/4}^{\pi/3} \int_0^1 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$, and then show the students how to write the volume of S as a triple integral in cylindrical coordinates.

- Give the students some three-dimensional regions and ask them which coordinate system would be most convenient for computing the volume of that region. Examples:

  1. $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \leq 4 \text{ and } -2 \leq z \leq 3\}$
  2. $\{(x, y, z) \in \mathbb{R}^3 \mid z^2 + y^2 \leq 4 \text{ and } |x| \leq 1\}$
  3. $\{(x, y, z) \in \mathbb{R}^3 \mid z^2 \geq x^2 + y^2 \text{ and } x^2 + y^2 + z^2 \leq 1\}$
  4. $\{(x, y, z) \in \mathbb{R}^3 \mid \frac{1}{4} \leq x^2 + y^2 + z^2 \leq 9, x \geq 0, y \geq 0, \text{ and } z \geq 0\}$

- Give the students some integrands and ask them which coordinate system would be most convenient for integrating that integrand.

  1. $f(x, y, z) = 1/(x^2 + y^2)$ over the solid enclosed by a piece of a circular cylinder $x^2 + y^2 = a$
  2. $f(x, y, z) = e^{2x^2 + 2y^2 + 2z^2}$ over a solid between a cone and a sphere

Group Work 1: Setting Up a Triple Integral

If a group finishes Problems 1–4 quickly, have them choose one of their integrals and compute it, and explain why they made the choice they did. Note that for Problem 5, the solid is a truncated piece of the cone $z = -r + R$ in cylindrical coordinates.

Group Work 2: A Partially Eaten Sphere

Notice that you are removing "ice cream cones" both above and below the xy-plane.

Homework Problems

Core Exercises: 1, 5, 8, 19, 28, 29, 30, 33

Sample Assignment: 1, 4, 5, 8, 14, 15, 19, 26, 28, 29, 30, 33

Note:
- Exercise 28(b) requires a CAS, or the students can be instructed to sketch the torus by hand.
- Exercise 31 makes a good lab project.
- Exercise 33 requires significant thought.

Exercise	C	A	N	G	V
1		×		×	×
4		×		×	×
5		×			×
8		×			
14		×			
15		×			

Exercise	C	A	N	G	V
19		×			
26		×			×
28		×		×	
29		×			×
30		×			×
33	×	×			

Group Work 1, Section 12.8
Setting Up a Triple Integral

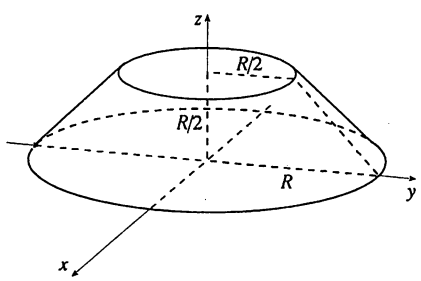
1. Set up the triple integral $\iiint_S (x^2 + y^2)\, dV$ in rectangular coordinates, where S is a solid sphere of radius R centered at the origin.

2. Set up the same triple integral in cylindrical coordinates.

3. Set up the same triple integral in spherical coordinates.

4. Set up an integral to compute the volume of the solid inside the cone $x^2 = y^2 + z^2$, $|x| \le 1$, in a coordinate system of your choice.

5. Compute $\iiint_E z\,(x^2 + y^2)\, dV$, where E is the solid shown below.



Group Work 2, Section 12.8
A Partially Eaten Sphere

1. Compute the volume of the solid S formed by starting with the sphere $x^2 + y^2 + z^2 = 9$, and removing the solid bounded below by the cone $z^2 = 2\left(x^2 + y^2\right)$.

2. Set up the triple integral $\iiint_S yz\, dV$ in the same coordinate system you used for Problem 1.

Applied Project: Roller Derby

This project has a highly dramatic outcome, in that a real race can be run whose results are predicted by mathematics. An in-class demonstration of the "roller derby" can be done before this project is assigned. If this project is to be assigned, it should be assigned in its entirety, in order to predict the outcome of the race.

Discovery Project: The Intersection of Three Cylinders

This discovery project extends the problem of finding the volume of two intersecting cylinders given in Exercise 62 in Section 6.2. This is a very thought-provoking project for students with good geometric intuition. There is value to be gained from having students work on it, even if they don't wind up getting a correct answer. A good solution is given in the Complete Solutions Manual.

12.9 Change of Variables in Multiple Integrals

Suggested Time and Emphasis

1–$1\frac{1}{4}$ classes   Recommended Material (essential if Chapter 17/*16* is to be covered)

Points to Stress

1. Reason for change of variables: to reduce a complicated multiple integration problem to a simpler integral or an integral over a simpler region in the new variables

2. What happens to area over a change in variables: The role of the Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)}$

3. Various methods to construct a change of variables

Text Discussion

- What is the area of the image R of the unit square S [with opposite corners at $(0,0)$ and $(1,1)$] in the uv-plane under the transformation $x = u + 2v$, $y = -6u - v$?

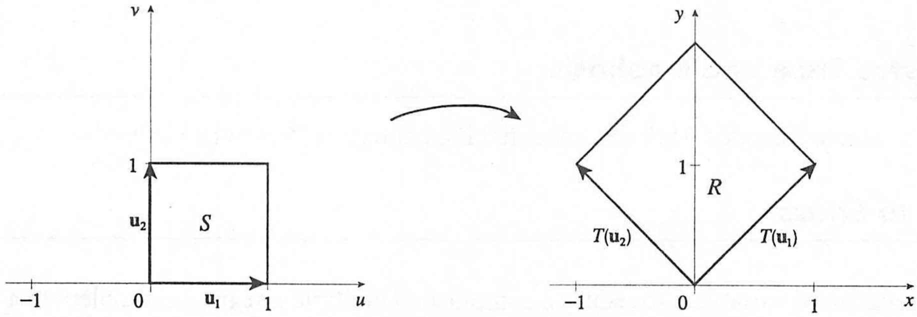
Materials for Lecture

- One good way to begin this section is to discuss u-substitution from a geometric point of view. For example, $\int (\sin^2 x) \cos x \, dx$ is a somewhat complicated integral in x-space, but using the change of coordinate $u = \sin x$ reduces it to the simpler integral $\int u^2 \, du$ in u-space. If the students are concurrently taking physics or chemistry, discuss how the semi-logarithmic paper that they use is an example of this type of coordinate transformation.

- Review the fact that to integrate $f(x,y)$ over a region R in the xy-plane, if we have a change of variables $\mathbf{r}(u,v)$ which transforms the rectangle R_1 in uv-space to R, then we have $\iint_R f(x,y)\, dA = \iint_{R_1} F(u,v) \left| r_u \times r_v \right| du\, dv$, where $F(u,v) = f(x(u,v), y(u,v))$ and $r_u \times r_v = C\mathbf{k}$, where C is the Jacobian determinant $\begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \dfrac{\partial x}{\partial u}\dfrac{\partial y}{\partial v} - \dfrac{\partial y}{\partial u}\dfrac{\partial x}{\partial v}$, denoted by $\dfrac{\partial(x,y)}{\partial(u,v)}$. Thus, $\left| r_u \times r_v \right| = \left| \dfrac{\partial(x,y)}{\partial(u,v)} \right|$.

- Note that it is very important that we take the absolute value of the Jacobian determinant. For example, point out that the Jacobian determinant for spherical coordinates is always negative (see Example 4). Another example of a negative Jacobian is the transformation $x = u + 2v$, $y = 3u + v$, which takes $\langle 1,0 \rangle$ to $\langle 1,3 \rangle$ and $\langle 0,1 \rangle$ to $\langle 2,1 \rangle$.

- Consider the linear transformation $x = u - v$, $y = u + v$. This takes the unit square S in the uv-plane into a square with area 2.

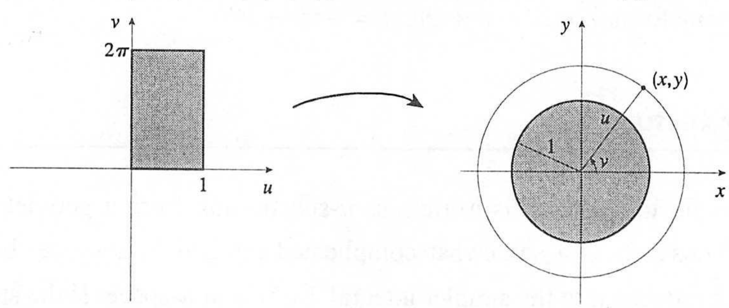


Note that the Jacobian of this transformation is $\dfrac{\partial(x,y)}{\partial(u,v)} = 2$. In general, we have

$$A(R) = \iint_R 1\,dA = \iint_S \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| A(S)$$

So for linear transformations, the Jacobian is the determinant of the matrix of coefficients, and the absolute value of this determinant describes how area in uv-space is magnified in xy-space under the transformation T.

- Pose the problem of changing the rectangle $[0,1] \times [0,2\pi]$ in the uv-plane into a disk in the xy-plane by a change of variable $\mathbf{r}(u,v) = x(u,v)\,\mathbf{i} + y(u,v)\,\mathbf{j}$. Show that $x = u\cos v$, $y = u\sin v$ will work.



Then $u^2 = x^2 + y^2$ and $\tan v = y/x$, so u can be viewed as the distance to the origin and v is the angle with the positive x-axis. This implies that the u, v transformation is really just the polar-coordinate transformation. The grid lines $u = c \geq 0$ go to circles, and $v = c$ go to rays. Therefore uv-rectangles go to polar rectangles in xy-space. Perhaps note that trying to look at this transformation in reverse leads to problems at the origin. See if the students can determine what happens to the line $u = 0$. Also note that there are other transformations that work, such as $x = u\sin v$, $y = u\cos v$.

- Pose the problem of changing a rectangle into the ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{9} \leq 1$. Posit that the answer might again be of the form $x = c_1 u\cos v$, $y = c_2 u\sin v$ and solve $\dfrac{x^2}{c_1^2} + \dfrac{y^2}{c_2^2} = u^2$. So chose $c_1 = 2$, $c_2 = 3$ and then the rectangle $[0,1] \times [0,2\pi]$ maps into the specified ellipse. This time, $\frac{2}{3}\tan v = \dfrac{y}{x}$ so $\tan v = \dfrac{3y}{2x}$. The grid line $u = c > 0$ goes to the ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{9} = c$ and $v = k$ goes to the ray $y = \frac{2}{3}x\tan k$. This means that (u,v) gives an elliptical coordinate system.

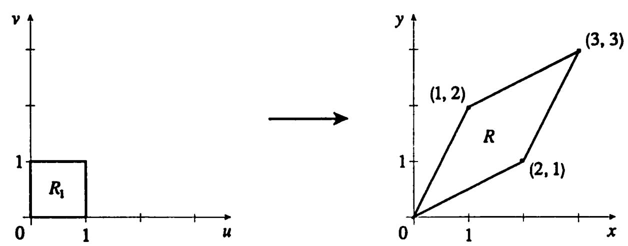
Workshop/Discussion

- Show that for the polar-coordinate representation discussed in the lecture suggestions above, the Jacobian is equal to u, and we get $u\,du\,dv$ as previously computed.

- For the "elliptical" coordinates described in the Materials for Lecture, the Jacobian is $\begin{vmatrix} 2\cos v & 3\sin v \\ -2u\sin v & 3u\cos v \end{vmatrix} = 6u$, leading to $6u\,du\,dv$. Using this new coordinate system to compute $\iint_R x^2\,dA$ where R is the region bounded by the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$, we have $\iint_R x^2 dA = \int_0^{2\pi} \int_0^1 (2u\cos v)^2\,|6u|\,du\,dv = \int_0^{2\pi} 6\cos^2 v\,dv = 6\pi$.

- Show how to compute the volume of the solid S bounded by the ellipsoid $\frac{x^2}{2} + \frac{y^2}{3} + \frac{z^2}{4} = 1$. In the xy-plane, we have the elliptic region R bounded by $\frac{x^2}{2} + \frac{y^2}{3} = 1$. The change of variables $x = \sqrt{2}u$, $y = \sqrt{3}v$ maps the unit disk D: $u^2 + v^2 \le 1$ to R. So we get $V = 2\iint_D 2\sqrt{1-(u^2+v^2)}\sqrt{6}\,du\,dv$. Now using polar coordinates on this uv-integral, we get $V = 4\sqrt{6}\int_0^{2\pi}\int_0^1 \sqrt{1-r^2}r\,dr\,d\theta = \frac{8\sqrt{6}}{3}\pi$. Another approach is to map the unit ball $u^2+v^2+w^2 \le 1$ to the ellipsoid using $x = \sqrt{2}u$, $y = \sqrt{3}v$, $z = 2w$ and then $V = \iiint_B 2\sqrt{6}\,dV = \frac{8\sqrt{6}}{3}\pi$, using the change of variables formula for triple integrals.

- Describe how to find a transformation which maps the uv-plane as follows:



  Show how it is sufficient to check what happens to $\langle 0, 1\rangle$ and $\langle 1, 0\rangle$.

- Consider the change of variables $x = u^2 - v^2$, $y = 2uv$ described in Example 1 of the text. Show that the grid lines $u = a$ give the parabolas $x = a^2 - \frac{y^2}{4a^2}$, and the grid lines $v = b$ give the parabolas $x = \frac{y^2}{4b^2} - b^2$.

Group Work 1: Many Changes of Variables

Group Work 2: Transformed Parabolas

Homework Problems

Core Exercises: 2, 9, 11, 17

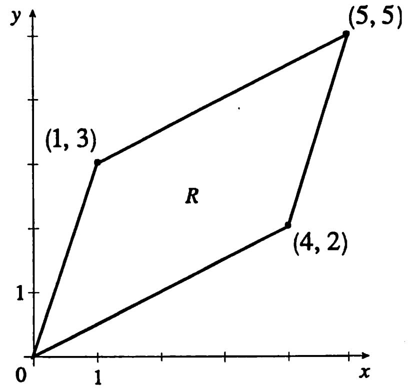
Sample Assignment: 2, 5, 9, 10, 11, 14, 17, 19

Exercise	C	A	N	G	V
2		×			
5		×			
9		×		×	×
10		×		×	×

Exercise	C	A	N	G	V
11		×			
14		×			
17		×	×		
19		×			×

Group Work 1, Section 12.9
Many Changes of Variables

1. Find $\iint_R xy^2\, dA$ where R is given below.



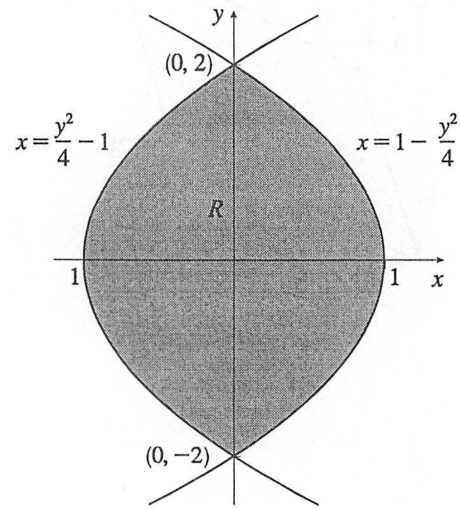
2. Find a mapping T which maps the triangle bounded by $(0,0)$, $(0,1)$, and $(1,0)$ to the triangle bounded by $(0,0)$, $(5,2)$, and $(5,-2)$. What is the Jacobian of T? What is the area of R? Compute $\iint_R xy\, dA$.

3. Find a mapping T which maps the unit disc $u^2 + v^2 \leq 1$ onto the region R enclosed by the ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$. Compute $\iint_R x^2\, dA$ and $\iint_R y^2\, dA$.

Group Work 2, Section 12.9
Transformed Parabolas

Consider the change of variables $x = u^2 - v^2$, $y = 2uv$ described in Example 1 of the text.

1. Compute $\iint_R x^2\, dA$, where R is the region shown below.
Hint: How is $\iint_R x^2\, dA$ related to $\iint_{R_1} x^2\, dA$, where R_1 is the portion of R above the x-axis?



2. Let S be the rectangle $\{(u, v) \mid 0 \le u \le 1, 0 \le v \le 1\}$. What is the image T in the xy-plane of S under this change of variables?

3. What is the area of T?

12 Sample Exam

Problems marked with an asterisk (*) are particularly challenging and should be given careful consideration.

1. Consider the function $f(x, y) = x^y$ on the rectangle $[1, 2] \times [1, 2]$.

(a) Approximate the value of the integral $\int_1^2 \int_1^2 f(x, y)\, dy\, dx$ by dividing the region into four squares and using the function value at the lower left-hand corner of each square as an approximation for the function value over that square.

(b) Does the approximation give an overestimate or an underestimate of the value of the integral? How do you know?

2. Given that $\int_0^{\pi/2} \dfrac{dx}{1 + \sin^2 x} = \dfrac{\pi}{2\sqrt{2}}$,

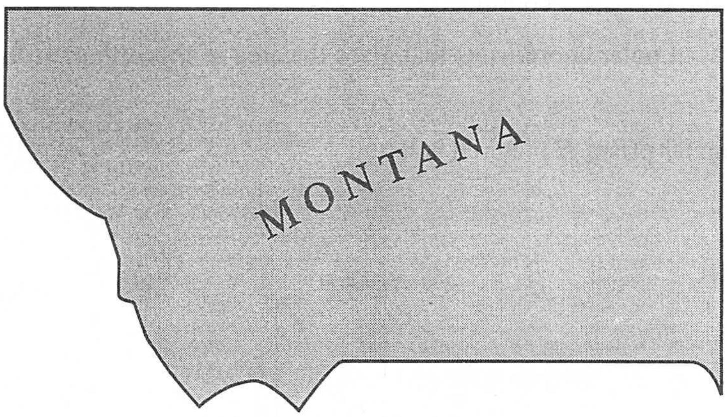
(a) evaluate the double integral

$$\int_0^{\pi/2} \int_0^{\pi/2} \frac{1}{(1 + \sin^2 x)(1 + \sin^2 y)}\, dx\, dy$$

(b) evaluate the triple integral

$$\int_0^{\pi/2} \int_0^{\pi/2} \int_{1/(1+\sin^2 x)}^{1/(1+\sin^2 y)} dz\, dx\, dy$$

3. Consider the region below:



(a) Divide the region into smaller regions, all of which are Type I.

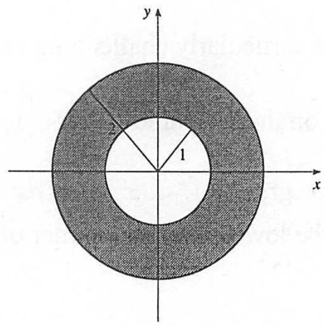
(b) Divide the region into smaller regions, all of which are Type II.

4. Rewrite the integral

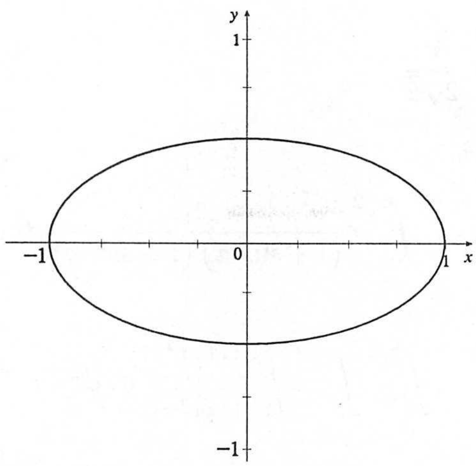
$$\int_0^{2\pi} \int_0^1 r^2\, dr\, d\theta$$

in rectangular coordinates.

5. Evaluate $\iint_D \cos\left(x^2+y^2\right)dA$, where $D=\left\{(x,y)\mid 1\le x^2+y^2\le 4\right\}$ is a washer with inner radius 1 and outer radius 2.



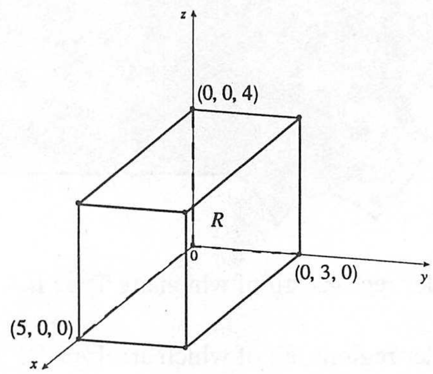
6. Consider the ellipse $x^2+2y^2=1$.



(a) Rewrite the equation in polar coordinates.

(b) Write an integral in polar coordinates that gives the area of this ellipse. *Note:* Your answer will not look simple.

7. Consider the rectangular prism R pictured below:



Compute $\iiint_R 10\,dV$ and $\iiint_R x\,dV$.

8. (a) Compute

$$\int_0^1\int_{-1}^1\int_0^{xy} 1\,dz\,dx\,dy$$

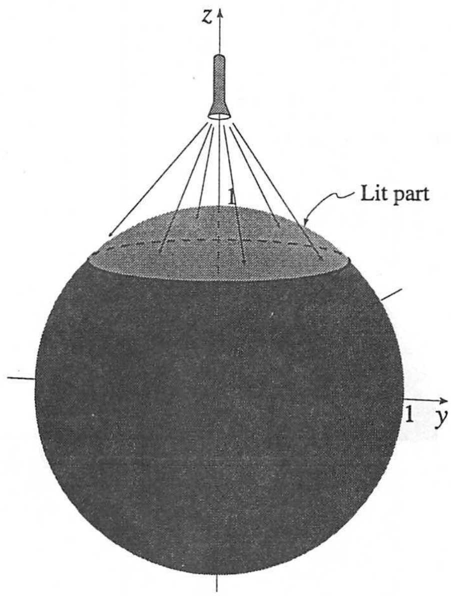
and give a geometric interpretation of your answer.

