9/30/2014 FIRST HOURLY **Math 21a, Fall 2014**

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- Start by printing your name in the above box and please **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3 or problem 9, we need to see **details** of your computation.
- All functions can be differentiated arbitrarily often unless otherwise specified.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) (20 points) No justifications are needed.

1) T **F** The surface $-x^2 + y^2 + z^2 = -1$ is a one-sheeted hyperboloid.

Solution:
Look at the traces

2) **T** F The equation $y = 3x + 2$ in space defines a plane.

Solution:
It looks like a line because only two variables are involved, but remember that we are in space.

3) T **F** Whenever $|\vec{r}\,'(t)| = 1$ then $|\vec{T}\,'(t)| = 1$.

Solution:
We know that $|\vec{T}'|$ is curvature. This can be arbitrary even if the speed is 1 at all times

4) **T** F The length of the vector projection $\text{Proj}_{\vec{v}}(\vec{w})$ is smaller than or equal to the length of $\vec{w}$.

Solution:
One can see this geometrically, or by expanding the dot product in the formula.

5) **T** F The velocity vector of $\vec{r}(t) = \langle t, t, t \rangle$ at time $t = 2$ is the same as the velocity vector at $t = 1$.

Solution:
It is $\langle 1, 1, 1 \rangle$ in both cases.

6) **T** F If $\vec{v} \times \vec{w} = \vec{w} \times \vec{v}$ then $\vec{v}, \vec{w}$ are parallel.

Solution:
[illegible]

7) T F

The vector $\langle -2, 1, 0\rangle$ is perpendicular to the line $\langle 1 + t, 2t, 3t\rangle$.

Solution:
Take the dot product with the velocity vector.

8) T F

The point given in spherical coordinates as $\rho = 3, \phi = 0, \theta = \pi$ is the same point as the point $\rho = 3, \phi = 0, \theta = 0$.

Solution:
It is the north pole.

9) T F

The parametrized curve $\vec{r}(t) = \langle 0, 3\cos(t), 5\sin(t)\rangle$ is an ellipse.

Solution:
Indeed, and it is contained in the xz-plane.

10) T F

The curvature of the line $\vec{r}(t) = \langle t, t, t\rangle$ is $\sqrt{3}$ everywhere.

Solution:
No, it is 0.

11) T F

If $|\vec{v} \times \vec{w}| = \vec{v} \cdot \vec{w}$ then either $\vec{v}$ is parallel to $\vec{w}$ or perpendicular to $\vec{w}$.

Solution:
They can be unit vectors of angle 45 degrees for example

12) T F

If the dot product between two unit vectors $\vec{v}, \vec{w}$ is -1, then $\vec{v} = -\vec{w}$.

Solution:
Yes, $\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos(\alpha) = \cos(\alpha) = -1$ shows that $\alpha = \pi$.

13) T F Writing $\vec{k} = \langle 0, 0, 1 \rangle$, we have $|(\vec{k} \times \vec{v}) \times \vec{w}| \leq |\vec{v}||\vec{w}|$ for all vectors $\vec{v}, \vec{w}$.

Solution:
Use the identity for the length of the cross product

14) T F The curvature of a curve $\vec{r}(t)$ is given by $\kappa(t) = |\vec{T}\,'(t)|/|\vec{r}\,'(t)|$. If $|\vec{r}\,'(t)| = 1$ for all times, then $\kappa(t) = |\vec{r}\,''(t)|$.

Solution:
Yes, under the assumption we have $\vec{T}(t) = \vec{r}'(t)$ and $\vec{T}' = \vec{r}''$ so that by definition of curvature we have $\kappa(t) = |\vec{r}\,''(t)|$.

15) T F The arc length of the curve $\langle \sin(t/2), 0, \cos(t/2) \rangle$ from $t = 0$ to $t = 2\pi$ is equal to 2π.

Solution:
We only trace half the circle.

16) T F If L, K are skew lines in space, there is a unique plane which is equidistant from L, K.