(b) Compute

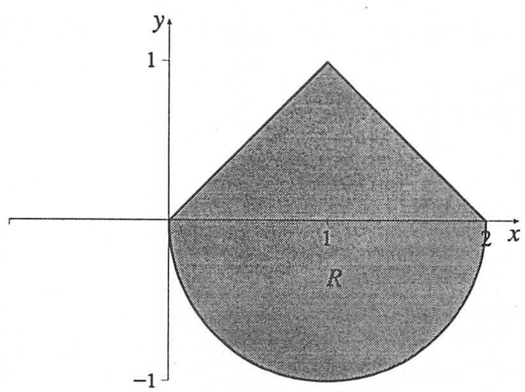
$$\int_0^1 \int_{-1}^1 \int_0^{|xy|} 1\, dz\, dx\, dy$$

and give a geometric interpretation of your answer.

9. A light on the z-axis, pointed at the origin, shines on the sphere $\rho = 1$ such that $\frac{1}{4}$ of the total surface area is lit. What is the angle ϕ?



10. Consider the region R enclosed by $y = x$, $y = -x + 2$, $y = -\sqrt{1 - (x-1)^2}$:

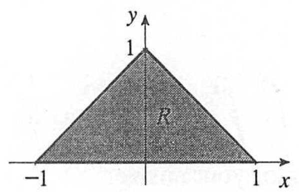


Set up the following integrals as one or more iterated integrals, but do not actually compute them:

(a) $\iint_R (x + y)\, dy\, dx$

(b) $\iint_R (x + y)\, dx\, dy$

11. Consider the region R enclosed by $y = x + 1$, $y = -x + 1$, and the x-axis.



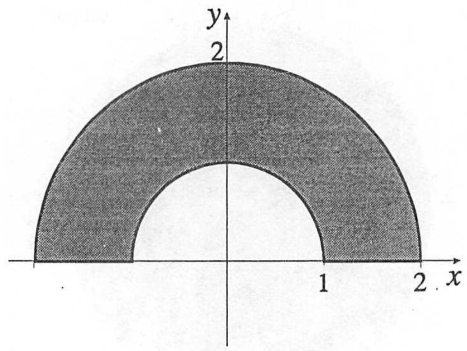
(a) Set up the integral $\iint_R xy\, dx\, dy$ in polar coordinates.

(b) Compute the integral $\iint_R xy\, dx\, dy$ using any method you know.

12. Consider the double integral

$$\iint_R \frac{1}{9 - (x^2 + y^2)^{3/2}}\, dA$$

where R is given by the region between the two semicircles pictured below:

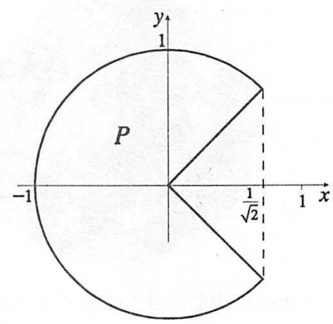


(a) Compute the shaded area.

(b) Show that the function $\dfrac{1}{9 - (x^2 + y^2)^{3/2}}$ is constant on each of the two bounding semicircles.

(c) Give a lower bound and an upper bound for the double integral using the above information.

13. Observe the following Pac-Man:



(a) Describe him in polar coordinates.

(b) Evaluate $\iint_{\text{Pac-Man}} x\, dA$ and $\iint_{\text{Pac-Man}} y\, dA$.

14. Set up and evaluate an integral giving the surface area of the parametrized surface $x = u + v$, $y = u - v$, $z = 2u + 3v$, $0 \le u \le 1$, $0 \le v \le 1$.

15. Consider the triple integral

$$\int_0^1 \int_{y^3}^{\sqrt{y}} \int_0^{xy} dz\, dx\, dy$$

representing a solid S. Let R be the projection of S onto the plane $z = 0$.

(a) Draw the region R.

(b) Rewrite this integral as $\iiint_S dz\, dy\, dx$.

16. Consider the transformation T: $x = 2u + v$, $y = u + 2v$.

(a) Describe the image S under T of the unit square $R = \{(u, v) \mid 0 \le u \le 1, 0 \le v \le 1\}$ in the uv-plane using a change of coordinates.

(b) Evaluate $\iint_S (3x + 2y)\, dA$

17. What is the volume of the following region, described in spherical coordinates: $1 \le \rho \le 9$, $0 \le \theta \le \frac{\pi}{2}$, $\frac{\pi}{6} \le \phi \le \frac{\pi}{4}$?

18. Consider the transformation $x = v \cos 2\pi u$, $y = v \sin 2\pi u$.

(a) Describe the image S under T of the unit square $R = \{(u, v) \mid 0 \le u \le 1, 0 \le v \le 1\}$.

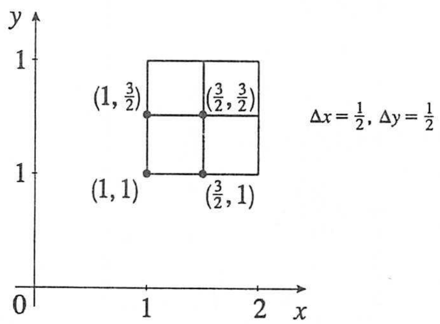
(b) Find the area of S.

19. Consider the function $f(x, y) = ax + by$, where a and b are constants. Find the average value of f over the region $R = \{(x, y) \mid -1 \le x \le 1, -1 \le y \le 1\}$.

12 Sample Exam Solutions

1. $f(x, y) = x^y$

(a)



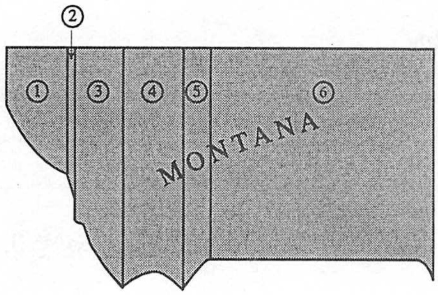
$$\int_1^2 \int_1^2 f(x, y)\, dy\, dx \approx f(1, 1) \cdot \tfrac{1}{2} \cdot \tfrac{1}{2} + f\left(\tfrac{3}{2}, 1\right) \cdot \tfrac{1}{2} \cdot \tfrac{1}{2} + f\left(1, \tfrac{3}{2}\right) \cdot \tfrac{1}{2} \cdot \tfrac{1}{2} + f\left(\tfrac{3}{2}, \tfrac{3}{2}\right) \cdot \tfrac{1}{2} \cdot \tfrac{1}{2}$$
$$= \tfrac{1}{4}\left[1 + \tfrac{3}{2} + 1 + \left(\tfrac{3}{2}\right)^{3/2}\right] \approx \tfrac{1}{4}(5.3375) \approx 1.344$$

(b) This estimate is an underestimate since the function is increasing in the x- and y-directions as x and y go from 1 to 2.

2. (a) $$\int_0^{\pi/2} \int_0^{\pi/2} \frac{1}{(1 + \sin^2 x)(1 + \sin^2 y)}\, dx\, dy = \left(\int_0^{\pi/2} \frac{dx}{1 + \sin^2 x}\right)\left(\int_0^{\pi/2} \frac{dy}{1 + \sin^2 y}\right)$$
$$= \left(\frac{\pi}{2\sqrt{3}}\right)^2 = \frac{\pi^2}{12}$$

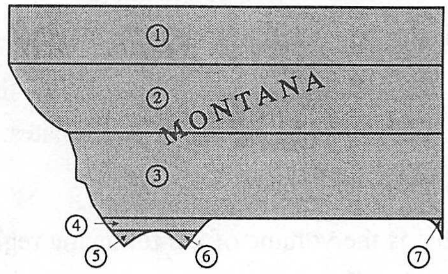
(b) $$\int_0^{\pi/2}\int_0^{\pi/2}\int_{1/(1+\sin^2 x)}^{1/(1+\sin^2 y)} dz\,dx\,dy = \int_0^{\pi/2}\int_0^{\pi/2}\left(\frac{1}{1+\sin^2 y}-\frac{1}{1+\sin^2 x}\right)dx\,dy$$
$$= \int_0^{\pi/2}\int_0^{\pi/2}\left(\frac{1}{1+\sin^2 y}\right)dx\,dy - \int_0^{\pi/2}\int_0^{\pi/2}\left(\frac{1}{1+\sin^2 x}\right)dx\,dy$$
$$= \frac{\pi^2}{4\sqrt{3}} - \frac{\pi^2}{4\sqrt{3}} = 0$$

3. (a)



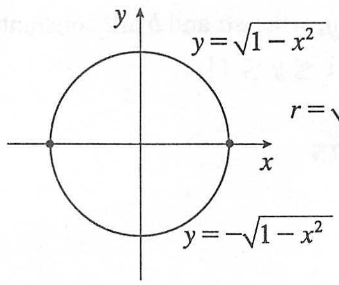
Type I

(b)



Type II

4. $\int_0^{2\pi}\int_0^1 r^2\,dr\,d\theta = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\sqrt{x^2+y^2}\,dy\,dx$



5. $\int_0^{2\pi}\int_1^2 \cos(r^2)\,r\,dr\,d\theta = \pi(\sin 4 - \sin 1)$

6. $x^2 + 2y^2 = 1$

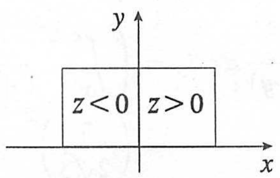
(a) $r^2(\cos^2\theta + 2\sin^2\theta) = r^2(1+\sin^2\theta) = 1,\ r \ge 0$

(b) $$\int_0^{2\pi}\int_0^{1/\sqrt{1+\sin^2\theta}} r\,dr\,d\theta$$

7. Since the parallelepiped has volume 60, we have $\iiint_R 10\,dV = 600$.

$\iiint_R x\,dV = 12\int_0^5 x\,dx = 12\left(\frac{25}{2}\right) = 150$

8. (a) $\int_0^1\int_{-1}^1\int_0^{xy} 1\,dz\,dx\,dy = \int_0^1\int_{-1}^1 xy\,dx\,dy = \int_0^1\left[\frac{1}{2}x^2y\right]_{-1}^{1}dy = 0$. The region between $z = 0$ and $z = xy$ in the first quadrant is above the xy-plane, while a symmetric region is below the xy-plane in the second quadrant.



(b) $\int_0^1 \int_{-1}^1 \int_0^{|xy|} 1\, dz\, dx\, dy = 2\int_0^1 \int_0^1 \int_0^{xy} dz\, dx\, dy = 2\int_0^1 \left[\frac{1}{2}x^2 y\right]_0^1 dy = 2\int_0^1 \frac{1}{2}y\, dy = \frac{1}{2}y^2\Big]_0^1 = \frac{1}{2}$. This is the total volume between $z = 0$ and $z = xy$. Because we take the absolute value, the volumes do not cancel.

9. Since the surface area is 4π, we need to find ϕ so that the area lit is π.
$\pi = \int_0^\phi \int_0^{2\pi} \sin\phi\, d\theta\, d\phi = 2\pi \int_0^\phi \sin\phi\, d\phi = 2\pi\left(-\cos\phi + \cos 0\right)$, so $\frac{1}{2} = 1 - \cos\phi \;\Rightarrow\; \cos\phi = \frac{1}{2}$ $\Rightarrow\; \phi = \frac{\pi}{3}$.

10. (a) $\int_0^1 \int_{-\sqrt{1-(x-1)^2}}^{x} (x+y)\, dy\, dx + \int_1^2 \int_{-\sqrt{1-(x-1)^2}}^{2-x} (x+y)\, dy\, dx$

(b) $\int_{-1}^0 \int_{1-\sqrt{1-y^2}}^{1+\sqrt{1-y^2}} (x+y)\, dx\, dy + \int_0^1 \int_y^{2y} (x+y)\, dx\, dy$. Note that the circular part of the curve is $y = -\sqrt{1-(x-1)^2}$ or $x = 1 \pm \sqrt{1-y^2}$.

11. (a) $\int_0^{\pi/2} \int_0^{1/(\sin\theta+\cos\theta)} r^3 \sin\theta\cos\theta\, dr\, d\theta + \int_{\pi/2}^{\pi} \int_0^{1/(\sin\theta-\cos\theta)} r^3 \sin\theta\cos\theta\, dr\, d\theta$

(b) 0

12. $\displaystyle\iint_R \frac{1}{9-(x^2+y^2)^{3/2}}\, dA$

(a) $\frac{1}{2}(4\pi - \pi) = \frac{3\pi}{2}$

(b) Since the semicircles satisfy $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$, we have on $x^2 + y^2 = 1$,
$\dfrac{1}{9\,(x^2+y^2)^{3/2}} = \dfrac{1}{8}$ and on $x^2 + y^2 = 4$, $\dfrac{1}{9\,(x^2+y^2)^{3/2}} = 1$.

(c) A lower bound is the minimum value times the area, that is, $\frac{1}{8}\cdot\frac{3\pi}{2} = \frac{3\pi}{16}$.
An upper bound is the maximum value times the area, that is, $1 \cdot \frac{3\pi}{2} = \frac{3\pi}{2}$.

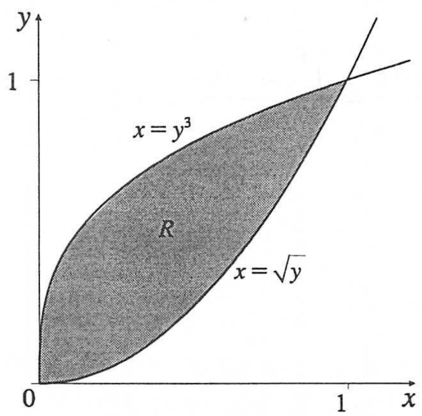
13. (a) $\left\{(r,\theta) \mid 0 \le r \le 1, \frac{\pi}{4} \le \theta \le \frac{7\pi}{4}\right\}$

(b) $\iint_{\text{Pac-Man}} x\, dA = \int_0^1 \int_{\pi/4}^{7\pi/4} r^2 \cos\theta\, d\theta\, dr = \int_0^1 \left[r^2 \sin\theta\right]_{\pi/4}^{7\pi/4} dr = -\sqrt{2}\int_0^1 r^2\, dr = -\frac{\sqrt{2}}{3}$

$\iint_{\text{Pac-Man}} y\, dA = \int_0^1 \int_{\pi/4}^{7\pi/4} r^2 \sin\theta\, d\theta\, dr = \int_0^1 \left[-r^2 \cos\theta\right]_{\pi/4}^{7\pi/4} dr = 0$

14. $\mathbf{F}(u,v) = (u+v)\,\mathbf{i} + (u-v)\,\mathbf{j} + (2u+3v)\,\mathbf{k}$, and so $\mathbf{F}_u = \mathbf{i}+\mathbf{j}+2\mathbf{k}$, $\mathbf{F}_v = \mathbf{i}-\mathbf{j}+3\mathbf{k}$. Then $\mathbf{F}_u \times \mathbf{F}_v = 5\mathbf{i} - \mathbf{j} - 2\mathbf{k}$, and the surface area is $\int_0^1 \int_0^1 |\mathbf{F}_u \times \mathbf{F}_v|\, du\, dv = \int_0^1 \int_0^1 \sqrt{30}\, du\, dv = \sqrt{30}$.

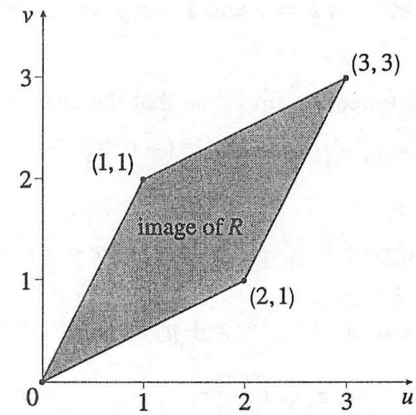
15. (a)



(b) $\int_0^1 \int_{y^3}^{\sqrt{y}} \int_0^{xy} dz\, dx\, dy = \int_0^1 \int_{x^2}^{\sqrt[3]{x}} \int_0^{xy} dz\, dy\, dx$

16. $x = 2u + v,\ y = u + 2v$

(a)

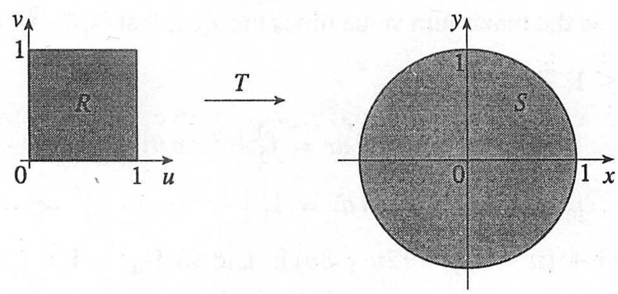


(b) The Jacobian is $\begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$, so

$$\begin{aligned} \iint_S (3x + 2y)\,dA &= \int_0^1 \int_0^1 [3(2u+v) + 2(u+2v)]\,3\,du\,dv \\ &= 3\int_0^1 \left[3u^2 + 3uv + u^2 + 4uv\right]_0^1 dv \\ &= 3\int_0^1 (3 + 3v + 1 + 4v)\,dv = 3\left[4v + \tfrac{7}{2}v^2\right]_0^1 = \tfrac{45}{2} \end{aligned}$$

17. $$\begin{aligned} \int_1^9 \int_0^{\pi/2} \int_{\pi/6}^{\pi/4} \rho^2 \sin\phi\,d\phi\,d\theta\,d\rho &= \tfrac{\pi}{2}\int_1^9 \left[-\rho^2\cos\phi\right]_{\pi/6}^{\pi/4} d\rho = \tfrac{\pi}{2}\int_1^9 \tfrac{\sqrt{3}-\sqrt{2}}{2}\rho^2\,d\rho \\ &= \tfrac{\sqrt{3}-\sqrt{2}}{2}\left[\tfrac{1}{3}\rho^3\right]_1^9 = \tfrac{182}{3}\left(\sqrt{3}-\sqrt{2}\right)\pi \end{aligned}$$

18. (a)



T maps the unit square in the uv-plane to the unit circle in the xy-plane.

(b) The area of S is π.

19. $$f_{\text{ave}} = \frac{\int_{-1}^1 \int_{-1}^1 (ax + by)\,dy\,dx}{\int_{-1}^1 \int_{-1}^1 1\,dy\,dx} = \frac{\int_{-1}^1 2ax\,dx}{4} = 0$$

13 Vector Calculus

13.1 Vector Fields

Suggested Time and Emphasis

1 class Essential material

Transparencies Available

- Transparency 56 (Figure 1, page 917)
- Transparency 57 (Figures 10–12, page 920)
- Transparency 58 (Exercises 11–14, graphs I–IV, page 923)
- Transparency 59 (Exercises 15–18, graphs I–IV, page 923)

Points to Stress

1. Two- and three-dimensional vector fields.
2. Vector fields can either be drawn "scaled," so that the lengths of the vectors are proportional to their magnitudes and the longest vectors in the field have a specified length, or "unscaled," so that the vectors appear at their true lengths.
3. Gradient fields in $\mathbb{R}^2$ and $\mathbb{R}^3$, and their relationships to level curves and surfaces.

Text Discussion

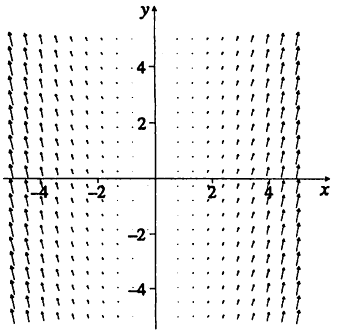
- Why is each vector in the vector field $\mathbf{F}(x,y) = -y\mathbf{i} + x\mathbf{j}$ tangent to the circle centered at the origin through the point (x,y)?
- Let $f(x,y)$ be a function of two variables, with level curves in the plane corresponding to $f(x,y) = k$. How is the gradient vector field ∇f related to these level curves? How does the length of ∇f vary with the spacing of the curves?

Materials for Lecture

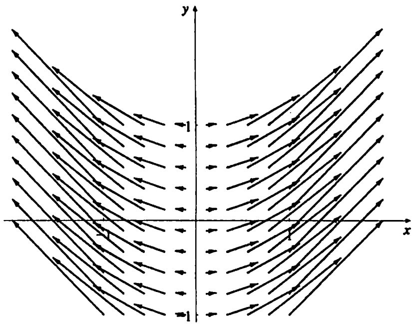
- Discuss various examples of vector fields on physical surfaces, such as wind speed and direction on the Earth, and temperature and altitude gradients.
- Point out that in order for $\mathbf{F}(x,y) = P(x,y)\,\mathbf{i} + Q(x,y)\,\mathbf{j}$ to be continuous at (x,y), both P and Q must be continuous at (x,y). Thus, for example, $\mathbf{F}(x,y) = (x/|x|)\,\mathbf{i} + xy\mathbf{j}$ is not continuous at $(0,0)$ since $P(x,y) = x/|x|$ is not continuous at $(0,0)$. Also define what is meant by a non-vanishing vector field: a vector field in which the zero vector does not appear.

- Show pictures of some interesting vector fields in $\mathbb{R}^2$, such as those shown below, and describe the process of scaling.

1. $\mathbf{F}(x,y) = x\,\mathbf{i} + x^2\,\mathbf{j}$

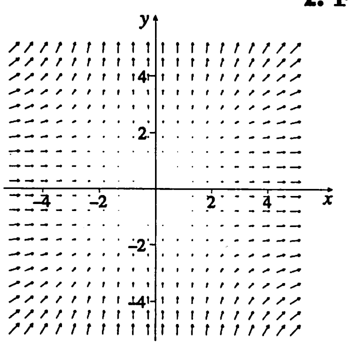


Scaled

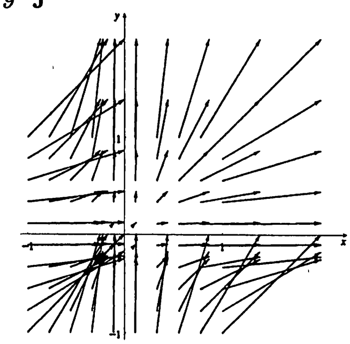


Unscaled

2. $\mathbf{F}(x,y) = x^2\,\mathbf{i} + y^2\,\mathbf{j}$

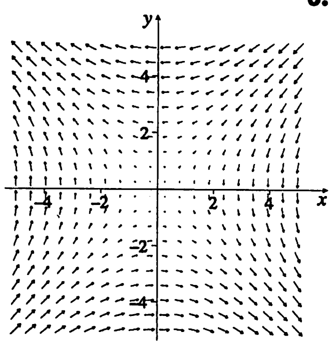


Scaled

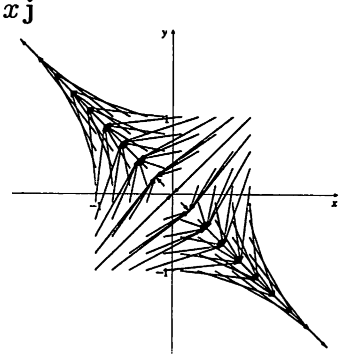


Unscaled

3. $\mathbf{F}(x,y) = -y\,\mathbf{i} - x\,\mathbf{j}$



Scaled



Unscaled

- Describe and sketch some elementary vector fields in $\mathbb{R}^3$:

  1. $\mathbf{F}(x,y,z) = -\frac{x}{2}\mathbf{i} - \frac{y}{2}\mathbf{j} - \frac{z}{2}\mathbf{k}$

  2. $\mathbf{F}(x,y,z) = -\frac{x}{\sqrt{x^2+y^2+z^2}}\mathbf{i} - \frac{y}{\sqrt{x^2+y^2+z^2}}\mathbf{j} - \frac{z}{\sqrt{x^2+y^2+z^2}}\mathbf{k}$, $(x,y,z) \neq (0,0,0)$

  Be sure to indicate what happens to these vector fields near the origin.

- Draw the contour map for $f(x,y) = x^2 + y^2$ and plot the gradient vector field $\nabla f(x,y) = 2x\,\mathbf{i} + 2y\,\mathbf{j}$. Point out how the spacing between the contour curves $f(x,y) = k$ decreases, and $\nabla f(x,y)$ gets longer, as k increases. Explain the connection between spacing and length of $\nabla f(x,y)$.

Workshop/Discussion

- Define the gradient vector field for $f(x,y)$: $\mathbf{F}(x,y) = \nabla f(x,y) = f_x(x,y)\,\mathbf{i} + f_y(x,y)\,\mathbf{j}$. Compute ∇f for $f(x,y) = x^2 + y^2$ and show that the vectors in the gradient field are all orthogonal to the circles $f(x,y) = k$. Then similarly analyze ∇f for $f(x,y) = -x^2 + y$, for which the level curves are the parabolas $y = x^2 + k$.
- Sketch some interesting vector fields in $\mathbb{R}^2$:
  1. $\mathbf{F}(x,y) = x^2\,\mathbf{i} + x^3\,\mathbf{j}$
  2. $\mathbf{F}(x,y) = y^3\,\mathbf{i} + y^2\,\mathbf{j}$
  3. $\mathbf{F}(x,y) = (x+y)\,\mathbf{i} + (x-y)\,\mathbf{j}$. (Plot along the line $y = mx$ for various values of m.)
- The following is a good way to demonstrate continuous vector fields and their flow lines (or streamlines) as in Exercises 33 and 34.

  Have a student point at some other randomly selected student. Now have all the students who are sitting adjacent to the first student point in a direction similar to, but not equal to the first student's direction. Have their neighbors similarly point, until the entire lecture room becomes a continuous vector field. Now start in the middle of the room, with some random student, and walk along the flow line determined by the student-vector field, stressing that at all times you are walking in the direction in which the nearest student is pointing. (If this is too ignoble, have a student do it for you.) Demonstrate that starting at a different initial student can result in an entirely different path. Then challenge the students to try to make a vector field that forces you to walk in a circle, by pointing appropriately. Finally, have them do it again, this time pointing in a random direction, not worrying about their neighbors. Show that it is now (probably) impossible to walk through the hall, because there are points where there isn't a clear direction to follow. Point out that in a true vector field, the speed at which you walk would be determined by the length of the students' arms.

Group Work 1: Sketching Vector Fields

Solutions are included with this group work. We recommend either handing them out to the students at the end of the activity, or displaying them with an overhead projector.

Group Work 2: Gradient Fields and Level Curves

The students should choose obvious points for the level curves for $f(x,y) = \frac{1}{4}x^2 + \frac{1}{9}y^2$ (ellipses), and the level curves for $f(x,y) = \dfrac{y}{x+y}$, $x \neq -y$ (straight lines). For the latter function, note that $\nabla f = \dfrac{-y\,\mathbf{i} + x\,\mathbf{j}}{(x+y)^2}$, and along the level curve $y = \dfrac{k}{1-k}x$, $k \neq 1$, $\nabla f = \dfrac{x}{k-1}\left[k\mathbf{i} + (k-1)\,\mathbf{j}\right]$.

Group Work 3 (Advanced): Points of Calm

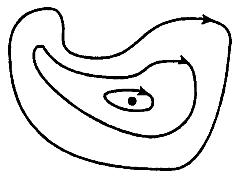
This is a difficult project which tries to show the non-existence of non-vanishing continuous vector fields on the sphere. The first part of this exercise is straightforward, the second is tricky, and the third is intended for a particularly motivated or talented group of students.

Set this activity up by having the students give examples of vector fields over the Earth, such as wind velocity or temperature gradients. Review the definition of a non-vanishing vector field, and give an intuitive idea of what is meant by a continuous vector field. A good example for part 2(a) is the function $2 + \sin(2\pi(x + y))$ or the function $3 + \sin(2\pi x) + \sin(2\pi y)$.

When the students are working on the second part, show them how one can create a torus out of the square by folding the sides together. Have the students figure out what kinds of vector fields on a square become continuous vector fields on a torus. Point out the basic topological idea that the vector field can now be viewed as a tangent vector field, since the torus becomes curved, but the tangent vectors stay "flat".

Part 3 is much harder than part 2. You may simply want to discuss what would happen if you tried to use an argument similar to the argument in part 2, that is, identifying the entire boundary with one point and trying to write a non-constant continuous function which lines up on the boundaries.

Another possible direction is to indicate that the answer to part 3 is "no," but that the proof is actually quite advanced. In an advanced class, you could provide an intuitive argument for the following special case: Assume that the solutions are a collection of nested closed curves, shrinking to a point as in the figure below. Since the solutions don't cross, you can keep on moving to the center point within all the nested closed curves. The vector field must vanish at this point; otherwise, the vector field would not be continuous there.



Conclude by discussing how this result shows that, at any given moment, there is at least one spot on the Earth at which no wind blows.

Homework Problems

Core Exercises: 2, 7, 11–14, 22, 25

Sample Assignment: 2, 6, 7, 11–14, 16, 19, 20, 22, 25, 27, 28, 29–32

Note: Problems 19, 20, 27, and 28 require a CAS.

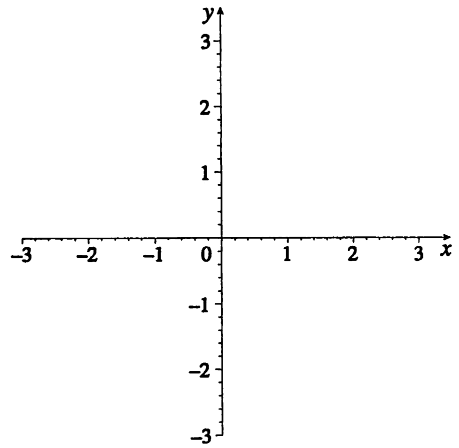
Exercise	C	A	N	G	V
1–10				×	
11–14					×
16					×
19		×		×	×
20		×		×	×

Exercise	C	A	N	G	V
22		×			
25		×		×	
27				×	×
28				×	×
29–32					×

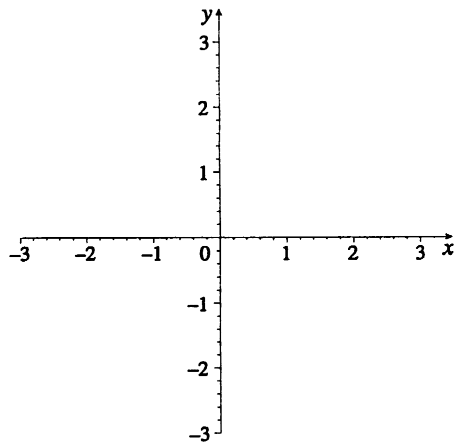
Group Work 1, Section 13.1
Sketching Vector Fields

Sketch each of the following vector fields.

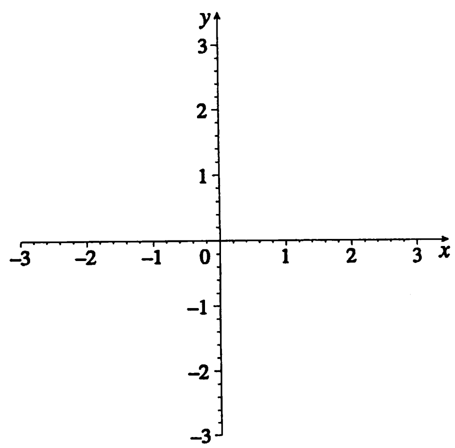
1. $x\,\mathbf{i} + y\,\mathbf{j}$



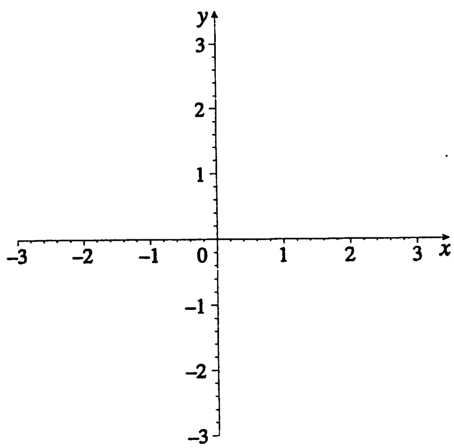
2. $y\,\mathbf{i} - x\,\mathbf{j}$



3. $\dfrac{x\,\mathbf{i} + y\,\mathbf{j}}{(x^2 + y^2)^{1/2}}$

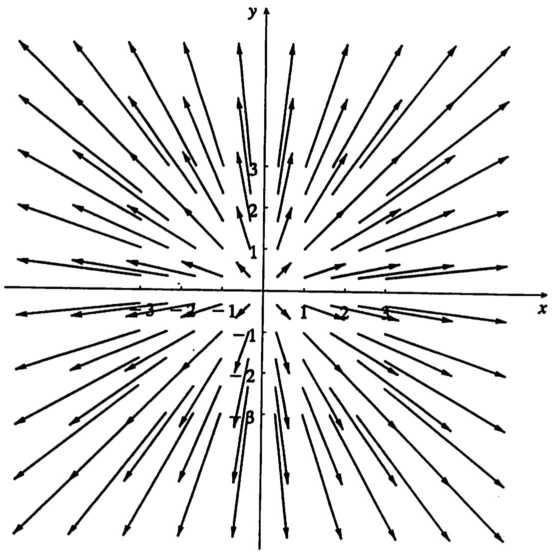


4. $y^2\,\mathbf{i} + x^2\,\mathbf{j}$

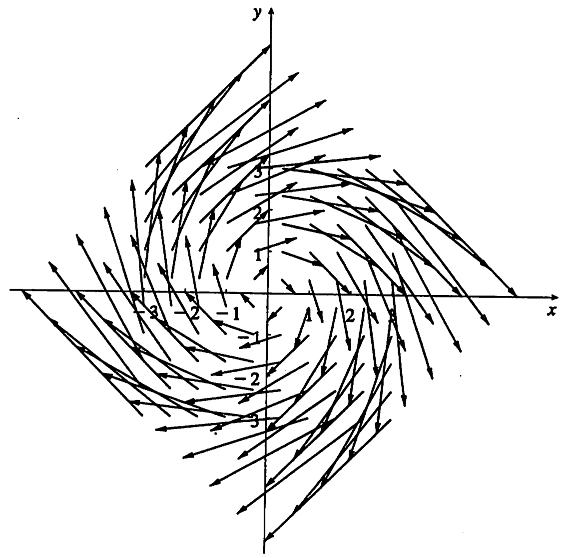


Group Work 1, Section 13.1
Sketching Vector Fields (Solutions)

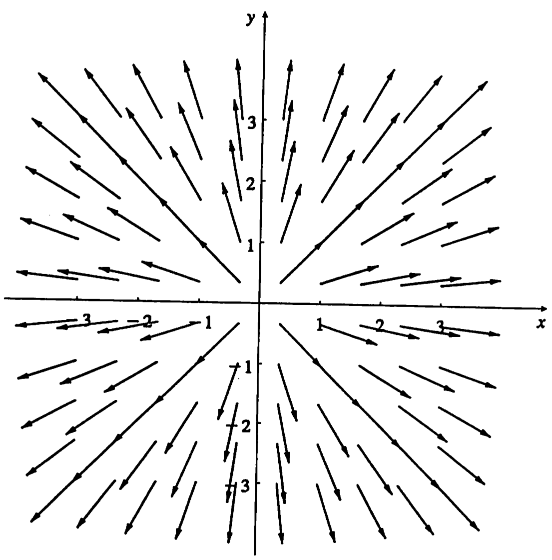
1. $x\,\mathbf{i} + y\,\mathbf{j}$



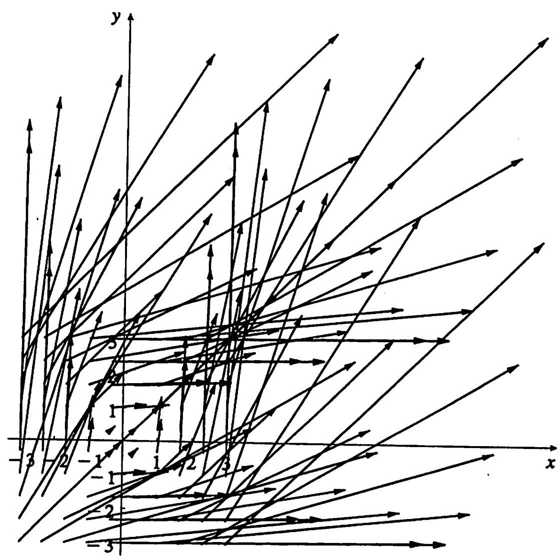
2. $y\,\mathbf{i} - x\,\mathbf{j}$



3. $\dfrac{x\,\mathbf{i} + y\,\mathbf{j}}{(x^2 + y^2)^{1/2}}$



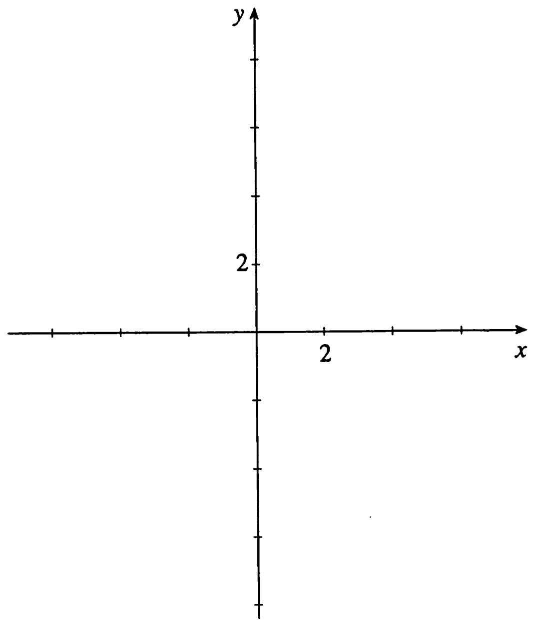
4. $y^2\,\mathbf{i} + x^2\,\mathbf{j}$



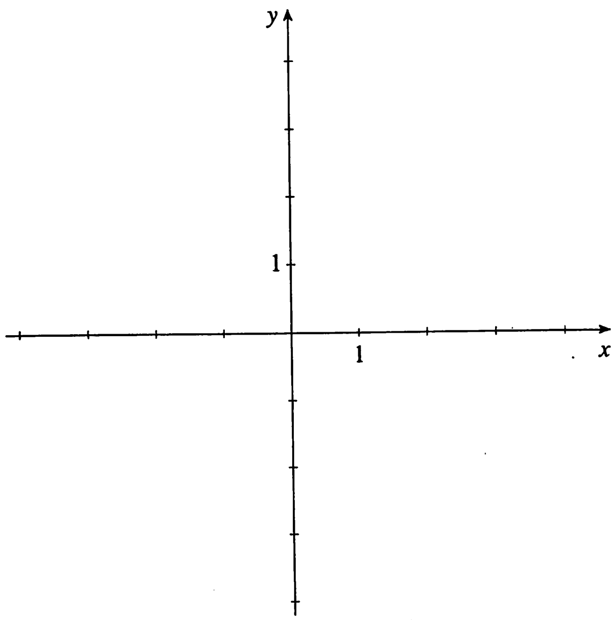
Group Work 2, Section 13.1
Gradient Fields and Level Curves

Compute the gradient fields for the following functions, and draw level curves $f(x, y) = k$ for the indicated values of k. Then sketch the gradient vector field at one or two points on each of these level curves.

1. $f(x, y) = \frac{x^2}{4} + \frac{y^2}{9}$; $k = 1, 2, 4$

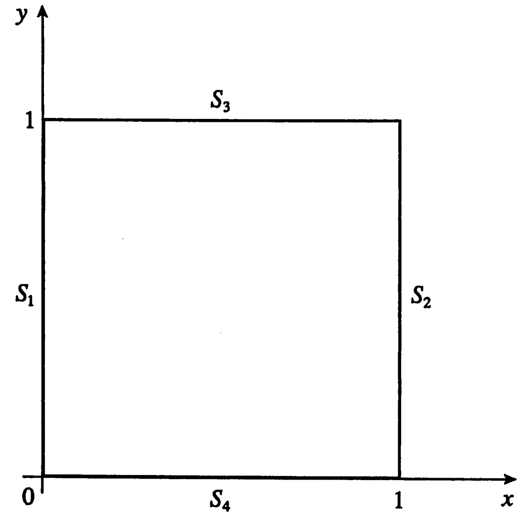


2. $f(x, y) = \frac{y}{x+y}$, $y \neq -x$; $k = \frac{1}{2}, \frac{3}{4}, 2$



Group Work 3, Section 13.1
Points of Calm

1. Draw a non-constant, non-vanishing, continuous vector field on the following unit square:



2. A torus (doughnut) can be obtained from a square by "gluing" the side S_1 to the side S_2, and then "gluing" S_3 to S_4.

(a) Describe a non-constant continuous function $f(x,y)$ such that $f(x,0) = f(x,1)$ for all x, and $f(0,y) = f(1,y)$ for all y. Notice that your function $f(x,y)$ can now be viewed as a continuous function on the torus.

(b) Describe a non-constant, non-vanishing, continuous tangent vector field on the torus.
Hint: Consider $\mathbf{F}(x,y) = f(x,y)(\mathbf{i}+\mathbf{j})$ where f is the function you found in part (a).

3. You have now described a non-constant, non-vanishing, continuous vector field on the torus. Is it possible to draw such a vector field on the unit sphere?

13.2 Line Integrals

Suggested Time and Emphasis

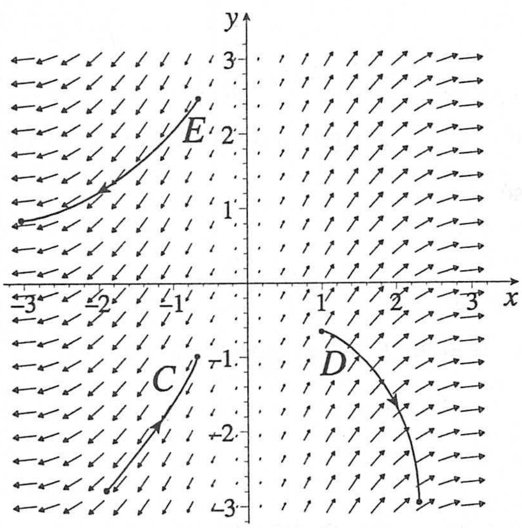
1–$1\frac{1}{2}$ classes   Essential Material

Points to Stress

1. The meaning of the line integral of a scalar function $f(x, y)$ along a curve C.
2. The meaning of $\int P\,dx + Q\,dy$ along a curve C.
3. Vector fields and work: the meaning of $\int_C \mathbf{F} \cdot d\mathbf{r}$.

Text Discussion

- Place the three line integrals $\int_C \mathbf{F} \cdot d\mathbf{r}$, $\int_D \mathbf{F} \cdot d\mathbf{r}$, and $\int_E \mathbf{F} \cdot d\mathbf{r}$ in order from largest (most positive) to smallest (most negative).



Materials for Lecture

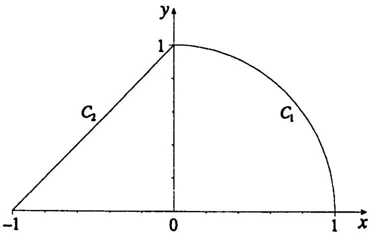
- Discuss the line integral of a scalar function as an extension of the ordinary single integral. Show in some detail why Formulas 3 and 9 actually work. In other words, partition the time interval, and show how the integral is approximated by the sum $\sum_{i=1}^{n} f(x_i^*, y_i^*)\,\Delta s_i$. In particular, show again why

$$\Delta s_i = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sqrt{\left(\frac{\Delta x_i}{\Delta t_i}\right)^2 + \left(\frac{\Delta y_i}{\Delta t_i}\right)^2}\,\Delta t_i$$

For the same formulas in three dimensions, simply show geometrically how the term under the square root involves Δz_i as well.

- Discuss the analytic and geometric interpretations of $\int_C f(x, y)\,ds = \int_a^b f(x(t), y(t))\,\frac{ds}{dt}\,dt$, where s is the arc length along the smooth curve C: $x = x(t)$, $y = y(t)$, $a \le t \le b$. This can be based on the area interpretation of $f(x,y) \geq 0$ along C, as shown in Figure 2 on page 925.