Solution:
Draw two planes through the lines which are spanned by vectors in L and K. These planes are parallel. There exists exactly one plane between. But there is a caveat. You can find many planes which intersect both lines and have distance zero to both. We graded both answers yes as we actually planned to assume the distance to be nonzero and the lines disjoint.

17) T F The curve $\vec{r}(t) = \langle t, t^2, 1-t \rangle$ is the intersection curve of a plane $x + z = 1$ and $y = x^2$

Solution:
Just plug in

18) T F The lines $\vec{r}_1(t) = \langle 5+t, 3-t, 2-t \rangle$ and $\vec{r}_2(t) = \langle 6-t, 2+t, 1-2t \rangle$ intersect at $(6, 2, 1)$ perpendicularly.

Solution:
First check that the point appears in both curves. Now compute the velocity vector.

19) | T | F | The vector $\langle 3/13, 12/13, 4/13 \rangle$ is a unit vector.

Solution:
Yes, its length is equal to 1.

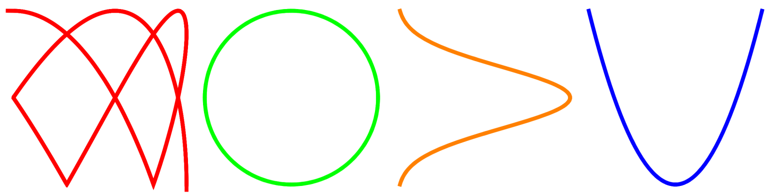
20) | T | F | $\vec{v} \times (\vec{v} \times \vec{u}) = \vec{0}$ for all vectors $\vec{u}, \vec{v}$.

Solution:
A counter example is $u = i, v = j$.

| | Total

Problem 2) (10 points) No justifications are needed in this problem.

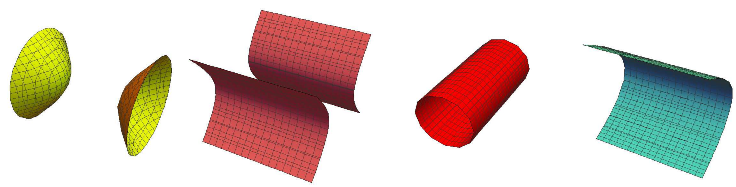
a) (2 points) Match the plane curves with their parametrizations $\vec{r}(t)$. Enter O, if there is no match. In each of the problems a) - e), every of the entries O, I, II, III, IV appears exactly once.



I II III IV

$\vec{r}(t) =$	Enter O-IV
$\langle \exp(-t^2), t \rangle$	
$\langle \cos(t), \sin(t) \rangle$	
$\langle \lvert\cos(3t)\rvert, \lvert\sin(5t)\rvert \rangle$	
$\langle 2t, 3t \rangle$	
$\langle t, t^2 + 1 \rangle$	

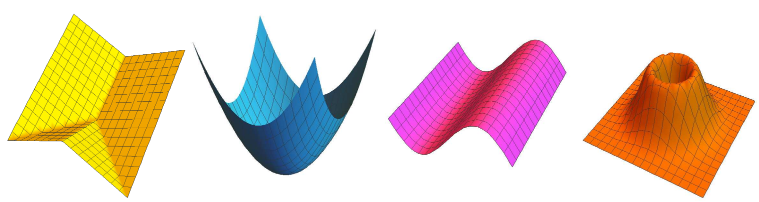
b) (2 points) Match the contour surfaces. Enter O, if there is no match.



I II III IV

$g(x, y, z) = 1$	Enter O-IV
$x + 2y = 1$	
$x^2 - y^2 - z^2 = 1$	
$z^2 + 2y = 1$	
$y^2 - z^2 = 1$	
$x^2 + z^2 = 1$	

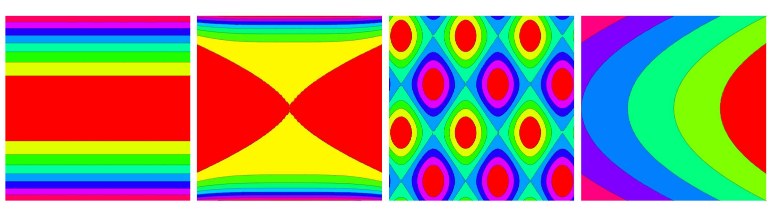
c) (2 points) Match the graphs of the functions $f(x, y)$. Enter O, if there is no match.