- Work through the following rich example: Consider the function $f(x,y) = x + y$ along the curve $C = C_1 \cup C_2$, where C_1 is parametrized by $x(t) = \cos t$, $y(t) = \sin t$, $0 \leq t \leq \frac{\pi}{2}$, and $C_2$is parametrized by $x(t) = -t$, $y(t) = 1 - t$, $0 \leq t \leq 1$.



$\int_C f(x,y)\,ds = \int_{C_1} (\cos t + \sin t)\sqrt{(-\sin t)^2 + (\cos t)^2}\,dt + \int_{C_2} [-t + (1-t)]\sqrt{(-1)^2 + (-1)^2}\,dt$

- If, instead of arc length, we just want to measure the (signed) distance traveled parallel to the x-axis, we can use the differential $dx = \dfrac{dx}{dt}\,dt$ for $x(t)$, and so

$$\int_C f(x,y)\,dx = \int_C f(x(t), y(t))\frac{dx}{dt}\,dt = \int_a^b f(x(t), y(t))\frac{dx}{dt}\,dt$$

Similarly, if we just want to measure the (signed) distance traveled parallel to the y-axis, we can use the differential $dy = \dfrac{dy}{dt}\,dt$, and so

$$\int_C f(x,y)\,dy = \int_C f(x(t), y(t))\frac{dy}{dt}\,dt = \int_a^b f(x(t), y(t))\frac{dy}{dt}\,dt$$

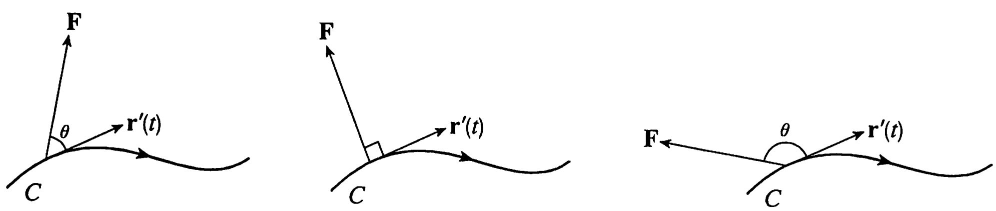
These are called the line integrals along C with respect to x and y. In the example above, we have

$$\int_C f(x,y)\,dx = \int_0^{\pi/2} (\cos t + \sin t)(-\sin t)\,dt + \int_0^1 (1 - 2t)\,dt = -\tfrac{1}{4}(\pi + 1)$$

and similarly $\int_C f(x,y)\,dy = \frac{1}{4}(\pi + 2)$.

- Show that if $\mathbf{F} = (x+y)\,\mathbf{i} + \mathbf{j}$, then the sum of the line integrals $\int_{-1}^{1} [(x+y)\,dx + dy]$ is not independent of path, by using the previous curve $C = C_1 \cup C_2$ and also using the line segment from $(1,0)$ to $(-1,0)$ as a curve C^*, parametrizing C^* as $x(t) = -t$, $y(t) = 0$, $-1 \leq t \leq 1$.

- In analyzing $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (\mathbf{F} \cdot \mathbf{T})\,ds$, show how the sign of $\mathbf{F} \cdot \mathbf{T}$ can be determined visually by looking at the angle θ between $\mathbf{F}$ and $\mathbf{T}$. Since $\mathbf{r}'(t)$ and $\mathbf{T}(t)$ are parallel and point in the same direction, the angle θ is also the angle between $\mathbf{F}$ and $\mathbf{r}'(t)$.



$\cos\theta > 0$ | $\cos\theta = 0$ because **F**' **r**'(t) | $\cos\theta < 0$

Here is an example treated algebraically using the previous curve $C = C_1 \cup C_2$ and $\mathbf{F}(x,y) = (-y+x)\,\mathbf{i} + y\,\mathbf{j}$: Along C_1, $\mathbf{F}\cdot\mathbf{r}'(t) = -\sin^2 t \le 0$, and along C_2, $\mathbf{F}\cdot\mathbf{r}'(t) = 3t-2$, which is positive for $0 \le t < \frac{2}{3}$, zero at $t = \frac{2}{3}$, and negative for $\frac{2}{3} < t \le 1$.

- If $\mathbf{F} = \mathbf{F}(x,y)$ is a vector field defined on a curve C parametrized by $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$, then we define $\int_C \mathbf{F}\cdot d\mathbf{r}$ as the line integral $\int_C (\mathbf{F}\cdot\mathbf{T})\,ds$, where $\mathbf{T}$ is a unit tangent to C at $(x(t), y(t))$. Recalling that $\mathbf{T}'(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$ and $ds = |\mathbf{r}'(t)|\,dt$, and substituting, we get the useful equation

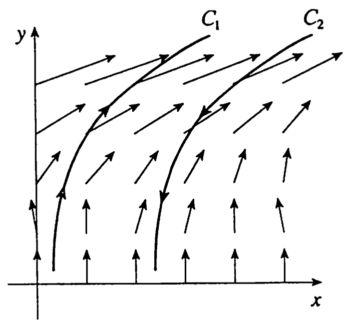
$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \frac{\mathbf{F}\cdot\mathbf{r}'(t)}{|\mathbf{r}'(t)|}|\mathbf{r}'(t)|\,dt = \int_a^b \mathbf{F}\cdot\mathbf{r}'(t)\,dt = \int_a^b F_1\frac{dx}{dt}\,dt + \int_a^b F_2\frac{dy}{dt}\,dt = \int_C F_1\,dx + F_2\,dy$$

If we consider $\mathbf{F}$ to be a force on a particle, then we can interpret $\int_C \mathbf{F}\cdot d\mathbf{r}$ to be the work done by the field $\mathbf{F}$ as the particle moves along the curve C. Similar results hold in $\mathbb{R}^3$.

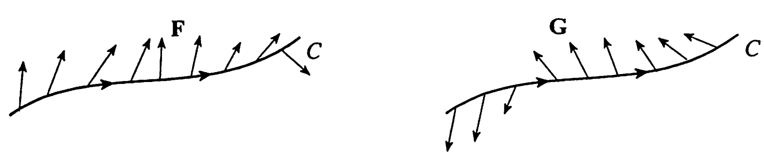
- Refer to Figure 12 and point out the difference between positive and negative work. The work done by the field in the figure is negative, but the work done by *you* is positive.

Workshop/Discussion

- Briefly discuss line integrals in $\mathbb{R}^3$.
- Consider the curve C parametrized by $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j}$, $0 \le t \le 1$. Draw the curve, then describe what is meant by $-\mathbf{r}$ and how replacing t by $1-t$ parametrizes $-\mathbf{r}$ as $(1-t)\,\mathbf{i} + (1-t)^2\,\mathbf{j}$, $0 \le t \le 1$..Explain why the curve given by this parametrization is denoted $-C$.
- Compute the integral $\int_C (x\cos y\,\mathbf{i} + \sin y\,\mathbf{j})\cdot d\mathbf{r}$ along the previous curve, obtaining $\frac{1}{2}\cos 1 + \sin 1 - 1$. Then compute $\int_C (x\cos y\,\mathbf{i} + \sin y\,\mathbf{j})\cdot d(-\mathbf{r}) = \int_0^1 (x\cos y\,\mathbf{i} + \sin y\,\mathbf{j})\cdot\left[(-\mathbf{r})'(t)\right]dt$ which turns out (after a u-substitution) to be $-\left(\frac{1}{2}\cos 1 + \sin 1 - 1\right)$, the negative of the line integral over $\mathbf{r}$. Conclude that $\int_{-C}\mathbf{F}\cdot d\mathbf{r} = -\int_C \mathbf{F}\cdot d\mathbf{r}$, in this case. Give an intuitive arguemnt as to why this is true in general.
- Consider the vector field $\mathbf{F}(x,y)$ and the curves C_1 and C_2 shown below. Explain why $\int_{C_1}\mathbf{F}\cdot d\mathbf{r} > 0$ and $\int_{C_2}\mathbf{F}\cdot d\mathbf{r} < 0$.



- Explain why $\int_C \mathbf{F}\cdot d\mathbf{r} > 0$ and $\int_C \mathbf{G}\cdot d\mathbf{r} < 0$ in the diagram below.



- Compute the line integral of $\mathbf{F} = x^2y\,\mathbf{i} + y^4\,\mathbf{j} + z^6\,\mathbf{k}$ along the curve $\mathbf{r}_1(t)$ given by $x = t^3$, $y = t$, $z = t^2$, $0 \le t \le 1$. Repeat for $\mathbf{r}_2(t)$, given by $x = t$, $y = t^2$, $z = t^3$, $0 \le t \le 1$. Note that both integrals go from $(0,0,0)$ to $(1,1,1)$, but the different paths led to different answers.
- Demonstrate that the value of a line integral is independent of the parametrization by considering the following parametrizations of the unit circle:
  1. $\boldsymbol{\alpha}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$, $0 \le t \le 2\pi$
  2. $\boldsymbol{\beta}(t) = \langle \cos(2t), \sin(2t) \rangle$, $0 \le t \le \pi$
  3. $\boldsymbol{\gamma}(t) = \langle \cos(t^2), \sin(t^2) \rangle$, $0 \le t \le \sqrt{2\pi}$

  Show that for each parametrization, the unit circle is traversed once and the arc length is 2π.

Group Work 1: Fun With Line Integrals

This activity should give students an intuitive feel for line integrals.

Group Work 2: Computing Line Integrals

Group Work 3: Line Integrals over Circles and Ellipses

Group works 2 and 3 anticipate the material on conservative vector fields and independence of path developed in Section 13.3.

Group Work 4: I Sing the Field Electric!

Homework Problems

Core Exercises: 1, 6, 8, 13, 17, 31, 37

Sample Assignment: 1, 4, 6, 7, 8, 13, 17, 22, 25, 31, 34, 37

Note: Exercise 22 requires a CAS.

Exercise	C	A	N	G	V
1–12		×			
13					×
17		×			
22		×		×	
25		×			
31		×			
34		×			
37			×		

Group Work 1, Section 13.2
Fun With Line Integrals

Determine if the following line integrals $\int_C f(x,y)\,ds$ and $\int_C \mathbf{F} \cdot d\mathbf{r}$ are positive, negative or 0 either by graphical analysis or by direct computation.

1. $f(x,y) = \dfrac{y}{x^2+y^2}$; C is the top half of the unit circle, starting at $(-1,0)$ and moving clockwise.

2. $f(x,y) = \dfrac{y}{x^2+y^2}$; C is the bottom half of the unit circle, starting at $(1,0)$ and moving clockwise.

3. $f(x,y) = x^2 \sin \pi y$; C is the curve parametrized by $x = t$, $y = t^3$, $-1 \le t \le 1$.

4. $\mathbf{F}(x,y) = x\mathbf{i} - y\mathbf{j}$; C is the top half of the unit circle, starting at $(1,0)$ and moving counterclockwise.

5. $\mathbf{F}(x,y) = x\mathbf{i} - \dfrac{1}{\sqrt{x}}\mathbf{j}$; C is the part of the parabola $y = x^2$ starting at $(1,1)$ and ending at $(2,4)$.

Group Work 2, Section 13.2
Computing Line Integrals

1. Compute the line integral of $\mathbf{F} = x^2\,\mathbf{i} + y^4\,\mathbf{j} + z^6\,\mathbf{k} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ over the paths $\mathbf{r}_1$: $x = t^3$, $y = t$, $z = t^2, 0 \leq t \leq 1$ and $\mathbf{r}_2$: $x = t$, $y = t^2$, $z = t^3$, $0 \leq t \leq 1$.

2. Compute the line integral of $\mathbf{F} = x^2\,\mathbf{i} + y^4\,\mathbf{j} + z^6\,\mathbf{k}$ over the path $\mathbf{r}_3$: $x = t$, $y = t$, $z = t$, $0 \leq t \leq 1$.

3. Find $g(x, y, z)$ so that $\mathbf{F} = \nabla g$. **Hint:** Assume $g(x, y, z) = h(x) + l(y) + k(z)$.

4. Compute $\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}$. Using the result from Problem 3, can you give a reason why Clairaut's Theorem could have been used to predict your answer?

5. Let $\mathbf{r} = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + 0\mathbf{k}$, $0 \le t \le 2\pi$ be a parametrization of the unit circle. First make a conjecture as to the value of $\int_C \mathbf{F} \cdot d\mathbf{r}$, and then compute it.

Group Work 3, Section 13.2
Line Integrals Over Circles and Ellipses

Let $\mathbf{F}(x, y) = x\,\mathbf{i} - y\,\mathbf{j}$. Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ if

1. C is the unit circle $\mathbf{r}$: $x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$.

2. C lies along the unit circle, starting at $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ and moving counterclockwise to $(-1, 0)$.

3. C lies along the unit circle, starting at $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ and moving clockwise to $(-1, 0)$.

4. C is the ellipse $\mathbf{r}$: $x = \cos t$, $y = 2\sin t$, $0 \le t \le 2\pi$.

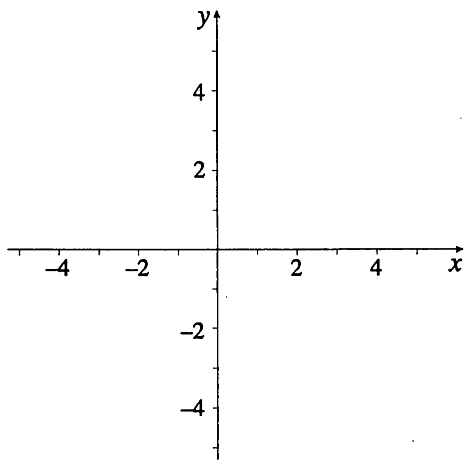
5. C is the ellipse $\mathbf{r}$: $x = 2\cos t$, $y = 3\sin t$, $0 \le t \le 2\pi$.

Group Work 4, Section 13.2
I Sing the Field Electric!

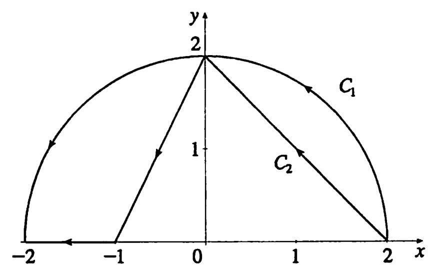
A charge q located at $(0,0)$ creates an electric field at (x,y) given by

$$\mathbf{F} = \frac{K\,(x\,\mathbf{i} + y\,\mathbf{j})}{(x^2+y^2)^{3/2}}, \; K \text{ constant}$$

1. Draw this vector field in the spirit of Figures 4 and 8 in Section 13.1, and then calculate the work required to move a charge around the circle $x^2 + y^2 = 25$ in this field.



2. Calculate the work required to move a charge along the path C_1, the top half of $x^2 + y^2 = 4$.



3. Calculate the work required to move the charge along the path C_2, which consists of three line segments (see above).

13.3 The Fundamental Theorem for Line Integrals

Suggested Time and Emphasis

1–$1\frac{1}{4}$ classes   Essential Material

Points to Stress

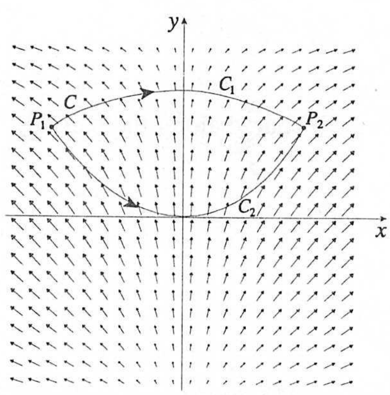
1. The path independence of $\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$ under suitable conditions.
2. The equivalence of path independence to the condition that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed curve C in the domain of $\mathbf{F}$.
3. The equivalence of the following three conditions on a simply-connected domain:
   - Path independence
   - $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j}$ being a conservative vector field ($\mathbf{F} = \nabla f$)
   - $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

Text Discussion

- Is it true that every integral of $\mathbf{F}(x, y) = (x - y)\,\mathbf{i} + (x - 2)\,\mathbf{j}$ is independent of path? Why or why not?
- Where is the Fundamental Theorem of Calculus used in the proof of Theorem 2 on page 936?

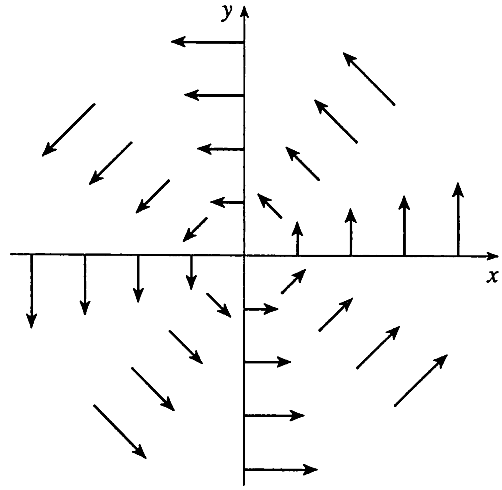
Materials for Lecture

- Give a proof for smooth curves of the Fundamental Theorem for line integrals, such as the one given on page 936.
- Give an outline of the proof that if every integral of $\mathbf{F}$ is independent of path and C is a closed curve in the domain of $\mathbf{F}$, then $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$. First write $C = C_1 \cup -C_2$ with each of C_1 and C_2 starting at P_1 and ending at P_2.



Then $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{-C_2} \mathbf{F} \cdot d\mathbf{r} = -\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$, so $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = 0$.

Explain why integrals of the vector field below are not generally independent of path, and hence that the field is not conservative.



- To show the geometry of conservative vector fields, look at the level sets of the potential function for some conservative vector fields, perhaps using Figure 9 on page 940. Explain why it is plausible that the line integral around a closed path is 0.

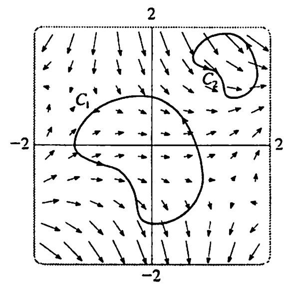


FIGURE 9

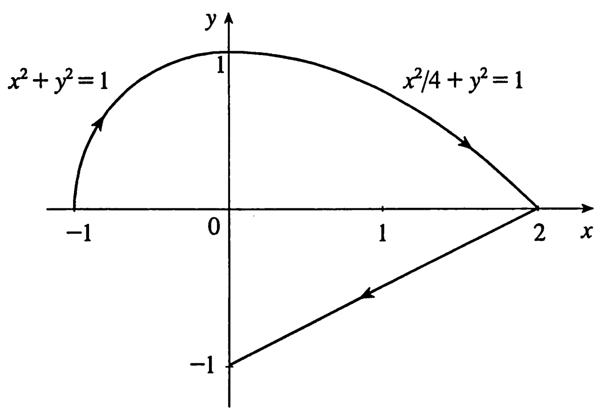
- Go through the following example:
  Consider $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ with $M(x,y) = \sin xy + xy\cos xy$ and $N(x,y) = x^2\cos xy$. Verify that $\partial M/\partial y = \partial N/\partial x$. We want to find a function f such that $\mathbf{F} = \nabla f$. So assume that $M = \partial f/\partial x$. Then $f(x,y) = \int (\partial f/\partial x)\,dx + k(y) = \int M\,dx + k(y) = x\sin xy + k(y)$. Now $N = \partial f/\partial y = x^2\cos xy + k'(y)$. This gives $k'(y) = 0$ and hence $k(y) = K$, a constant. So $f(x,y) = x\sin xy + K$ is a function that satisfies $\nabla f = \mathbf{F}$.

- Repeat the same procedure with $M(x,y) = x^2y$ and $N(x,y) = xy^2$. This time $\partial M/\partial y \neq \partial N/\partial x$, and the procedure doesn't yield a $k(y)$ that works. So when $\mathbf{F}$ is not conservative, we cannot find a function f such that $\nabla f = \mathbf{F}$.

Workshop/Discussion

- Consider $\mathbf{F} = yz^2\,\mathbf{i} + xz^2\,\mathbf{j} + 2xyz\,\mathbf{k}$. Let $M = yz^2$, $N = xz^2$, and $R = 2xyz$. Check that $\partial M/\partial z = \partial R/\partial x$, $\partial M/\partial y = \partial N/\partial x$, and $\partial N/\partial z = \partial R/\partial y$, and so by Clairaut's Theorem, this is possibly a gradient field. Now try the procedure outlined in Materials for Lecture above:
$$f(x,y,z) = \int (\partial f/\partial z)\,dz = \int (2xyz)\,dz = xyz^2 + g(x,y)$$
and now
$$N = \partial f/\partial y = xz^2 + \partial g/\partial y = xz^2, \text{ by definition of } N$$
so $\partial g/\partial y = 0$, meaning that $g(x,y)$ is a function only of x, that is, $g(x)$. Then
$$M = \partial f/\partial x = yz^2 + g'(x) = yz^2$$
Thus $g(x) = k$ is a constant, and $f(x,y,z) = xyz^2 + k$.

- Show students why it is very easy to evaluate $\int_{C_1} \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F} = 2xy\,\mathbf{i} + x^2\,\mathbf{j}$ and C_1 is the curve shown below, by noting that $\mathbf{F}$ is conservative, and then either computing f such that $\nabla f = \mathbf{F}$ or by using the straight line path from $(-1,0)$ to $(0,-1)$.



Group Work 1: Think Before You Compute

In Problem 2, even after recognizing that $\mathbf{F}$ is conservative, the direct path from $(2,0,0)$ to $(0,1,2)$ is not the easiest choice for computations. For example, the path $(2,0,0) \longrightarrow (0,0,0) \longrightarrow (0,0,2) \longrightarrow (0,1,2)$ makes for a simpler calculation.

Group Work 2: Finding the Gradient Fields

Have each group try one of the first three problems below, and give Problem 4 to groups that finish early.
The following vector fields are conservative. Find the function $f(x,y)$ or $f(x,y,z)$ for which they are gradient fields.

1. $\mathbf{F}(x,y) = 3xy^2\,\mathbf{i} + 3x^2y\,\mathbf{j}$
2. $\mathbf{F}(x,y) = y\sin(xy)\,\mathbf{i} + x\sin(xy)\,\mathbf{j}$
3. $\mathbf{F}(x,y) = (2x+y)\,\mathbf{i} + (x+3y^2)\,\mathbf{j}$
4. $\mathbf{F}(x,y,z) = yze^{xyz}\,\mathbf{i} + xze^{xyz}\,\mathbf{j} + xye^{xyz}\,\mathbf{k}$

Extended Group Work 3: The Winding Number

Parts 1–5 of this extended activity can be done independently of the remaining parts, and may be suitable as a challenging in-class group work. The concept of a winding number is completely developed in this activity.

Homework Problems

Core Exercises: 1, 2, 4, 7, 11, 15, 23

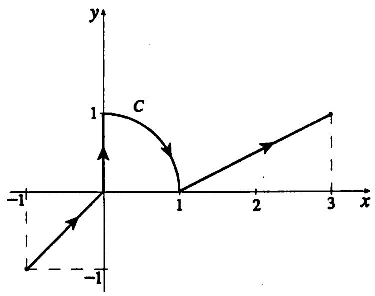
Sample Assignment: 1, 2, 4, 7, 10, 11, 12, 15, 21, 23, 27, 28

Exercise	C	A	N	G	V
1			×		×
2			×		
3–10		×			
11	×	×			×
12		×			

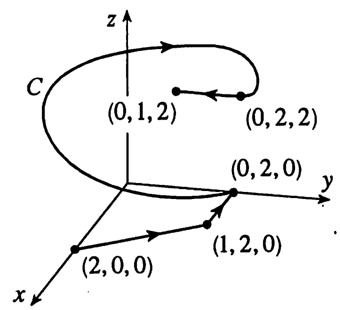
Exercise	C	A	N	G	V
15		×			
21		×			
23	×				×
27		×			
28		×			

Group Work 1, Section 13.3
Think Before You Compute

1. Compute $\int_C (ye^{xy}\,\mathbf{i} + xe^{xy}\,\mathbf{j}) \cdot d\mathbf{r}$ for the curve C shown below.



2. Compute $\int_C \left(yz^2\,\mathbf{i} + xz^2\,\mathbf{j} + 2xyz\,\mathbf{k}\right) \cdot d\mathbf{r}$ for the curve C shown below.



Group Work 3, Section 13.3
The Winding Number

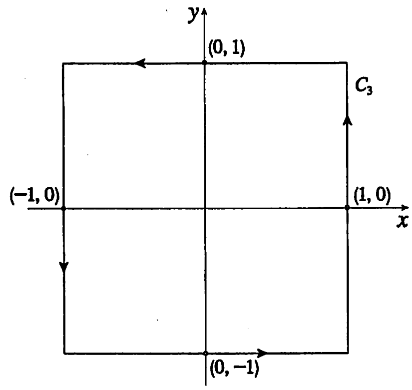
In this activity we consider the vector field $\mathbf{F}(x, y) = -\dfrac{y}{x^2+y^2}\,\mathbf{i} + \dfrac{x}{x^2+y^2}\,\mathbf{j}$.

1. Show that $\dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y} = 0$ where $\mathbf{F}$ is defined.

2. Compute $\oint_{C_1} \mathbf{F} \cdot d\mathbf{r}$ where C_1 is the unit circle centered at the origin, oriented counterclockwise.

3. Compute $\oint_{C_2} \mathbf{F} \cdot d\mathbf{r}$ where C_2 is the circle $(x-2)^2 + y^2 = 1$, oriented counterclockwise.

4. Compute $\oint_{C_3} \mathbf{F} \cdot d\mathbf{r}$ where C_3 is the square shown below.



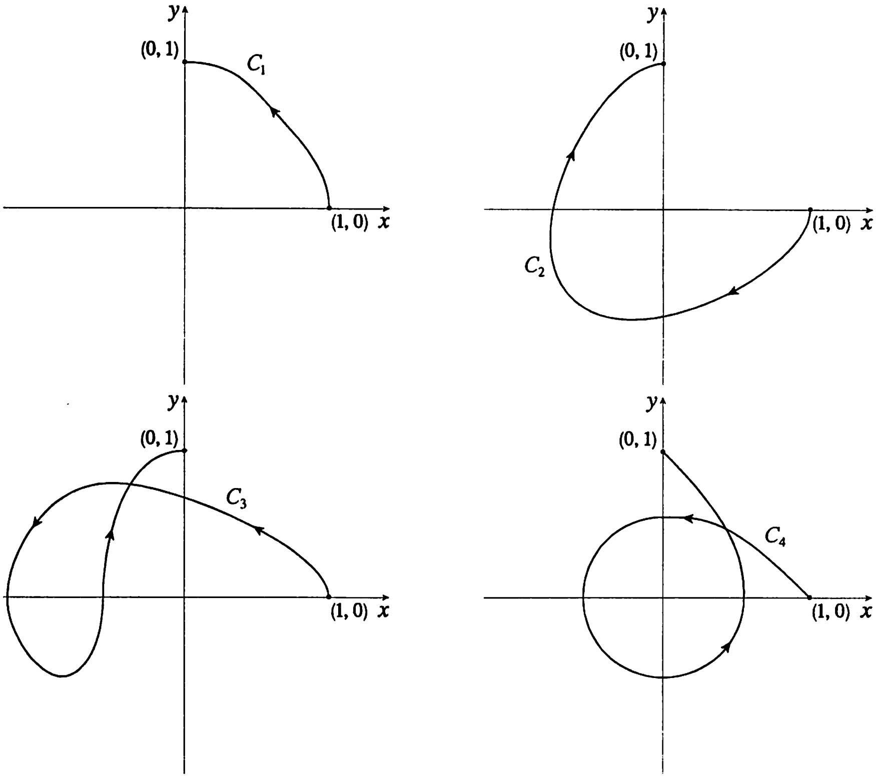
5. For what closed paths will you get zero for $\int_C \mathbf{F} \cdot d\mathbf{r}$, and under what conditions will you get a nonzero answer?

6. One meaning of $\oint_C \mathbf{F} \cdot d\mathbf{r}$ for a closed curve C and any vector field $\mathbf{F}$ is the net circulation of $\mathbf{F}$ around C. Suppose we take an arbitrarily small path around a point (not the origin). What is the net circulation of $\mathbf{F}$ around this small path?

7. What is the net circulation of $\mathbf{F}$ around *any* path which encloses the origin?

8. Letting θ be the angle in polar coordinates for a point (x, y), show that $d\theta = -\dfrac{y}{x^2+y^2}\,dx + \dfrac{x}{x^2+y^2}\,dy$ and hence the vector field $\mathbf{F}$ is the gradient vector field for θ. Conclude that $\oint_C \mathbf{F} \cdot d\mathbf{r} = \theta(B) - \theta(A)$ where C connects the point A to the point B, and thus we can write $\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C d\theta$.

9. Use the previous result to calculate $\oint_C \mathbf{F} \cdot d\mathbf{r}$ for the following paths.



10. The number $\frac{1}{2\pi} \oint_C \mathbf{F} \cdot d\mathbf{r} = \frac{1}{2\pi} \oint_C d\theta$ is called the **winding number** for any closed curve C. It measures the number of times C "winds" counterclockwise around the origin. Find parameterizations for closed paths with winding numbers of 1, 2, -1, and 4.

13.4 Green's Theorem

Suggested Time and Emphasis

1 class Essential Material

Points to Stress

1. The statement of Green's Theorem over a region D with boundary curve $C = \partial D$:

$$\oint_C P\,dx + Q\,dy = \oint_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

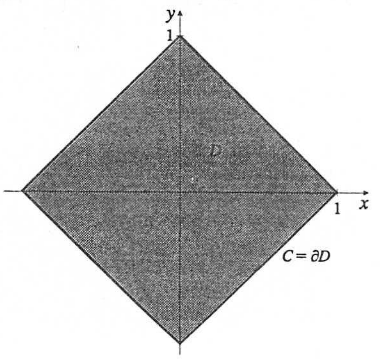
2. The extension of Green's Theorem to domains with holes.
3. The importance of Green's Theorem, in that it allows us to replace a difficult line integration by an easier area integration, or a difficult area integration by an easier line integration.

Text Discussion

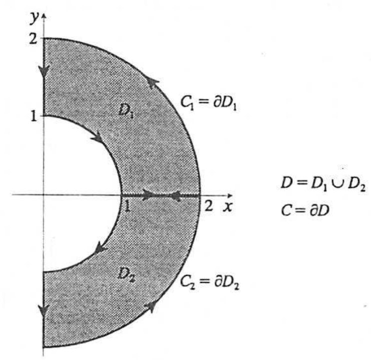
- If we know that $P(x,y) \equiv 0$ and $Q(x,y) \equiv 0$ on the boundary $C = \partial D$ of a region D, what is $\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$?
- How do holes in a region affect $\oint_C P\,dx + Q\,dy$?
- Express $\oint_C y\,dx$ in terms of the region D enclosed by C.

Materials for Lecture

- Have a discussion of terminology: What is meant by "positive orientation". What is meant by "$C = \partial D$"?
- Give a careful statement of Green's Theorem. Indicate that its use is primarily to replace a difficult integral of one type (area or line) with a simpler integral of the other type.
- Go through some rich examples such as the following:



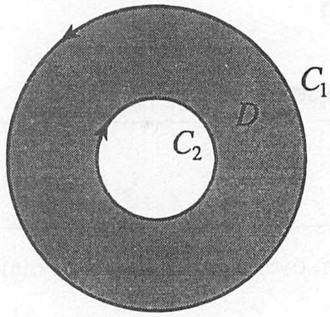
$\oint_C (x^4 + 2y)\,dx + (5x + \sin y)\,dy$
$= \iint_D 3\,dA = 6$, by geometry



$\oint_C (x^2 y)\,dx + (x^3 + 2xy^2)\,dy$
$= \oint_C = \oint_{C_1} + \oint_{C_2} = 2\iint_{D_1} (x^2 + y^2)\,dA$
Evaluating this integral using polar coordinates gives 15π.

- For arbitrary regions D, compute $\oint_{\partial D} -y\,dx + x\,dy$ using Green's Theorem, obtaining twice the area of D.

- Demonstrate Green's Theorem for regions with holes:

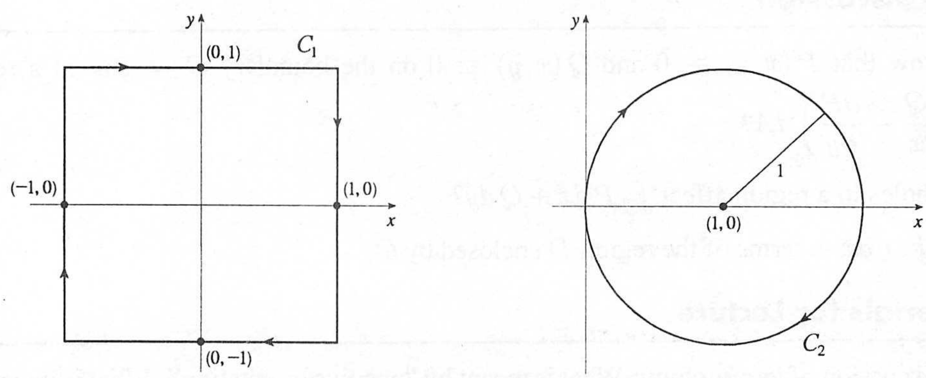


$$C = \partial D = C_1 \cup C_2$$

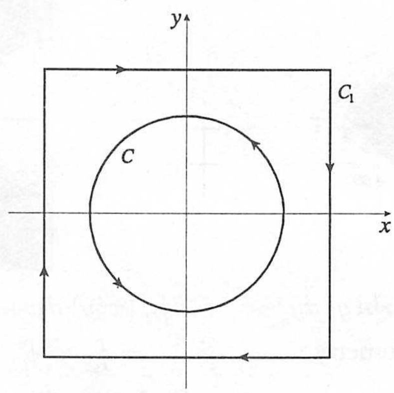
- Use Green's Theorem to set up a line integral to compute the area of the astroid $x^{2/3} + y^{2/3} = 1$.

Workshop/Discussion

- Compute $\oint_C \left(y^2 - 2y + 2xy\right) dx + \left(x^2 + 3x + 2xy\right) dy$ for the following closed curves C_1 and C_2:

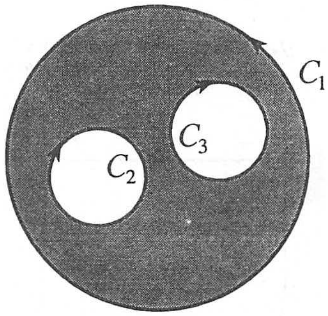


- Compute $\displaystyle\oint_{C_1} \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$ in two ways: (a) by direct computation and (b) using Green's Theorem, where C_1 is the first closed curve shown above. This is equivalent to integrating around the curve C given by the unit circle oriented counterclockwise, since $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = 0$ in the region between the two curves.



- Suppose we know that $P(x,y) \equiv 1$ and $Q(x,y) \equiv 2$ on a boundary circle $C = \partial D$ of radius R. Ask students how to compute $\displaystyle\iint_D \left(\frac{\partial Q}{\partial y} - \frac{\partial P}{\partial x}\right) dA$. Point out that on a closed curve C, $\int_C dx + 2\,dy = 0$.

- If time permits, show Green's Theorem for a region with 2 holes, showing that $C = C_1 \cup C_2 \cup C_3 = \partial D$ needs the positive orientation.



Group Work 1: Using Green's Theorem

These problems may be too hard for students to do without a few hints. Here are some hints that might prove helpful:

Problem 1: Green's Theorem can be used to show that the required integral is equal to $\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{1} 4xy^3 \, dy \, dx$.

Problem 2: Green's Theorem can be used to replace the line integral with

$$\iint_D \left(6 - 3x^2 - 3y^2 + 6\right) dx\, dy = 3 \iint_D \left(4 - x^2 - y^2\right) dx\, dy$$

This integrand is positive until $x^2 + y^2 = 4$, and then remains negative. Thus letting C be the circle of radius 2 gives the maximum value of the integral, namely 24π.

An extension of Problem 2 is given in Problem 2 from Focus on Problem Solving after Chapter 13 (page 989).

Group Work 2: Green's Theorem and the Area of Plane Regions

In Problem 2, the natural parametrization does not give positive orientation, so we need to use $-C$ in its place.

Homework Problems

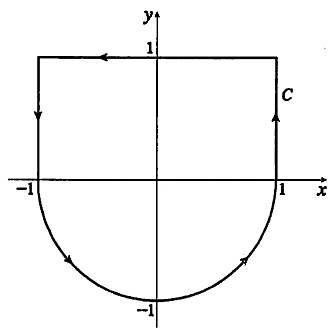
Core Exercises: 1, 4, 7, 11, 17, 19

Sample Assignment: 1, 4, 7, 8, 11, 14, 17, 19, 21, 27

Exercise	C	A	N	G	V
1		×		×	
4		×		×	
7		×		×	
8		×		×	
11		×		×	
14		×		×	
17		×		×	
19		×			
21	×	×			
27	×	×			

Group Work 1, Section 13.4
Using Green's Theorem

1. Compute $\oint_C \left(-\frac{xy^4}{2}\right) dx + \left(x^2y^3\right) dy$, where C is as follows:



2. What simple closed curve $C = \partial D$ gives the maximal value of $\oint_C \left(x^5 - 6y + y^3\right) dx + \left(y^4 + 6x - x^3\right) dy$?

Group Work 2, Section 13.4
Green's Theorem and the Area of Plane Regions

1. Let $C = \partial D$, where the area of the region D is A. Compute $\oint_C (a_1 x + a_2 y + a_3)\,dx + (b_1 x + b_2 y + b_3)\,dy$ where the a_i and b_i are constants.

2. Find the area under one arch of the cycloid with parametric equations $x(t) = 2(t - \sin t)$, $y(t) = 2(1 - \cos t)$, $0 \le t \le 2\pi$.

13.5 Curl and Divergence

Suggested Time and Emphasis

1 class   Essential Material

Points to Stress

1. The definition of curl: $\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F}$
2. If $\mathbf{F}$ has continuous partial derivatives, $\mathbf{F}$ is conservative if and only if $\operatorname{curl}\mathbf{F} = 0$
3. The definition of divergence: $\operatorname{div}\mathbf{F} = \nabla \cdot \mathbf{F}$
4. Physical interpretations of curl and divergence

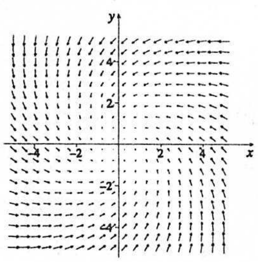
Text Discussion

- What theorem about mixed partial derivatives is used to show that if $\mathbf{F}$ is conservative, then $\operatorname{curl}\mathbf{F} = 0$?
- What do we know about $\operatorname{div}(\operatorname{curl}\mathbf{F})$?

Materials for Lecture

- Make sure to point out that an expression such as $\dfrac{\partial}{\partial x}\mathbf{i}$ refers to an operator which, when applied to a function f, gives a vector, in this case $\dfrac{\partial f}{\partial x}\mathbf{i}$. Thus, ∇ maps a scalar function to its gradient, which is a vector function.
- Given $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$, define $\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix} = \nabla \times \mathbf{F}$ where ∇ is the operator $\dfrac{\partial}{\partial x}\mathbf{i} + \dfrac{\partial}{\partial y}\mathbf{j} + \dfrac{\partial}{\partial z}\mathbf{k}$ and $\operatorname{div}\mathbf{F} = \nabla \cdot \mathbf{F} = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} + \dfrac{\partial R}{\partial z}$.
- Give examples illustrating rotation, and how it is reflected in the curl. Point out that if $\operatorname{curl}\mathbf{F} = 0$, $\mathbf{F}$ is called irrotational.

1. $\mathbf{F} = (-x - y)\,\mathbf{i} + (x - y)\,\mathbf{j} + 0\,\mathbf{k}$. $\nabla \times \mathbf{F} = 2\,\mathbf{k}$, and the vector field is a rotation of each vector $x\,\mathbf{i} + y\,\mathbf{j}$ by $\frac{3}{4}\pi$ coupled with a stretch of $\sqrt{2}$.



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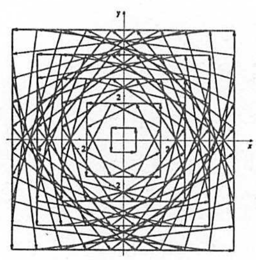

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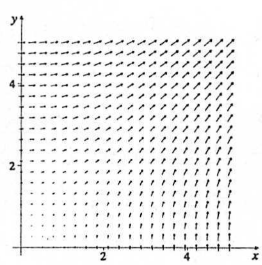
2. $\mathbf{F} = y\,\mathbf{i} + x\,\mathbf{j}$. $\nabla \times \mathbf{F} = 0$, and the vector field has no rotation. Notice that $\mathbf{F}$ is conservative since $\dfrac{\partial M}{\partial y} = 1 = \dfrac{\partial N}{\partial x}$.



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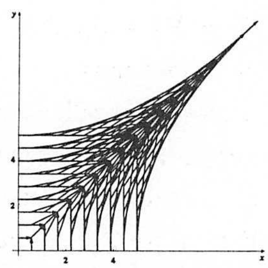

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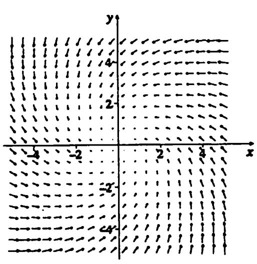
- If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + 0\,\mathbf{k}$ is an extension to $\mathbb{R}^3$ of the two-dimensional field $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$, then $\operatorname{curl}\mathbf{F} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$ and so Green's Theorem can be written as $\oint_{C=\partial D}\mathbf{F}\cdot d\mathbf{r} = \iint_D(\operatorname{curl}\mathbf{F}\cdot\mathbf{k})\,dA$.

- Point out how Clairaut's Theorem shows that $\operatorname{curl}\nabla f = \mathbf{0}$, and also that $\operatorname{div}(\operatorname{curl}\mathbf{F}) = \nabla\cdot(\nabla\times\mathbf{F}) = 0$.

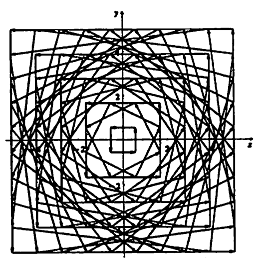
Workshop/Discussion

- The text shows that $\mathbf{F} = \nabla f(x,y,z)$ gives $\operatorname{curl}\mathbf{F} = 0$. Point out that the converse is also true under "normal" circumstances. First use the vector field $\mathbf{F} = yz\,\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k}$ and show that $\mathbf{F} = \nabla(xyz)$. Then note that $\mathbf{F} = y^2e^{xyz}(1+xyz)\,\mathbf{i} + xye^{xyz}(2+xyz)\,\mathbf{j} + x^2y^3e^{xyz}\,\mathbf{k}$ has $\nabla\times\mathbf{F} = 0$. So $\mathbf{F} = \nabla f$. Then $f = \int\frac{\partial f}{\partial z}\,dz + k(x,y) = xy^2e^{xyz} + k(x,y)$. Now compute that $k(x,y) = k$, a constant, and so $f(x,y,z) = xy^2e^{xyz} + k$.

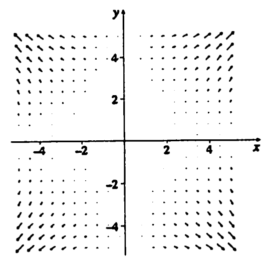
- If $\mathbf{F}_1 = (-x-y)\,\mathbf{i} + (x-y)\,\mathbf{j}$, $\nabla\cdot\mathbf{F}_1 = -1-1 = -2$. Thus the flow is tending to compress and is not diverging anywhere. If $\mathbf{F}_2 = xy^2\,\mathbf{i} + yx^2\,\mathbf{j}$, $\nabla\cdot\mathbf{F}_2 = x^2+y^2$ which is greater than zero if $(x,y)\neq(0,0)$. So in this case the flow is tending to diverge everywhere except at the origin. If $\nabla\cdot\mathbf{F} = 0$, then $\mathbf{F}$ is neither tending to compress nor tending to diverge, and $\mathbf{F}$ is called incompressible. Point out that for any vector field $\mathbf{F}$, $\operatorname{curl}\mathbf{F}$ is incompressible. Note that $\nabla\times\mathbf{F}_1 = 2\mathbf{k}$ and $\nabla\times\mathbf{F}_2 = \mathbf{0}$, both of which are clearly incompressible.