I II III IV

$f(x, y) =$	Enter O-IV
$\exp(-x^2 - y^2)(x^2 + y^2)$	
$\sin(x)$	
$\exp(-x^2 - y^2)\sin(x^2)$	
$\lvert\lvert x\rvert + y\rvert$	
$x^2 + y^2$	

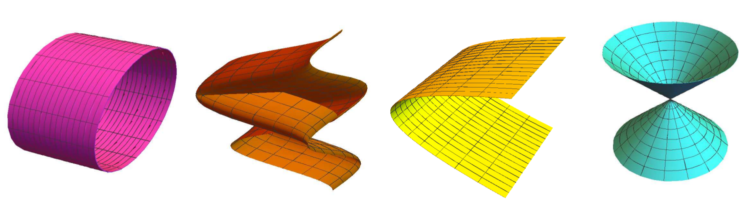
d) (2 points) Match functions $g(x, y)$ with contour maps. Enter O, if no match.



I II III IV

$g(x, y) =$	Enter O-IV
$\sin(3x) + \sin(2y)$	
$y^2 + x^2$	
$y^2 - 2x$	
y^2	
$y^6 - x^4$	

e) (2 points) Match the surface parametrization. Enter O, where is no match.



I II III IV

$\vec{r}(u, v) =$	Enter O,I-IV
$\langle u^2, v, u \rangle$	
$\langle u\cos(v), u\sin(v), u \rangle$	
$\langle \cos(u), \sin(v), u + v \rangle$	
$\langle v, \cos(u), \sin(u) \rangle$	
$\langle v, u, v \rangle$	

Solution:
a) III,II,I,O,IV
b) O,I,IV,II,III
c) IV,III,O,I,II
d) III,O,IV,I,II
e) III,IV,II,I,O

Problem 3) (10 points) No justifications are needed

In this problem $\vec{v}, \vec{w}$ are arbitrary vectors in space, $\vec{r}(t)$ is an arbitrary space curve.
The vectors $\vec{v}, \vec{w}, \vec{r}\,', \vec{r}\,'', \vec{T}, \vec{T}\,'$ are assumed to be nonzero where $\vec{N}$ is the normal vector and $\vec{B}$ the bi-normal vector.
All these vectors $\vec{r}, \vec{T}, \vec{B}, \vec{N}$ and its derivatives are evaluated at the fixed time $t = 0$.

first vector	second vector	always parallel	always perpendicular	depends
$\vec{r}\,'$	$\vec{r}\,''$			
$\vec{B}$	$\vec{N}$			
$\text{Proj}_{\vec{v}}(\vec{w})$	$\vec{v}$			
$\text{Proj}_{\vec{v}}(\vec{w})$	$\vec{w}$			
$\vec{v} \times \vec{w}$	$\vec{v}$			
$\vec{w} + \vec{v}$	$\vec{v} - \vec{w}$			
$\vec{v} \times \vec{w}$	$\vec{w} \times \vec{v}$			
$(\vec{v} + \vec{w}) \times \vec{w}$	$\vec{v} \times \vec{w}$			
$\vec{T}$	$\vec{r}\,'$			
$\vec{T}$	$\vec{T}\,'$			

Solution:

first vector	second vector	always parallel	always perpendicular	depends
$\vec{r}\,'$	$\vec{r}\,''$			X
$\vec{B}$	$\vec{N}$		X	
$\text{Proj}_{\vec{v}}(\vec{w})$	$\vec{v}$	X		
$\text{Proj}_{\vec{v}}(\vec{w})$	$\vec{w}$			X
$\vec{v} \times \vec{w}$	$\vec{v}$		X	
$\vec{w} + \vec{v}$	$\vec{v} - \vec{w}$			X
$\vec{v} \times \vec{w}$	$\vec{w} \times \vec{v}$	X		
$(\vec{v} + \vec{w}) \times \vec{w}$	$\vec{v} \times \vec{w}$	X		
$\vec{T}$	$\vec{r}\,'$	X		
$\vec{T}$	$\vec{T}\,'$		X	

Problem 4) (10 points)

A parallelepiped has vertices at $A = (0,0,0), B = (1,1,1), C = (2,3,4)$ and $A' = (3,4,8)$ and contains the sides AB, AC and AA'.