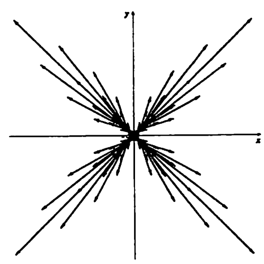
$\mathbf{F}_1$ (scaled)



$\mathbf{F}_1$ (unscaled)



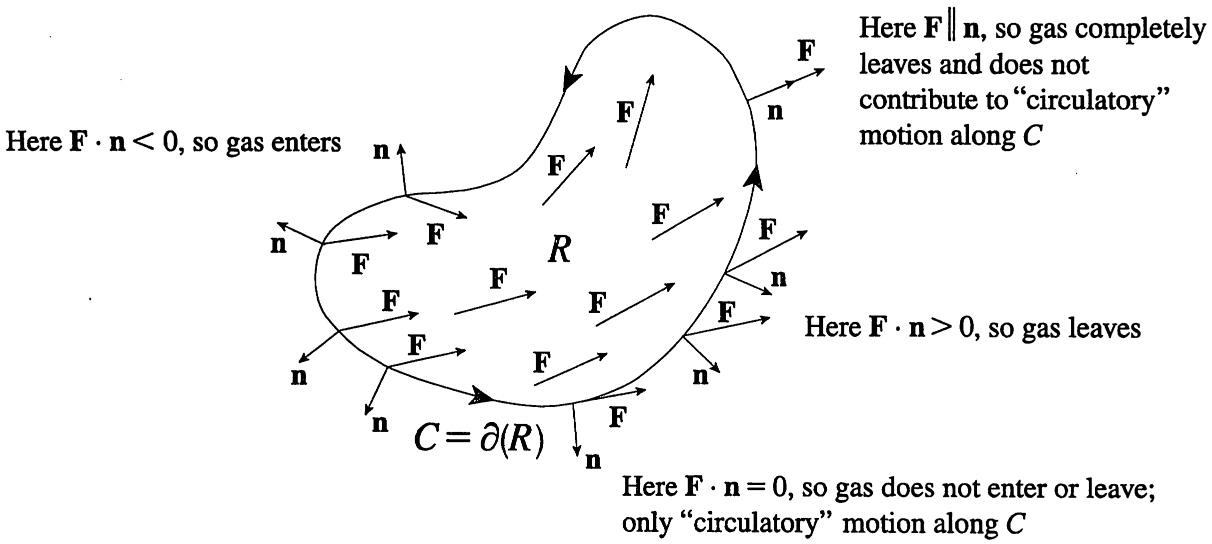
$\mathbf{F}_2$ (scaled)



$\mathbf{F}_2$ (unscaled)

- State the divergence form of Green's Theorem:
If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$, then $\oint_{C=\partial D}(\mathbf{F}\cdot\mathbf{n})\,ds = \iint_D \operatorname{div}\mathbf{F}\,dA$.
The following is a physical interpretation of the theorem. Picture a gas in a thin box, all of whose particles are moving parallel to the xy-plane. Suppose that we can approximate the box by a plane, and consider a region R in the plane with boundary $C = \partial R$. At any point (x,y), if $\mathbf{F}(x,y)$ represents the velocity vector of the gas, then $\operatorname{div}\mathbf{F}(x,y)$ measures the net movement from (x,y). By summing up (integrating) $\operatorname{div}\mathbf{F}(x,y)$ over the region R, we get the net change in the amount of gas contained in R. But another way to measure the net change is to stand on C, and measure how much gas leaves at each point. Here you need the normal component $\mathbf{F}\cdot\mathbf{n}$ of $\mathbf{F}$ to C, where $\mathbf{n}$ is a unit normal to C. This is precisely another

statement of Green's Theorem, using div $\mathbf{F}(x, y)$.



- Do Exercise 33 to illustrate the relationships between curl $\mathbf{F}$ and rotations.

Group Work 1: Gradient Fields Revisited

Problem 2 shows that the result of Problem 1(b) is always true. Problem 2 is a somewhat abstract exercise, suitable for more advanced students.

Group Work 2: Divergence and Curl

Group Work 3: An Essential, Incompressible Fluid

Homework Problems

Core Exercises: 1, 5, 7, 10, 13, 19, 20

Sample Assignment: 1, 5, 7, 8, 10, 13, 16, 19, 20, 23, 24, 26

Exercise	C	A	N	G	V
1		×			
5		×			
7				×	
8				×	
10	×	×			
13		×			

Exercise	C	A	N	G	V
16		×			
19		×			
20		×			
23		×			
24		×			
26		×			

Group Work 1, Section 13.5
Gradient Fields Revisited

1. Let $\mathbf{F} = -2x\,\mathbf{i} - 3y\,\mathbf{j} + 5z\,\mathbf{k}$.

(a) Compute $\nabla \cdot \mathbf{F}$ and give a geometric description of $\mathbf{F}$.

(b) Is $\mathbf{F}$ a gradient vector field? If so, find $f(x, y, z)$ such that $\mathbf{F} = \nabla f$.

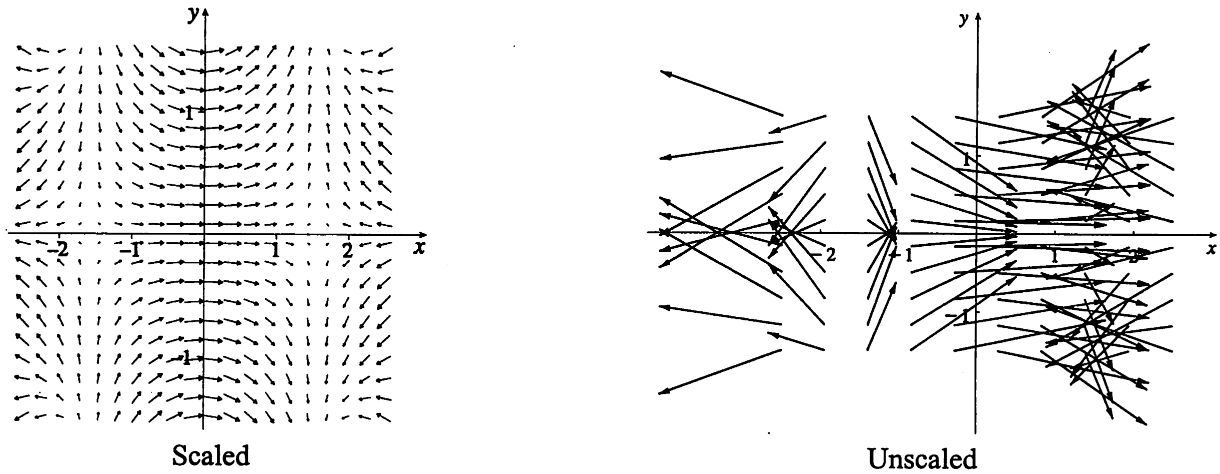
2. Let $\mathbf{F} = P(x)\,\mathbf{i} + Q(y)\,\mathbf{j} + R(z)\,\mathbf{k}$.

(a) Is $\mathbf{F}$ always a gradient vector field?

(b) Explain how you would find $f(x, y, z)$ such that $\mathbf{F} = \nabla f$, if you had explicit functions $P(x)$, $Q(y)$, and $R(z)$.

Group Work 2, Section 13.5
Divergence and Curl

Consider the vector field $f(x, y) = 2\cos x\,\mathbf{i} + \sin xy\,\mathbf{j}$ shown below.



1. Find formulas for the divergence and curl of f.

2. Show that the divergence is 0 everywhere along the y-axis. How is this apparent in the graph?

3. Find the divergence at $\left(\frac{\pi}{4}, 1\right)$, $\left(-\frac{\pi}{4}, 1\right)$, $\left(\frac{\pi}{4}, -1\right)$, and $\left(-\frac{\pi}{4}, -1\right)$. How can the signs of the answers be seen in the graph?

4. Find the curl at $\left(\frac{\pi}{3}, 1\right)$ and at $\left(\frac{2\pi}{3}, 1\right)$. Relate the sign difference in your answers to the direction of the curl.

Group Work 3, Section 13.5
An Essential, Incompressible Fluid

Water is an essentially incompressible fluid, that is, the divergence of a velocity field representing a flow of water is 0. For each of the following vector fields, compute $\nabla \cdot \mathbf{F}$ and determine if $\mathbf{F}$ could represent the velocity vector field for water flowing. Then compute curl $\mathbf{F}$ and describe the axis of rotation (direction of the curl) of the fluid at the origin and at $(1, 1, 1)$.

1. $\mathbf{F}(x, y, z) = xy\,\mathbf{i} + xz\,\mathbf{j} - yz\,\mathbf{k}$

2. $\mathbf{F}(x, y, z) = (2x - y)\,\mathbf{i} + (2z - y)\,\mathbf{j} + (2x - z)\,\mathbf{k}$

3. $\mathbf{F}(x, y, z) = \dfrac{1}{y^2 + z^2}\,\mathbf{i} - \dfrac{2xy}{(y^2 + z^2)^2}\,\mathbf{j} - \dfrac{2xz}{(y^2 + z^2)^2}\,\mathbf{k}$

13.6 Surface Integrals

Suggested Time and Emphasis

$1\text{–}1\frac{1}{4}$ classes   Essential Material

Points to Stress

1. The definition of the surface integral of a scalar function $f(x, y, z)$ viewed as an extension of the surface area integral.

2. The intuitive idea of an oriented surface with orientation given by a unit normal vector. The concept of positive orientation.

3. The surface integral of a vector field over an oriented surface

Text Discussion

- What do we know about the unit normal vector to a closed surface if that surface has positive orientation?
- Give an intuitive explanation as to why it isn't possible to choose an orientation for the Möbius strip.

Materials for Lecture

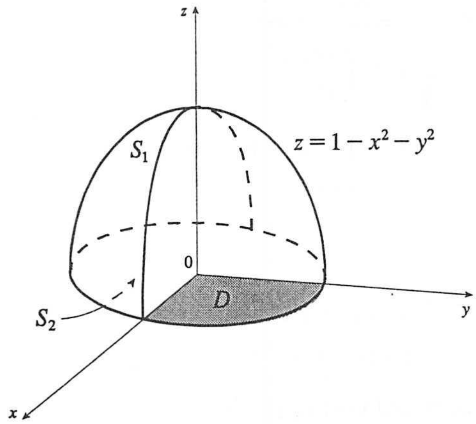
- Describe the meaning of the surface integral $\iint_S f(x, y, z)\, dS = \iint_D f(\mathbf{r}(u, v))\, |\mathbf{r}_u \times \mathbf{r}_v|\, dA$ for $f(x, y, z)$ defined over the parametric surface $S = \mathbf{r}(u, v)$, $(u, v) \in D$.
- If S is given by $z = g(x, y)$, show how the surface integral $\iint_S f(x, y, z)\, dS$ becomes

$$\iint_D f(x, y, g(x, y)) \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}\, dA$$

- Do an extended example such as the following: Let $S = S_1 + S_2$ as in the figure (S_2 is the unit disk in the xy-plane). We want to compute $\iint_S xy\, dS = \iint_{x^2+y^2\le 1} xy\sqrt{1 + 4x^2 + 4y^2}\, dA + \iint_{x^2+y^2\le 1} xy\, dA$ [here $g(x, y) = 1 - x^2 - y^2$ for the first integral, $g(x, y) = 0$ for the second.] Using polar coordinates, we compute that this integral is $0 + 0 = 0$. We now use $f(x, y) = |xy|$ and some symmetry arguments. If the shaded region is D, then the surface integral becomes

$$4\left(\iint_D xy\sqrt{1 + 4x^2 + 4y^2}\, dA + \iint_D xy\, dA\right)$$

which is not 0.



- Give examples of oriented surfaces with upward orientation and closed surfaces with positive (outward) orientation (indicated by unit normals). For example, the paraboloid $z = x^2+y^2$ has $\mathbf{N} = -2x\,\mathbf{i}-2y\,\mathbf{j}+\mathbf{k}$. The upward unit normal is $\mathbf{n} = \dfrac{\mathbf{N}}{|\mathbf{N}|}$. At $(1,1,2)$, $\mathbf{n} = \frac{1}{3}(-2\,\mathbf{i} - 2\,\mathbf{j} + \mathbf{k})$. Notice that this vector points inward to the paraboloid. The outward unit normal is $\mathbf{n}_1 = -\mathbf{n}$. Use the ellipsoid $\dfrac{x^2}{2} + \dfrac{y^2}{3} + z^2 = 1$ for a positively-oriented closed surface.

- Return to the ellipsoid $\dfrac{x^2}{2} + \dfrac{y^2}{3} + z^2 = 1$. Parametrize the surface by $x = \sqrt{2}\sin u \sin v$, $y = \sqrt{3}\cos u \sin v$, $z = \cos v$. Then $\mathbf{r}_u = \dfrac{\partial x}{\partial u}\mathbf{i} + \dfrac{\partial y}{\partial u}\mathbf{j} + \dfrac{\partial z}{\partial u}\mathbf{k} = \sqrt{2}\cos u \sin v\,\mathbf{i} - \sqrt{3}\sin u \sin v\,\mathbf{j}$ and $\mathbf{r}_v = \dfrac{\partial x}{\partial v}\mathbf{i} + \dfrac{\partial y}{\partial v}\mathbf{j} + \dfrac{\partial z}{\partial v}\mathbf{k} = \sqrt{2}\sin u \cos v\,\mathbf{i} - \sqrt{3}\cos u \cos v\,\mathbf{j} - \sin v\,\mathbf{k}$. The unit normal vector $\mathbf{n} = \dfrac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}$ is difficult to compute in general. However, it is reasonable to compute at certain points; the point $\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$ in uv-space gives the point $\left(\frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{2}, \frac{1}{\sqrt{2}}\right)$ on the ellipsoid, and there we have outward unit normal $\mathbf{n} = \frac{1}{\sqrt{17}}\left(\sqrt{3}\,\mathbf{i} + \sqrt{2}\,\mathbf{j} + 2\sqrt{3}\,\mathbf{k}\right)$.

- Give examples of surface integrals $\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D \mathbf{F}\cdot(\mathbf{r}_u \times \mathbf{r}_v)\,dA$ for a vector field $\mathbf{F}$ over the parametric surface $S = \mathbf{r}(u,v)$ where $(u,v) \in D$. For example, letting $\mathbf{F} = y^2\,\mathbf{i} - z^2\,\mathbf{j} + \mathbf{k}$ over the ellipsoid $\dfrac{x^2}{2} + \dfrac{y^2}{3} + z^2 = 1$ gives

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iint_{0 \le u,v \le 2\pi} \left(3\cos^2 u \sin^2 v\,\mathbf{i} - \cos^2 v\,\mathbf{j} + u\,\mathbf{k}\right)\cdot(\mathbf{r}_u \times \mathbf{r}_v)\,dA \\ &= \iint_{0 \le u,v \le 2\pi} \left(3\sqrt{3}\sin u \cos^2 u \sin^4 v - \sqrt{2}\cos u \cos^2 v \sin^2 v + \sqrt{6}\sin v \cos v\right)dA\end{aligned}$$

This integral will be easy to evaluate when we learn the Divergence Theorem.

Workshop/Discussion

- Point out that the general unit normal vector for a parametrized surface $S = \mathbf{r}(u,v)$ is $\dfrac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}$, and for the surface $z = g(x, y)$ becomes $\dfrac{-\dfrac{\partial g}{\partial x}\mathbf{i} - \dfrac{\partial g}{\partial y}\mathbf{j} + \mathbf{k}}{\sqrt{1 + \left(\dfrac{\partial g}{\partial x}\right)^2 + \left(\dfrac{\partial g}{\partial y}\right)^2}}$.

- Consider $\iint_S \mathbf{F} \cdot d\mathbf{S}$ with $\mathbf{F} = y^2\,\mathbf{i} - z^2\,\mathbf{j} + \mathbf{k}$, where S is the piece of the paraboloid $z = x^2 + y^2$ above the unit disk. Perhaps just set up the integral

$$\begin{aligned}\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_{x^2+y^2\le 1} \left[-2xy^2 - \left(x^2+y^2\right)(-2y) + 1\right] dA \\ &= \iint_{x^2+y^2\le 1} \left(2x^2y - 2xy^2 + 2y^3 + 1\right) dA\end{aligned}$$

  and note that the answer is π, since the first three terms integrate to zero by symmetry.
- Compute both the upward and outward unit normals to the closed surface $S = S_1 + S_2$, where S_1 is the piece of the plane $3x + 2y + 4z = 0$ inside the sphere $x^2 + y^2 + z^2 = 4$, and S_2 is the part of that sphere above S_1. It is important to be able to set up these unit normal vectors for Stokes' Theorem and the Divergence Theorem.
- Let $\mathbf{F} = 5x\,\mathbf{i} + 3y\,\mathbf{j} + 2z\,\mathbf{k}$ and let S be the surface $x^2 + y^2 + z^2 = 4$. Compute $\iint_S (\mathbf{F} \cdot \mathbf{n})\, dS$, where $\mathbf{n}$ is the outward unit normal vector.

Group Work 1: Up and Out

Group Work 2: The Flux of a Vector Field

Homework Problems

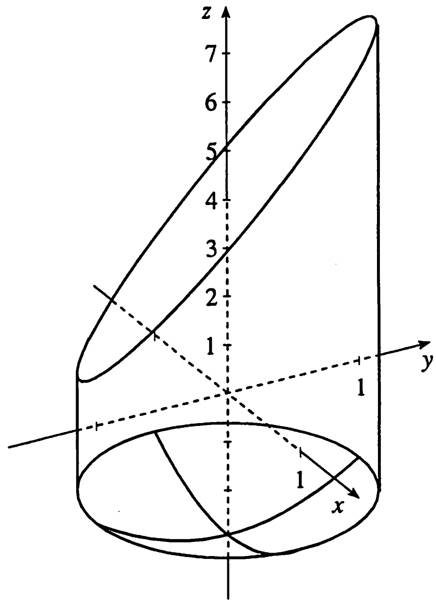
Core Exercises: 1, 7, 18, 25, 38

Sample Assignment: 1, 6, 7, 12, 13, 18, 22, 25, 34, 38

Exercise	C	A	N	G	V
1			×		
5–18		×			
22		×			
25		×			
34		×			
38		×			

Group Work 1, Section 13.6
Up and Out

1. Consider the piecewise smooth surface $S = S_1 \cup S_2 \cup S_3$ which is bounded on top by S_1: $z = 2x + 3y + 4$, on the bottom by S_3: $z = x^2 + y^2 - 3$ and on the side by S_2: $x^2 + y^2 = 1$ (see picture below).



(a) Compute unit normal vector fields $\mathbf{n}_1$, $\mathbf{n}_2$, and $\mathbf{n}_3$ that are outward on S_1, S_2, and S_3.

(b) Compute unit normal vector fields $\mathbf{m}_1$, $\mathbf{m}_2$, and $\mathbf{m}_3$ that are upward on S_1 and S_3, and inward on S_2.

2. Now consider the surface $S = S_1 \cup S_2$ where S_1 is given by $z = x^4 + y^4$, and S_2 is the piece of the plane $z = x + 2$ inside S_1.

(a) Compute unit normal vector fields $\mathbf{n}_1$ and $\mathbf{n}_2$ that are upward everywhere.

(b) Compute unit normal vector fields $\mathbf{m}_1$ and $\mathbf{m}_2$ that are outward everywhere.

(c) If we wanted to "walk" around the intersecting curve in a counterclockwise orientation, which choices of the surface normal vector fields on $\mathbf{S}_1$ and $\mathbf{S}_2$ are both consistent with this assignment of orientation?

Group Work 2, Section 13.6
The Flux of a Vector Field

Consider the sphere $x^2 + y^2 + z^2 = R^2$ as a level surface of the function $G(x, y, z) = x^2 + y^2 + z^2$.

1. Compute the gradient $\nabla G(x, y, z)$ to this surface.

2. Does $\nabla G(x, y, z)$ point inward or outward from the surface of the sphere?

3. Compute an outward unit normal vector $\mathbf{n}$ to the sphere.

Now consider the vector field $\mathbf{F}(x, y, z) = \dfrac{1}{(x^2 + y^2 + z^2)^{3/2}}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$.

4. Where is $\mathbf{F}$ defined?

5. Compute the flux $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S (\mathbf{F} \cdot \mathbf{n})\, dS$, where S is the sphere $x^2 + y^2 + z^2 = R^2$.

13.7 Stokes' Theorem

Suggested Time and Emphasis

1 class   Essential Material

Points to Stress

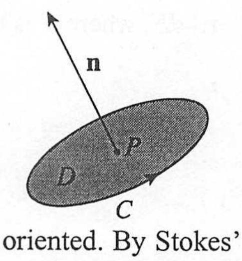
1. The statement of Stokes' Theorem
2. The connection between the curl and the circulation of a velocity field

Text Discussion

- Why can Stokes' Theorem be regarded as a three-dimensional version of Green's Theorem?
- Is it possible for a closed oriented curve C to be the boundary of more than one smooth oriented surface?