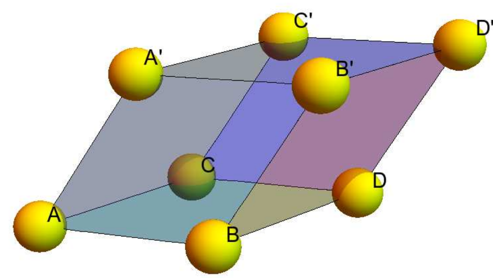
a) (2 points) Find a fourth point D so that A, B, C, D is a parallelogram.

b) (2 points) What is the area of that parallelogram $ABCD$?

c) (2 points) What is the volume of the parallelepiped?

d) (2 points) Find the height of the parallelepiped with floor $ABCD$ and roof A', B', C', D'.

e) (2 points) Find the distance between the face diagonals AD and $B'C'$.



Solution:

a) $\vec{A} + \vec{AB} + \vec{AC} = \langle 0,0,0 \rangle + \langle 1,1,1 \rangle + \langle 2,3,4 \rangle = \boxed{\langle 3,4,5 \rangle}$

b) $\vec{AB} \times \vec{AC} = \langle 1,1,1 \rangle \times \langle 2,3,4 \rangle = \langle 1,-2,1 \rangle$. Its length is $\boxed{\sqrt{6}}$.

c) $\vec{AA'} \cdot (\vec{AB} \times \vec{AC}) = \langle 3,4,8 \rangle \cdot \langle 1,-2,1 \rangle = 3$.

d) The height is Volume/area $= \boxed{\sqrt{6}/3}$.

e) The distance is equal to the height and therefore also $\boxed{\sqrt{6}/3}$.

Problem 5) (10 points)

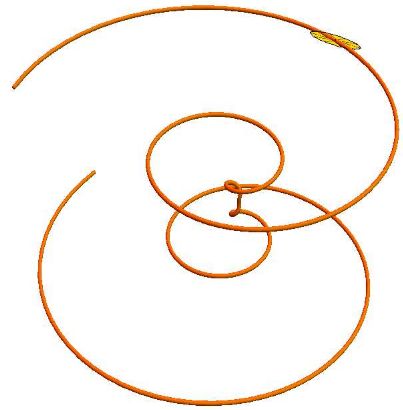
Autumn is here. A leaf tumbles down along the curve

$$\vec{r}(t) = \langle t^2\cos(t), t^2\sin(t), 16 - 2t\rangle$$

in space.

a) (3 points) What is the speed of the leaf at $t = \pi$?

b) (7 points) Find the arc length of the curve traced in the time interval $-8 \le t \le 8$.



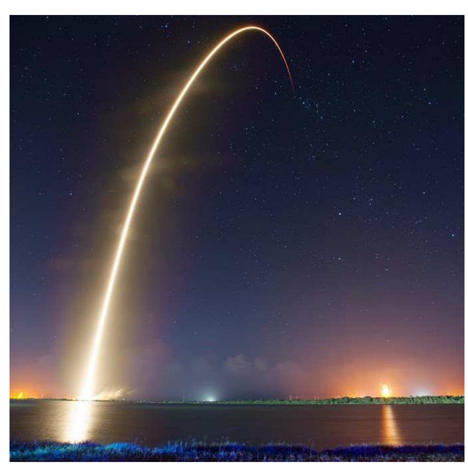
Solution:
a) $\vec{r}\,'(t) = \langle 2t\cos(t) - t^2\sin 9t), 2t\sin(t) + t^2\cos(t), -2\rangle$. The length $|\vec{r}\,'(t)| = \sqrt{(2+t^2)^2}$ simplifies to $2 + t^2$. At $t = \pi$ we have the speed $\boxed{2 + \pi^4}$.
b) $\int_{-8}^{8} 2 + t^2 = 2t + t^3/3|_{-8}^{8} = \boxed{1120/3}$.

Problem 6) (10 points)

On September 21, 2014, SpaceX launched a Dragon capsule with tons of supplies and experiments including a 3D printer to the space station. Assume the rocket experiences an acceleration

$$\vec{r}\,''(t) = \langle 2t, 0, 3t^2 - 5t^4\rangle$$

starts at Cape Canaveral Air force station $\vec{r}(0) = \langle 2, 3, 0\rangle$ with zero velocity $\vec{r}\,'(0) = \langle 0, 0, 0\rangle$.



a) (5 points) Where is the capsule at time $t = 1$?

b) (5 points) What is the curvature of the path at $t = 1$? You can use the formula $\kappa(t) = |\vec{r}\,'(t) \times \vec{r}\,''(t)|/|\vec{r}\,'(t)|^3$.

Solution:
a) Integrate twice and fix the initial condition

$$\vec{r}\,''(t) = \langle 2t, 0, 3t^2 - 5t^4 \rangle$$

$$\vec{r}\,'(t) = \langle t^2, 0, t^3 - t^5 \rangle$$

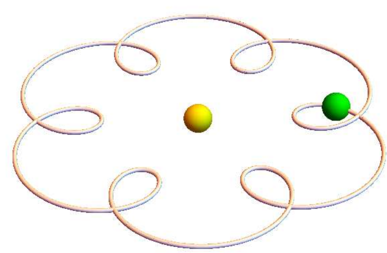
$$\vec{r}\,(t) = \langle t^3/3, 0, t^4/4 - t^6/6 \rangle + \langle 2, 3, 0 \rangle$$

At $t = 1$, this is $\langle 2 + 1/3, 3, 1/4 - 1/6 \rangle = \langle 7/3, 2, 1/12 \rangle$.
b) Compute the velocity and acceleration at $t = 1$ to get $\vec{r}\,'(1) = \langle 1, 0, 0 \rangle$ and $\vec{r}\,''(1) = \langle 1/3, 0, 1/4 - 1/6 \rangle$ which have the cross product $\langle 0, -1/12, 0 \rangle$. The curvature is $\kappa(1) = |\vec{r}\,'(1) \times \vec{r}\,''(1)|/|\vec{r}\,'(1)|^3 = 1/12$.

Problem 7) (10 points)

Before Kepler and Newton clarified planetary motion, there was the **Ptolemaic universe** which was based on the idea that planets move on epicycles like

$$\vec{r}(t) = \langle 3\cos(t) + \cos(7t), \sin(t) + \sin(7t), 3 \rangle \ .$$



a) (2 points) What is the velocity $\vec{v} = \vec{r}\,'(t)$ at $t = \pi$?

b) (2 points) What is the velocity $\vec{w} = \vec{r}\,'(t)$ at $t = \pi/2$?

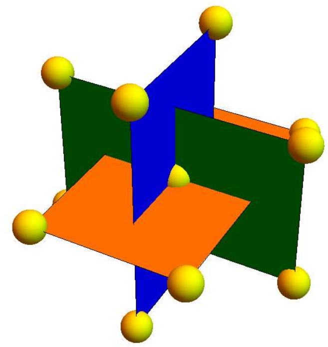
c) (2 points) Yes or no? Is $\vec{v} \times \vec{w}$ parallel to the binormal vector $\vec{B}(t)$ for all times t?

d) (4 points) Parametrize the line tangent to the curve at the point $A = \vec{r}(\pi)$.

Solution:
a) $\langle 0, -8, 0 \rangle$.
b) $\langle 4, 0, 0 \rangle$.
c) Yes, the binormal vector is perpendicular to the velocity vector and the acceleration. In this case, both the velocity and acceleration are in the xy plane.
d) $\vec{r}(\pi) = \langle -4, 0, 3 \rangle$. Now the parametrization is $\langle -4, 0, 3 \rangle + t\langle 0, -8, 0 \rangle = \langle -4, -8t, 3 \rangle$.