Materials for Lecture

- Stress the meaning of oriented smooth surfaces and bounding simple closed curves (with the notation $C = \partial S$). Use the hemisphere $x^2 + y^2 + z^2 = 1$, $z \geq 0$ and the top half of the ellipsoid $x^2 + y^2 + \frac{z^2}{4} = 1$ to illustrate that a closed curve (here the circle $x^2 + y^2 = 1$) can be the boundary of many oriented smooth surfaces.
- State Stokes' Theorem in the case where S is the surface $z = f(x, y)$ with upper unit normal $\mathbf{n}$ and boundary curve $C = \partial S$.
- Verify Stokes' Theorem for $\mathbf{F} = z^2 y\,\mathbf{i} + 2x\,\mathbf{j} + x^2 y z^3\,\mathbf{k}$ on S, where S is the top half of the sphere $x^2 + y^2 + z^2 = 4$. Obtain $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \oint_{C=\partial S} \mathbf{F} \cdot d\mathbf{r}$, where C: $x^2 + y^2 = 4$, or $\oint_{C=\partial S} z^2 y\,dx + 2x\,dy + x^2 y z^3\,dz = \oint_{x^2+y^2=4} 2x\,dy = 2\,(\text{area of a circle of radius } 2) = 8\pi$, by Green's Theorem. Also, if S is the top half of the ellipsoid $\frac{x^2}{4} + \frac{y^2}{4} + \frac{z^2}{9} = 1$, then we still get $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = 8\pi$, since $\partial S = C$ is the same circle, $x^2 + y^2 = 4$.
- The following is an intuitive justification of why the curl is a measure of circulation per unit area. Let $\mathbf{v} = \mathbf{v}(x, y, z)$ be the velocity of a fluid flow. Define the circulation of $\mathbf{v}$ around a circle C as $\oint_C (\mathbf{v} \cdot \mathbf{T})\,ds$. Point out that for velocities of a given magnitude, the circulation measures the extent to which $\mathbf{v}$ maintains the direction of the unit tangent vector $\mathbf{T}$, which is to say the extent to which the flow is rotating in the direction of C. Now take a point P within the flow, and let D be a very small disk centered at P with unit normal $\mathbf{n}$ at P.

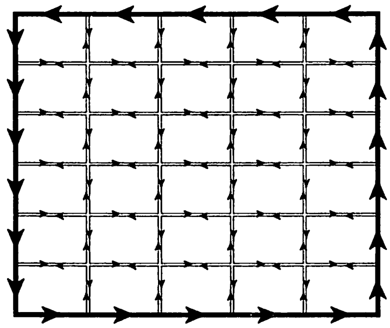


Let C be the boundary of D, positively oriented. By Stokes' Theorem, the circulation of $\mathbf{v}$ around C is approximately equal to the average $\mathbf{n}$-component of $\operatorname{curl} \mathbf{v}$ on D times the area of D. It follows that the average n-component of curl v on D equals

$$\frac{\text{the circulation of } v \text{ around } C}{\text{the area of } D}$$

Now vary D by letting the radius shrink to zero. This process describes at each point P the component of curl $\mathbf{v}$ in the direction of $\mathbf{n}$ as the circulation of $\mathbf{v}$ per unit area in the plane normal to $\mathbf{n}$.

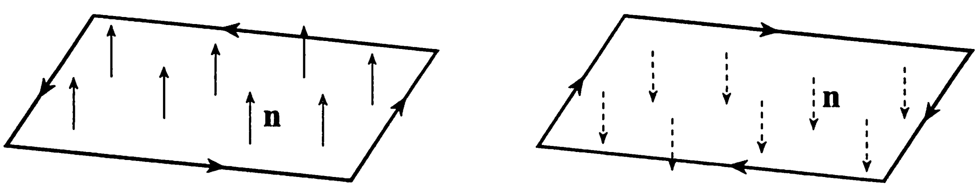
- Let S be a surface and let $C = \partial S$. The following is an intuitive explanation of the equation $\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \oint_C \mathbf{F} \cdot d\mathbf{r}$. Consider the region R chopped up into a bunch of smaller regions as shown, and suppose we want to integrate $\int_R (\nabla \times \mathbf{F})\, dA$ for some field $\mathbf{F}$.



Since $\nabla \times \mathbf{F}$ is the curl of the field, which measures the local rotation of the field, the arrows in the small rectangles above represent the curl of $\mathbf{F}$. Notice that all of the rotations inside the square cancel out, and the only rotation left is the part along the boundary. So, when we compute $\int_S (\nabla \times \mathbf{F})\, dA$, we are really measuring the rotational movement along the boundary. However, there is another way to measure the movement along the boundary: a line integral. The work done by the field in moving a particle along the boundary is given by $\oint_C \mathbf{F} \cdot d\mathbf{r}$. Therefore $\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \oint_C \mathbf{F} \cdot d\mathbf{r}$.

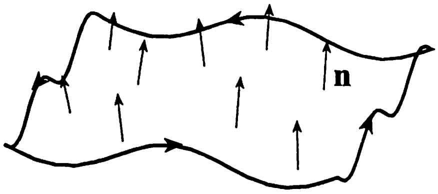
Workshop/Discussion

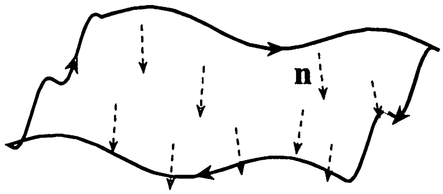
- Use the following to discuss the idea of an oriented surface with a positively oriented boundary. Start with a planar region R, such as $0 \le x \le \pi$, $0 \le y \le \pi$. Thinking of this region as lying in $\mathbb{R}^3$, there are clearly two ways to continuously assign normal vectrs, either in the positive z-direction or the negative z-direction. For each case, we get a different positive orientation on $C_1 = \partial R$, as shown below.



Now suppose we "wrinkle" the surface slightly. Let S be the surface $z = \frac{1}{5}\sin x + \frac{1}{5}\cos y$, $0 \le x \le \pi$, $0 \le y \le \pi$. Although the normal vectors no longer all point in the same direction, there are still two distinct ways to continuously assign the normal vectors, each one giving a different positive orientation on

$C_2 = \partial S$.





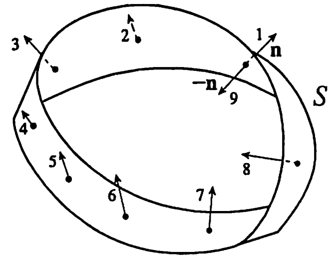
- Use Stokes' Theorem to show that $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = -2\pi$, where S is the surface formed by the lower hemisphere of $x^2 + y^2 + z^2 = 1$ and $\mathbf{F} = z^2 y\,\mathbf{i} + 2x\,\mathbf{j} + x^2 y z^3\,\mathbf{k}$. Explain how the negative result arises from the orientation given to the boundary circle $x^2 + y^2 = 1$.

- Evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = -y^3\,\mathbf{i} + x^3\,\mathbf{j} + \cos z^3\,\mathbf{k}$ and C is the curve generated by the intersection of the cylinder $x^2 + y^2 = 4$ and the plane $x + y + z = 1$. One approach is to create a surface S such that $C = \partial S$. To do this, choose S to be the portion of the plane with normal $\mathbf{N} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and unit normal $\mathbf{n} = \frac{1}{\sqrt{3}}(\mathbf{i} + \mathbf{j} + \mathbf{k})$ over the circle $x^2 + y^2 = 4$. Then by Stokes' Theorem,

$$\begin{aligned}\oint_C \mathbf{F} \cdot d\mathbf{r} &= \iint_S (\operatorname{curl} \mathbf{F} \cdot \mathbf{n})\,dS = \iint_{x^2+y^2 \le 4} (\operatorname{curl} \mathbf{F} \cdot \mathbf{n})\,dA = \iint_{x^2+y^2 \le 4} 3\left(x^2 + y^2\right) dA \\ &= 3 \iint_{x^2+y^2 \le 4} \left(x^2 + y^2\right) dx\,dy = 3 \int_0^{2\pi} \int_0^2 r^3\,dr\,d\theta \\ &= 24\pi\end{aligned}$$

Note that trying to compute this integral without Stokes' Theorem is very difficult.

- Show that the Möbius strip S is not orientable, as follows. If you start at P with unit normal $\mathbf{n}_1$ on S_1 and move around continuously in the direction indicated, you need to chose $-\mathbf{n}_1$ for consistency, a contradiction.



Group Work 1: The Silo

Group Work 2: Plane Surfaces

Problems 1 and 2 are straightforward. In Problem 3, students need to realize that because we don't have a formula for $C = \partial S$, since it is arbitrary, the only possible way to compute the line integral is to use Stokes' Theorem. Since $\operatorname{curl} \mathbf{F} = \mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$, this surface integral turns out to be a very easy calculation.

Homework Problems

Core Exercises: 1, 2, 8

Sample Assignment: 1, 2, 5, 8, 9, 11, 17

Note:
- Exercise 11 parts (b) and (c) require a CAS.
- Exercise 16 makes an excellent group work. Students may need the hint that we need to find conditions on **F** so that Stokes' Theorem applies.

Exercise	C	A	N	G	V
1	×				
2		×			
5		×			×
8		×			
9		×			
11		×		×	
17		×			

Group Work 1, Section 13.7
The Silo

Let S be the surface formed by capping the piece of the cylinder $x^2 + y^2 = 2$, $0 \le z \le 4$ with the top half of the sphere $x^2 + y^2 + (z-4)^2 = 2$.

1. Draw a rough sketch of S.

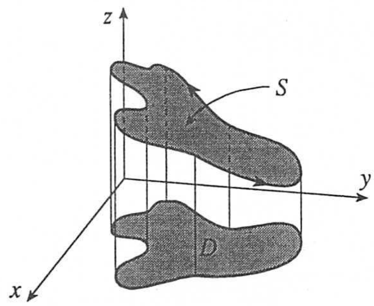
2. Show that the outward normal gives a smooth orientation to S.

3. What is $C = \partial S$? Parametrize C so that it has a positive orientation with respect to the outward normal.

4. Evaluate $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = \left(zx + z^2y + x\right)\mathbf{i} + \left(z^3yx + y\right)\mathbf{j} + z^4x^2\,\mathbf{k}$.

Group Work 2, Section 13.7
Plane Surfaces

Consider the surface S formed by the piece of the plane $ax + by + cz + d = 0$ above the region D in the xy-plane with area A_D.



1. Show that the surface area of S is $A_S = \frac{A_D}{|c|}\sqrt{a^2 + b^2 + c^2}$.

2. Show that a unit normal $\mathbf{n}$ to S is $\mathbf{n} = \frac{a\mathbf{i} + b\mathbf{j} + c\mathbf{k}}{\sqrt{a^2 + b^2 + c^2}}$.

3. Consider the plane $2x + 3y + 4z + 5 = 0$ and S the surface over D as above with boundary curve $C = \partial S$ having positive orientation.

(a) If $\mathbf{F}(x, y, z) = (x^3 + z + 2y)\,\mathbf{i} + 2x\,\mathbf{j} + (-4x + y)\,\mathbf{k}$, compute curl $\mathbf{F}$.

(b) Compute $\oint_C \left[(x^3 + z + 2y)\,\mathbf{i} + 2x\,\mathbf{j} + (-4x + y)\,\mathbf{k}\right] \cdot d\mathbf{r}$ in terms of A_D.
Hint: Can you use Stokes' Theorem here?

Writing Project: Three Men and Two Theorems

The story behind Green's and Stokes' Theorems turns out to be quite fascinating. This project should be thoroughly enjoyable for any student interested in the history of mathematics.

13.8 The Divergence Theorem

Suggested Time and Emphasis

1 class Essential Material

Points to Stress

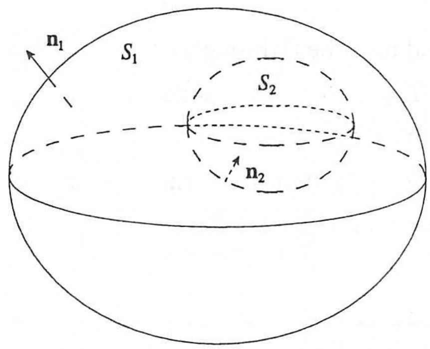
1. The meaning of a simple closed solid region R and its boundary surface $S = \partial R$
2. A careful statement of the Divergence Theorem.

Text Discussion

- Why is it that if we apply the Divergence Theorem to the region between two surfaces with one inside the other, we choose a normal for the inner surface that points toward the inside of the inner surface?
- "Source" and "sink" are defined on page 982. Give some intuitive reasons why these names are appropriate.

Materials for Lecture

- Provide a statement of the Divergence Theorem and stress the importance of an outward positive orientation.
- Point out that if $\mathbf{F}$ is incompressible, then $\operatorname{div} \mathbf{F} = 0$ and hence $\iint_{S=\partial R} \mathbf{F} \cdot d\mathbf{S} = 0$.
- Note that the value of the Divergence Theorem is that it allows us to reduce a surface integral to a triple integral.
- State the extension of the Divergence Theorem to regions between two closed surfaces, as shown:



- Check again that $\mathbf{F} = r(y,z)\,\mathbf{i} + s(x,z)\,\mathbf{j} + t(x,y)\,\mathbf{k}$ is incompressible on any simple closed region R and so, by the Divergence Theorem, $\iint_{S=\partial R} \mathbf{F} \cdot d\mathbf{S} = 0$.
- Perhaps give the following intuitive interpretation of $\nabla \cdot \mathbf{F}$:
  Choose a point P and surround it by a closed ball N with small radius r. According to the Divergence Theorem, the flux of $\mathbf{v}$ out of N is given by $\iiint_N (\operatorname{div} \mathbf{v})\, dx\, dy\, dz$. Thus, the Average Value Theorem tells us that the flux of $\mathbf{v}$ out of N is the average divergence of $\mathbf{v}$ on N times the volume of N. Dividing by the volume gives the average divergence of $\mathbf{v}$ on N to be $\dfrac{\text{flux of } \mathbf{v} \text{ out of } N}{\text{volume of } N}$. Letting the radius of the

ball shrink to 0 says that the divergence of $\mathbf{v}$ at P is $\lim_{r\to 0} \frac{\text{flux of } \mathbf{v} \text{ out of } N}{\text{volume of } N}$. In other words, divergence can be regarded as flux per unit volume. Now view $\mathbf{v}$ as the velocity of a fluid in steady-state motion. A positive divergence at a point indicates a net flow of liquid away from that point, since $\operatorname{div} \mathbf{v} > 0$ at P means that for some ball N, the flux out of N is positive. Similarly, a negative divergence indicates a net flow of liquid toward the point.

Points at which the divergence is positive are called sources; points at which the divergence is negative are called sinks. If the divergence of $\mathbf{v}$ is 0 throughout, then the flow has no source and no sink, and $\mathbf{v}$ is called incompressible.

- If there is time, recall the Laplacian $\nabla^2 = \operatorname{div} \cdot \nabla$ defined in Section 13.5. We can now use the Divergence Theorem and the following argument to show that a steady-state temperature distribution T satisfies $\nabla^2 T = 0$: Assume that on the surface of a hotplate, a temperature distribution is maintained which varies from point to point, but does not change over time. (We do *not* assume that the hot-plate is two dimensional. It is a three-dimensional piece of metal with a heating element on one side, and insulation on the other.) Then, in many cases, the temperature distribution inside the metal of the plate will also reach a steady state, again independent of time. Let $T(x, y, z)$ be the temperature at (x, y, z) over this solid. We will show that T satisfies the partial differential equation $\nabla^2 T = T_{xx} + T_{yy} + T_{zz} = 0$.

  At each point in the solid, ∇T points in the direction of most rapid increase in temperature. Since heat flows from warmer to cooler regions, the heat flows in the direction of $-\nabla T$. We will assume that the rate of flow (as a function of time) is proportional to the magnitude of the vector $-\nabla T$.

  Pick any point (x, y, z) and let R be a solid ball of metal containing (x, y, z), with surface S. Since the temperature is in a steady state in the entire region, heat neither enters nor leaves R. Since the flow is parallel to ∇T, this means that $\int_S (\nabla T) \cdot \mathbf{n}\, dS = 0$. By the Divergence Theorem, $\int_R \nabla \cdot \nabla T\, dV = 0$, or

$$\int_R \nabla \cdot \left(\frac{\partial T}{\partial x}\mathbf{i} + \frac{\partial T}{\partial y}\mathbf{j} + \frac{\partial T}{\partial z}\mathbf{k} \right) dV = \int_R \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) dV = 0$$

  We can conclude that the integrand must be 0 throughout R, and since R can be chosen to be any ball, we conclude that the Laplacian of T, $T_{xx} + T_{yy} + T_{zz}$, is identically 0 throughout the solid. So the problem of finding the temperature distribution in a metal object such as a hotplate reduces to the problem of finding a solution to Laplace's equation $\nabla^2 T = 0$ (a harmonic function) that satisfies certain conditions on the boundary. Give examples of harmonic functions such as $T(x, y, z) = e^x \cos y + z$.

Workshop/Discussion

- Compute $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = xz\,\mathbf{i} + yx\,\mathbf{j} + xyz\,\mathbf{k}$ and S is the surface of the unit cube. (The Divergence Theorem gives $\frac{5}{4}$.)
- Show that if $\mathbf{F}$ and $\mathbf{G}$ are given as

$$\begin{aligned} \mathbf{F} &= (8x + 3y)\,\mathbf{i} + (5x + 4z - 2y)\,\mathbf{j} + \left(9y^2 - \sin x + 7z\right)\mathbf{k} \\ \mathbf{G} &= (12y + 8z)\,\mathbf{i} + (e^z + \sin x + 9y)\,\mathbf{j} + \left(xy^2 e^{xy} + 4z\right)\mathbf{k} \end{aligned}$$

  then $\iint_{S=\partial R} \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{G} \cdot d\mathbf{S}$, where S is the surface of a region R for which the Divergence Theorem holds.

- Evaluate $\iint_{S=\partial R}(x+y^2+2z)\,dS$, where R is the solid sphere $x^2+y^2+z^2\le 4$. Note that to apply the Divergence Theorem, we need to "guess" a vector field $\mathbf{F}$ such that $\mathbf{F}\cdot\mathbf{n}=x+y^2+2z$. Set $\mathbf{F}=P\,\mathbf{i}+Q\,\mathbf{j}+W\,\mathbf{k}$. Since the outward unit normal vector on S is $\mathbf{n}=\frac{1}{2}(x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k})$, we have $\frac{1}{2}xP=x$, $\frac{1}{2}yQ=y^2$, and $\frac{1}{2}zW=2z$. Thus we need $P=2$, $Q=2y$, and $W=4z$. So one natural choice is $\mathbf{F}=2\mathbf{i}+2y\,\mathbf{j}+4z\,\mathbf{k}$. Then $\nabla\cdot\mathbf{F}=2$ and $\iint(x+y^2+2z)\,dS=\iiint_R 2\,dV=\frac{32}{3}\pi$.
- Discuss harmonic functions and give some additional examples, such as $f(x,y,z)=2x^2+3y^2-5z^2$ and $g(x,y,z)=e^{\sqrt{2}z}\sin x\cos y$.

Group Work 1: A Handy Way to Find Flux

By using the Divergence Theorem and noting that $\operatorname{div}\mathbf{F}>0$, the students can answer the first question without having to compute an integral.

Group Work 2: Finding Surface Integrals

The surface in this activity is similar to the surface in Group Work 1, making it a good supplement to that exercise. The volume of the region R can easily be shown to be π using geometry.

Group Work 3: The Position Vector

This activity is a good warmup to Exercise 18 in the text.

Group Work 4: When Are Surface Integrals Always Zero?

Problem 1 of this activity is related to Exercise 19, since the vector field is a scalar multiple of $\mathbf{E}(x)$ defined on page 984. Problem 2 is easily answered using the Divergence Theorem.

Homework Problems

Core Exercises: 1, 2, 7, 12

Sample Assignment: 1, 2, 7, 8, 9, 12, 17, 22, 23

Exercise	C	A	N	G	V
1					×
2		×			×
7–15		×			
17		×			
22		×			
23		×			

Group Work 1, Section 13.8
A Handy Way to Find Flux

Consider $\mathbf{F} = \dfrac{xy^2}{2}\,\mathbf{i} + \dfrac{y^3}{6}\,\mathbf{j} + zx^2\,\mathbf{k}$ over the surface S, where S is the cylinder $x^2 + y^2 = 1$ capped by the planes $z = \pm 1$.

1. Is the net flux of **F** from the surface positive or negative?

2. What is the value of the flux?

Group Work 2, Section 13.8
Finding Surface Integrals

Compute

$$\iint_{S=\partial R} \mathbf{F} \cdot d\mathbf{S}$$

where

$$\mathbf{F} = (x - z)\,\mathbf{i} + (y - x)\,\mathbf{j} + (z - y)\,\mathbf{k}$$

and S is the cylinder $x^2 + y^2 = 1$ capped by the planes $2z = 1 - x$ and $2z = x - 1$.

Group Work 3, Section 13.8
The Position Vector

Let $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, the position vector at (x, y, z).

1. Compute $\iint_{S=\partial R} \mathbf{F} \cdot \mathbf{n}\, dS$ for any region R.

2. Find a vector field $\mathbf{G}$ such that $\iint_{S=\partial R} \mathbf{G} \cdot \mathbf{n}\, dS$ is equal to the volume of R, for any region R.

Group Work 4, Section 13.8
When Are Surface Integrals Always Zero?

Let U be the solid interior of a closed surface S, and assume that the origin does not lie in the set U or on its boundary S.

1. Show that $\iint_S \frac{\mathbf{x}}{|\mathbf{x}|^3} \cdot d\mathbf{S} = 0$, where $\mathbf{x} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

2. If S is the surface of the sphere $x^2 + y^2 + (z-2)^2 = 1$, then is it true that $\iint_S \frac{\mathbf{x}}{|\mathbf{x}|^2} \cdot d\mathbf{S} = 0$?

13 Sample Exam

Problems marked with an asterisk (*) are particularly challenging and should be given careful consideration.

1. Match up each entry in the first column to one in the second. Note that a given entry in the second column can be used once, more than once, or not at all.