Problem 8) (10 points)

In this problem, the symbol φ is used to represent the golden ratio $\varphi = (\sqrt{5}+1)/2 \sim 1.618$ which satisfies the equation $\varphi^2 = \varphi + 1$.
The centers of four unit spheres are placed in the xy-plane at $A = (1, \varphi, 0), C = (-1, \varphi, 0), B = (1, -\varphi, 0)$ and $D = (-1, -\varphi, 0)$. 8 further points are located in the same way in the yz and xz plane so that we get 12 points which form the vertices of an **icosahedron** and surround a unit sphere centered at $(0, 0, 0)$.



a) (3 points) Consider the distances between the points A and B. Verify that the unit spheres centered at A and B do not intersect. Likewise, verify that the unit spheres centered at A and C do just intersect in a point.
b) (2 points) Using the concept of an icosahedron, explain why all 12 spheres either pairwise do not intersect or intersect in a point.

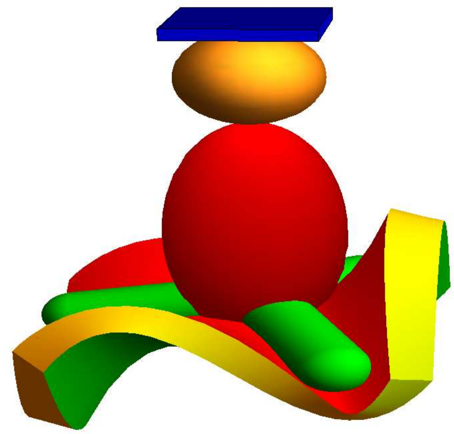
c) (3 points) The centers of all spheres have equal distance d from $(0, 0, 0)$. What is d in terms of φ?

d) (2 points) Why does the central unit sphere intersect all other unit spheres?

Solution:
a) $d(A, B) = 2\varphi$ is larger than 2. We also have $d(A, C) = 2$. They just touch.
b) Symmetry. All other distances are the same. Either 2φ or 2.
c) $\sqrt{1+\varphi^2} = \sqrt{2+\varphi}$ is the distance of the center to the origin so that $\sqrt{1+\varphi^2} - 1$ is the distance of the center to the sphere.
d) The distance between the centers is smaller than 2 so that any of the 12 spheres intersects with the central sphere

Isaac Newton and **James Gregory** argued whether 13 unit spheres can be placed around a central unit sphere just "kissing the central sphere". They knew that 12 work. Newton believed 13 is impossible, but it was only proven in 1954 that the **kissing number** is 12. Here we have seen how to place 12 spheres: by pushing the 12 spheres constructed here a bit so that they just touch the central sphere, you showed that they have positive distance from each other and solve the 12 sphere kissing problem. It is known since 2003 that the kissing number in 4 dimensions is 24 but nobody has any clue what the kissing number in 5 dimensions is! It is only known that the answer is between 40 and 44.

Problem 9) (10 points)



As a souvenir for this exam, we build a Monkey riding a "Monkey saddle" and 3D print it. No explanations are necessary.

a) (2 points) Parametrize the hat $z = 5$.

b) (2 points) Parametrize the saddle $z = yx^2 - x^3$.

c) (2 points) Parametrize the torso $x^2 + y^2 + \frac{(z-1)^4}{4} = 1$.

d) (2 points) Parametrize the head $4x^2 + y^2 + (z-4)^2 = 1$.

e) (2 points) Parametrize the monkey tail $x^2 + z^2 = \frac{1}{4}$.

Solution:
In each case we have a function $\vec{r}$ of two variables.
a) $\langle x, y, 5 \rangle$.
b) $\langle x, y, yx^2 - x^3 \rangle$.
c) $\langle \cos(\theta)\sqrt{1 - \frac{(z-1)^4}{4}}, \sin(\theta)\sqrt{1 - \frac{(z-1)^4}{4}}, z \rangle$.
d) $\langle \cos(\theta)\sin(\phi)/2, \sin(\theta), \cos(\phi), 4 + \sin(\phi) \rangle$.
e) $\langle \cos(\theta)/2, y, \sin(\theta)/2 \rangle$.