If a vector field $\mathbf{F}$ is the gradient of some scalar function, then $\mathbf{F}$ is ______________ .	conservative
If a curve C is the union of a finite number of smooth curves, then C is ______________ .	curl
If $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path C in D, then $\int_C \mathbf{F} \cdot d\mathbf{r}$ is ______________ in D.	divergence
If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$, $\mathbf{F}$ is defined everywhere in $\mathbb{R}^2$ and $\partial P/\partial y = \partial Q/\partial x$, then $\mathbf{F}$ is ______________ .	flux
If a curve C doesn't intersect itself anywhere between its endpoints, then C is ______________ .	irrotational
If $\mathbf{F}$ is a vector field on $\mathbb{R}^3$ then $\nabla \times \mathbf{F}$ is called the ______________ .	path independent
If $\mathbf{F}$ is a vector field on $\mathbb{R}^3$ then $\nabla \cdot \mathbf{F}$ is called the ______________ .	piecewise smooth
If $\mathbf{F}$ is a continuous vector field defined on an oriented surface S then $\iint_S \mathbf{F} \cdot d\mathbf{S}$ is the ______________ .	simple
If $\mathbf{F}$ is a vector field and $\operatorname{curl} \mathbf{F} = \mathbf{0}$ at a point P, then $\mathbf{F}$ is ______________ at P.	simply-connected

2. Consider the oriented surface S for $z \geq 0$, consisting of the portion of the surface of the paraboloid $z = 4 - \left(x^2 + y^2\right)$ above the xy-plane and with outward normal.

(a) What is the boundary curve $C = \partial S$ and what direction is its positive orientation?

(b) What surface S_1 in the xy-plane with what assignment of a normal has the same boundary curve $C = \partial S_1$ with the same orientation?

(c) Compute $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$, if $\mathbf{F} = \left(xe^z - 3y\right)\mathbf{i} + \left(ye^{z^2} + 2x\right)\mathbf{j} + \left(x^2y^2z^3\right)\mathbf{k}$.

3. Parametrize the boundary curve $C = dS$ of the surface S: $\frac{x^2}{9} + \frac{y^2}{9} + \frac{z^2}{16} = 1$, $z \le 0$, so that it has positive orientation with respect to S.

4. (a) Find a counterclockwise parametrization of the ellipse $x^2 + \frac{y^2}{4} = 1$.

(b) Compute the double integral

$$\iint_{0 \le x^2 + y^2/4 \le 1} 3x^2 y \, dA$$

Hint: Can you find a vector function $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ such that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 3x^3 y$?

5. Consider $\mathbf{F}(x, y, z) = -\frac{y}{x^2 + y^2}\mathbf{i} + \frac{x}{x^2 + y^2}\mathbf{j} + z\,\mathbf{k}$.

(a) Compute $\oint_C \mathbf{F} \cdot d\mathbf{r}$, where C is the circle $x^2 + y^2 = 1$ in the xy-plane, oriented counterclockwise.

(b) Show that $\operatorname{curl} \mathbf{F} = \langle 0, 0, 0 \rangle$ everywhere that $\mathbf{F}$ is defined.

(c) Indicate why you cannot use Stokes' Theorem on this problem. [That is, explain why your answers to (a) and (b) don't contradict one another.]

6. (a) Use the Divergence Theorem to show that, for a closed surface S with an outward normal which encloses a solid region B,

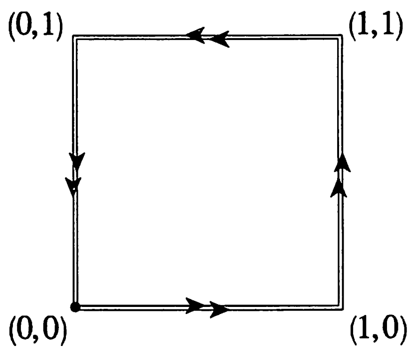
$$\text{Volume}\,(B) = \iint_S \mathbf{F} \cdot d\mathbf{S}$$

where $\mathbf{F}(x, y, z) = \langle x, 0, 0 \rangle$.

(b) Use part (a) to show that the volume enclosed by the unit sphere is $\frac{4}{3}\pi$.

(c) Compute $\iint \mathbf{F} \cdot d\mathbf{S}$ if $\mathbf{F}(x, y, z) = \langle 3x, 4y, 5z \rangle$.

7. Compute the work done by the vector field $\mathbf{F}(x, y) = \left(\sin x + xy^2\right)\mathbf{i} + \left(e^y + \frac{1}{2}x^2\right)\mathbf{j}$ in $\mathbb{R}^2$, where C is the path that goes around the unit square twice.



8. Consider the vector field $\mathbf{F}(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$.

(a) Compute $\operatorname{curl} \mathbf{F}$.

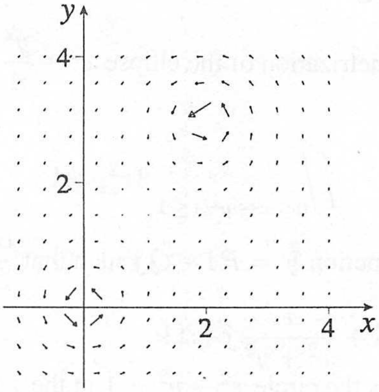
(b) If C is any path from $(0, 0, 0)$ to (a_1, a_2, a_3) and $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, show that $\int_C \mathbf{F} \cdot d\mathbf{r} = \mathbf{a} \cdot \mathbf{a}$.

9. Consider the vector fields $\mathbf{F}(x, y) = \frac{-y}{x^2 + y^2}\mathbf{i} + \frac{x}{x^2 + y^2}\mathbf{j}$ and

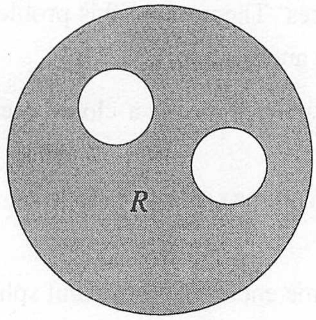
$\mathbf{G}(x, y) = \frac{-(y-3)}{(x-2)^2 + (y-3)^2}\mathbf{i} + \frac{x-2}{(x-2)^2 + (y-3)^2}\mathbf{j}$.

(a) Given that $\operatorname{curl} \mathbf{F}(x, y) = \mathbf{0}$ for $(x, y) \ne (0, 0)$, compute $\operatorname{curl} \mathbf{G}(x, y)$ for $(x, y) \ne (2, 3)$.

(b) Below is the plot of the vector field $\mathbf{F}(x,y) + \mathbf{G}(x,y)$. Describe where this vector field is defined. Describe where it is irrotational.



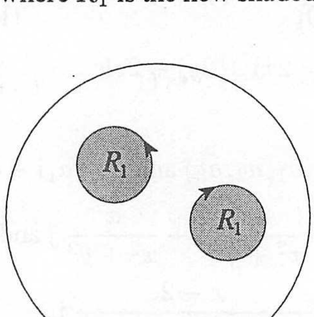
10. Consider the shaded region below.



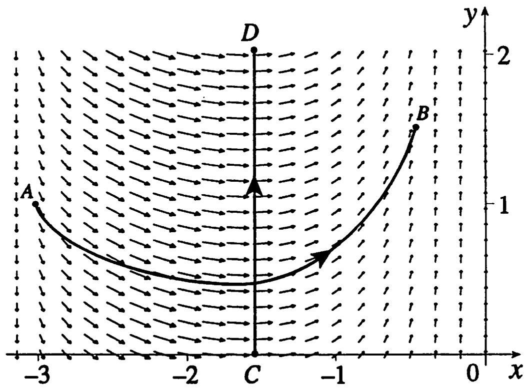
(a) Draw arrows on the boundaries ∂R of R to give it a positive orientation.

(b) If the outer circle has radius 4 and the two smaller circles have radius 1, evaluate $\frac{1}{2}\left(\int_{\partial R} y\,dx - x\,dy\right)$.

(c) Compute $\frac{1}{2}\left(\int_{\partial R_1} y\,dx - x\,dy\right)$, where R_1 is the new shaded region in the figure below. Each smaller circle has radius 1.



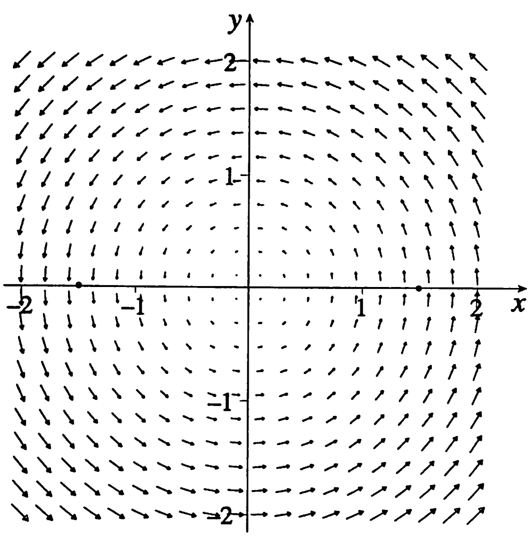
11. Consider the following vector field.**F**.



(a) Is the line integral of **F** along the path from A to B positive, negative, or zero? How do you know?

(b) Is the line integral of **F** along the path from C to D positive, negative, or zero? How do you know?

12. Consider the vector field below.

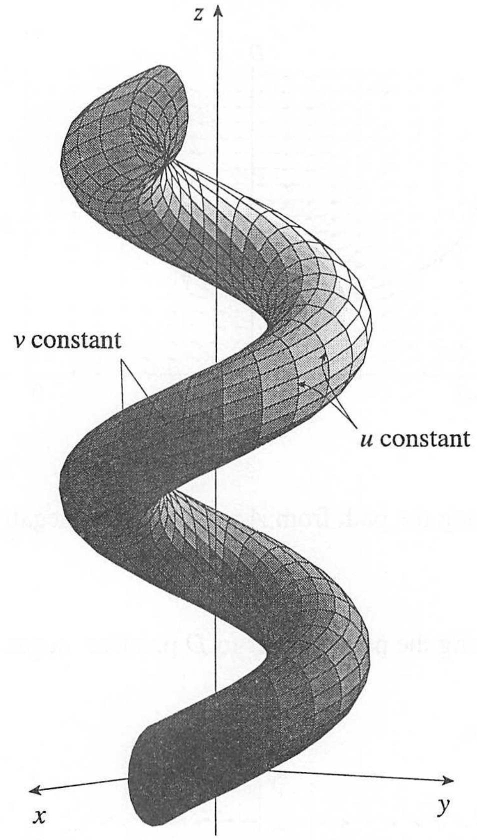


(a) Draw and label a curve C_1 from $(-1.5, 0)$ to $(1.5, 0)$ such that $\int_{C_1} \mathbf{F} \cdot ds > 0$.

(b) Draw and label a curve C_2 from $(-1.5, 0)$ to $(1.5, 0)$ such that $\int_{C_2} \mathbf{F} \cdot ds < 0$.

(c) Draw and label a curve C_3 from $(-1.5, 0)$ to $(1.5, 0)$ such that $\int_{C_3} \mathbf{F} \cdot ds \approx 0$.

13. The following parametric surface has grid curves which can be shown to be circles when u is constant.



$$x = (2 + \sin v)\cos u \qquad y = (2 + \sin v)\sin u \qquad z = u + \cos v$$

(a) Find the center and radius of the circle at $u = \frac{\pi}{2}$.

(b) Find the normal vector to S at the point P generated when $u = v = \frac{\pi}{2}$.

14. Consider the surfaces S_1: $\dfrac{x^2}{9} + \dfrac{y^2}{9} + \dfrac{z^2}{4} = 1$, $z \geq 0$ and S_2: $4z = 9 - x^2 - y^2$, $z \geq 0$. Let $\mathbf{F}$ be any vector field with continuous partial derivatives defined everywhere. Show that $\iint_{S_1} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_2} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$.

15. Set up and evaluate the integral for the surface area of the parametrized surface

$$x = u + v \qquad y = u - v \qquad z = 2u + 3v$$
$$0 \leq u \leq 1 \qquad 0 \leq v \leq 1$$

13 Sample Exam Solutions

1. Conservative; piecewise smooth; path independent; conservative; simple; curl; divergence; flux; irrotational

2. (a) C is the circle of radius 2 centered at the origin in the xy-plane. It has positive orientation if it is parametrized in the counterclockwise direction as viewed from above.

(b) If S_1 is the disk of radius 2 centered at the origin with upward normal, then $C = \partial S_1$ with the same orientation.

(c) By Stokes' Theorem, $\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \oint_{C=\partial S}\mathbf{F}\cdot d\mathbf{r} = \oint_{C=\partial S_1}\mathbf{F}\cdot d\mathbf{r} = \iint_{S_1}\operatorname{curl}\mathbf{F}\cdot d\mathbf{S}$. Since $z=0$ on S_1, $\mathbf{F} = (x-3y)\,\mathbf{i} + (y+2x)\,\mathbf{j} + 0\mathbf{k}$, $\mathbf{n} = \mathbf{k}$, and $\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x-3y & y+2x & 0 \end{vmatrix} = 5\mathbf{k}$. So $\iint_{S_1}\operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \iint_{x^2+y^2\le 4}(\operatorname{curl}\mathbf{F}\cdot\mathbf{k})\,dA = 5\left(\text{area of disk } x^2+y^2\le 4\right) = 20\pi$.

3. $\mathbf{r}(\phi,\theta) = 3\sin\phi\cos\theta\,\mathbf{i} + 3\sin\phi\sin\theta\,\mathbf{j} + 4\cos\phi\,\mathbf{k}$

4. (a) $x^2 + \dfrac{y^2}{4} = 1$ can be parametrized counterclockwise by $\mathbf{F}(t) = \langle\cos t, 2\sin t\rangle$, $0\le t\le 2\pi$.

(b) Note that if $\mathbf{F} = 0\mathbf{i} + x^3y\mathbf{j}$, then $\partial Q/\partial x - \partial P/\partial y = 3x^2y$. So

$$\begin{aligned}\iint_{0\le x^2+y^2/4\le 1} 3x^2y\,dA &= \int_{\text{boundary}} P\,dx - Q\,dy = \int_0^{2\pi}\cos^3 2 + 2\sin t\,(2\cos t\,dt)\\ &= 4\int_0^{2\pi}\cos^4 t\sin t\,dt \quad (\text{Let } u=\cos t,\ du = -\sin t\,dt)\\ &= -4\int_{-1}^{1}u^4\,du = 0\end{aligned}$$

This can also be found directly, as follows:

$$\int_0^1\int_{-2\sqrt{1-x^2}}^{2\sqrt{1-x^2}} 3x^2y\,dy\,dx = \int_0^1\left[\tfrac{3}{2}x^2y^2\right]_{-2\sqrt{1-x^2}}^{2\sqrt{1-x^2}}dx = 0$$

5. $\mathbf{F}(x,y,z) = -\dfrac{y}{x^2+y^2}\mathbf{i} + \dfrac{x}{x^2+y^2}\mathbf{j} + z\,\mathbf{k}$.

(a) $\mathbf{r}(t) = \langle\cos t,\sin t,0\rangle$, $0\le t\le 2\pi$.
$\oint_C\mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi}\langle-\sin t,\cos t,0\rangle\cdot\langle-\sin t,\cos t,0\rangle\,dt = \int_0^{2\pi}1\,dt = 2\pi$

(b) $\operatorname{curl}\mathbf{F}(x,y,z) = \left\langle 0-0, 0-0, \dfrac{y^2-x^2}{(x^2+y^2)^2} - \dfrac{y^2-x^2}{(x^2+y^2)^2}\right\rangle = \langle 0,0,0\rangle$ everywhere except the z-axis (where $\mathbf{F}$ is undefined).

(c) Since $\mathbf{F}$ is not defined along the z-axis, we cannot find a surface such that C is its boundary and $\mathbf{F}$ is defined everywhere on the surface.

Another reason: If $P = -\dfrac{y^2}{x^2+y^2}$, then $\dfrac{\partial P}{\partial y} = \dfrac{y^2-x^2}{(x^2+y^2)^2}$, which does not have a limit at $(0,0)$ and is discontinuous there.

6. (a) If we assume an outward normal, then by the Divergence Theorem,
$\iint_S\mathbf{F}\cdot d\mathbf{S} = \iiint_B\operatorname{div}\mathbf{F}\cdot d\mathbf{V} = \iiint_B d\mathbf{V}$ (since $\operatorname{div}\mathbf{F} = 1$), which is simply the volume of B.

(b) Parametrize the sphere by $r(\theta,\phi) = \langle\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi\rangle$. Then
$r_\theta\times r_\phi = \left\langle\sin^2\phi\cos\theta, \sin^2\phi\sin\theta, -\sin\phi\cos\phi\right\rangle$, which points outward, and
$f(r(\theta,\phi)) = \langle\cos\theta\sin\phi, 0, 0\rangle$, so
$\iint_S\mathbf{F}\cdot d\mathbf{S} = \int_0^{\pi}\int_0^{2\pi}\cos^2\theta\sin^3\phi\,d\theta\,d\phi = \int_0^{2\pi}\cos^2\theta\,d\theta\int_0^{\pi}\sin^3\phi\,d\phi = \frac{4}{3}\pi$.

(c) $\operatorname{div}\mathbf{F} = 12$, so $\iint_S\mathbf{F}\cdot d\mathbf{S} = 12\cdot\text{Volume}(B) = 16\pi$.

7. Using Green's Theorem with $P = \sin x + xy^2$ and $Q = e^y + \frac{1}{2}x^2$, we get

$$\text{Work} = \int_C P\,dx + Q\,dy = 2\int_{\text{Square}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = 2\int_0^1 \int_0^1 (x - 2xy)\,dx\,dy$$

$$= 2\int_0^1 \left[\tfrac{1}{2}x^2 - x^2 y\right]_0^1 dy = 2\int_0^1 \left(\tfrac{1}{2} - y\right) dy = 2\left[\tfrac{1}{2}y - \tfrac{1}{2}y^2\right]_0^1 = 0$$

8. (a) $\operatorname{curl}\mathbf{F} = 0$

(b) Let $f(x, y, z) = x^2 + y^2 + z^2$. Then $\mathbf{F} = \nabla f$ and by the Fundamental Theorem for line integrals, $\int_C \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{a}) - f(0) = a_1^2 + a_2^2 + a_3^2 = \mathbf{a} \cdot \mathbf{a}$.

9. (a) If $\mathbf{G} = P\mathbf{i} + Q\mathbf{j}$, then computation gives

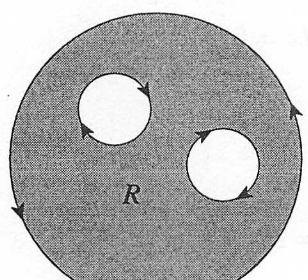
$$\operatorname{curl}\mathbf{G} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k} = \left[\frac{(y-3)^2 - (x-2)^2}{(x-2)^2 + (y-3)^2} - \frac{(y-3)^2 - (x-2)^2}{(x-2)^2 + (y-3)^2}\right]\mathbf{k} = \mathbf{0} \text{ for}$$

$(x, y) \neq (2, 3)$.

Or: $\operatorname{curl}\mathbf{G} = \mathbf{0}$ since the vector field $\mathbf{G}$ is just $\mathbf{F}$ translated to the right 2 units and up 3 units.

(b) $\mathbf{F} + \mathbf{G}$ is defined at all points except $(0, 0)$ and $(2, 3)$, since $\mathbf{F}$ is not defined at $(0, 0)$ and $\mathbf{G}$ is not defined at $(2, 3)$. At all other points, $\operatorname{curl}(\mathbf{F} + \mathbf{G}) = \operatorname{curl}\mathbf{F} + \operatorname{curl}\mathbf{G} = \mathbf{0}$, and $\mathbf{F} + \mathbf{G}$ is irrotational.

10. (a)



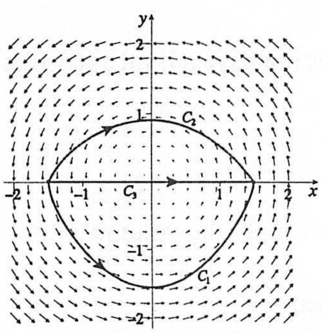
(b) $\frac{1}{2}\left(\int_{\partial R} y\,dx - x\,dy\right) = \text{area}(R) = \pi \cdot 4^2 - 2 \cdot \pi \cdot 1^2 = 14\pi$

(c) $\frac{1}{2}\left(\int_{\partial R_1} y\,dx - x\,dy\right) = 0$, since the two smaller circles have equal areas and opposite orientations.

11. (a) Since $\mathbf{F}$ points in almost the same direction as vectors tangent to the path from A to B, $\mathbf{F}(t) \cdot \mathbf{r}'(t) > 0$ everywhere along the path, and hence the line integral $\int \mathbf{F} \cdot d\mathbf{r} > 0$.

(b) Since $\mathbf{F}$ is perpendicular to the path from C to D at every point, we have $\mathbf{F}(t) \cdot \mathbf{r}'(t) = 0$ everywhere along the path, and hence the line integral $\int \mathbf{F} \cdot d\mathbf{r} = 0$.

12.



13. (a) When $u = \frac{\pi}{2}$, $x = 0$, $y = 2 + \sin v$, and $z = \frac{\pi}{2} + \cos v$, so the center is $\left(0, 2, \frac{\pi}{2}\right)$ and the radius is 1.

(b) The normal vector at $P\left(0, 3, \frac{\pi}{2}\right)$ is $3\mathbf{j}$.

14. Both surfaces have the same boundary curve C: $x^2 + y^2 = 9$, $z = 0$. By Stokes' Theorem, $\iint_{S_1} \operatorname{curl}\mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r} = \iint_{S_2} \operatorname{curl}\mathbf{F} \cdot d\mathbf{S}$.

15. $\mathbf{F}(u, v) = \langle u + v, u - v, 2u + 3v \rangle \Rightarrow \mathbf{F}_u = \langle 1, 1, 2 \rangle$, $\mathbf{F}_v = \langle 1, -1, 3 \rangle$, and $\mathbf{F}_u \times \mathbf{F}_v = \langle 5, -1, -2 \rangle$. Thus the surface area is $\int_0^1 \int_0^1 |\mathbf{F}_u \times \mathbf{F}_v|\,du\,dv = \int_0^1 \int_0^1 \sqrt{30}\,du\,dv = \sqrt{30}$.