Name:

MWF 9 Oliver Knill
MWF 9 Chao Li
MWF 10 Gijs Heuts
MWF 10 Yu-Wen Hsu
MWF 10 Yong-Suk Moon
MWF 11 Rosalie Belanger-Rioux
MWF 11 Gijs Heuts
MWF 11 Siu-Cheong Lau
MWF 12 Erick Knight
MWF 12 Kate Penner
TTH 10 Peter Smillie
TTH 10 Jeff Kuan
TTH 10 Yi Xie
TTH 11:30 Jeff Kuan
TTH 11:30 Jameel Al-Aidroos

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3, we need to see details of your computation.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
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9		10
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11		10
12		10
13		10
14		10
Total:		150

Problem 1) True/False questions (20 points)

1) T **F** The distance from $(1, 2, -1)$ to $(3, -2, 1)$ is $(-2, 4, -2)$.

Solution:
The distance is a number, not a vector.

2) T **F** The plane $y = 3$ is perpendicular to the xz plane.

Solution:
The plane is parallel to the xz plane.

3) T **F** All functions $u(x, y)$ that obey $u_x = u$ at all points obey $u_y = 0$ at all points.

Solution:
The function $u(x, y) = e^x y$ satisfies $u_x = u$ but $u_y = e^x$.

4) T **F** The best linear approximation at $(1, 1, 1)$ to the function $f(x, y, z) = x^3 + y^3 + z^3$ is the function $L(x, y, z) = 3x^2 + 3y^2 + 3z^2$

Solution:
The linear approximation is a linear function in x, y, z. The correct linear approximation would be $L(x, y, z) = 3 + 3(x - 1) + 3(y - 1) + 3(z - 1)$.

5) T **F** If $f(x, y)$ is any function of two variables, then $\int_0^1 \left(\int_x^1 f(x, y)\, dy\right) dx = \int_0^1 \left(\int_y^1 f(x, y)\, dx\right) dy$.

Solution:
The correct identity would be $\int_0^1 \left(\int_x^1 f(x, y)\, dy\right) dx = \int_0^1 (\int_0^y f(x, y)\, dx)\, dy$.

6) **T** F Let $C = \{(x, y) \mid x^2 + y^2 = 1\}$ be the unit circle in the plane and $\vec{F}(x, y)$ a vector field satisfying $|\vec{F}| \leq 1$. Then $-2\pi \leq \int_C \vec{F} \cdot dr \leq 2\pi$.

Solution:
By definition: $\int \vec{F} \cdot dr = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$ and so $|\int \vec{F} \cdot \vec{dr}| = \int_0^{2\pi} |\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)|\, dt \leq \int_0^{2\pi} |\vec{F}(\vec{r}(t))||\vec{r}'(t)|\, dt \leq \int_0^{2\pi} |\vec{r}\,'(t)|\, dt = 2\pi$.

7) | T | **F** | Let $\vec{a}$ and $\vec{b}$ be two nonzero vectors. Then the vectors $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ always point in different directions.

Solution:
Take $\vec{a} = \langle 4, 2 \rangle$ and $\vec{b} = \langle 2, 1 \rangle$. Then $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ point in the same direction.

8) | T | **F** | If all the second-order partial derivatives of $f(x, y)$ vanish at (x_0, y_0) then (x_0, y_0) is a critical point of f.

Solution:
Take $f(x, y) = x + x^3 + y^3$. Then $(0, 0)$ not a critical point even so all second-order partial derivatives are zero.

9) | **T** | F | If $\vec{a}, \vec{b}$ are vectors, then $|\vec{a} \times \vec{b}|$ is the area of the parallelogram determined by $\vec{a}$ and $\vec{b}$.

Solution:
$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|| \sin(\phi)|$. $|\vec{a}|$ is the length of the base of the parallelogram and $|\vec{b}|| \sin(\phi)|$ is the height of the parallelogram.

10) | **T** | F | The distance between two points A, B in space is the length of the curve $\vec{r}(t) = A + t(B - A),\ t \in [0, 1]$.

Solution:
$|\vec{r}'(t)| = |B - A|$ and $\int_0^1 |r'(t)|\, dt = |B - A|$.

11) | T | **F** | The function $f(x, y) = xy$ has no critical point.

Solution:
$(0, 0)$ is a critical point of f.

12) T F The length of a curve does not depend on the chosen parameterization.

Solution:
This is a consequence of the chain rule: if $r(s(t))$ is a new parameterization, then $\int_a^b |r(s(t))'|\, dt = \int_a^b |r'(s(t))||s'(t)|dt = \int_{s(a)}^{s(b)} |r'(s)|ds$.

13) T F There exists a non-zero function $f(x, y, z)$ and non-zero vector field $\vec{F}(x, y, z)$ so that $\vec{F} = \text{grad}(f)$ and $f = \text{div}\vec{F}$.

Solution:
We need a solution to $\text{graddiv}(F) = F$. Trying $F = \langle P, 0, 0\rangle$ we get $F = \langle e^x, 0, 0\rangle$.

14) T F For any numbers a, b satisfying $|a| \neq |b|$, the vector $\langle a - b, a + b\rangle$ is perpendicular to $\langle a + b, b - a\rangle$.

Solution:
The dot product is 0.

15) T F The line integral of $\vec{F}(x, y) = \langle -y, x\rangle$ along the counterclockwise oriented boundary of a region R is twice the area of R.

Solution:
The curl of $\vec{F}$ is 2. The statement is a consequence of Greens theorem.

16) T F There is no surface for which both the parabola and the hyperbola appear as traces.

Solution:
The hyperbolic paraboloid is an example.

17) T F If $(u, v) \mapsto \vec{r}(u, v)$ is a parameterization for a surface, then $\vec{r}_u(u, v) + \vec{r}_v(u, v)$ is a vector which lies in the tangent plane to the surface.

Solution:
Both $\vec{r}_u(u, v)$ as well as as $\vec{r}_v(u, v)$ are tangent to the surface. Therefore, also the sum is tangent.

18) T **F** When using spherical coordinates in a triple integral, one needs to include the volume element $dV = \rho^2 \cos(\phi)\ d\rho d\phi d\theta$.

Solution:
The correct factor would be $\rho^2 \sin(\phi)$.

19) **T** F A surface in space for which all normal vectors are parallel to each other must be part of a plane.

Solution:
Assume all normal vectors are parallel to $\vec{n} = \langle a, b, c \rangle$ and assume $\vec{x}_0$ is a point in the plane. Then $\vec{n}(\vec{x} - \vec{x}_0) = 0$. But this is the equation of a plane. (This TF question implicitly assumes the surface to be connected. Also a union of parallel planes has the property that all normal vectors are parallel.)

20) **T** F A vector field $\vec{F} = \langle P(x, y), Q(x, y) \rangle$ is conservative in the plane if and only if $P_y(x, y) = Q_x(x, y)$ for all points (x, y).

Solution:
This is a consequence of Green's theorem: the line integral along a closed curve is the double integral of $Q_x - P_y$ over the enclosed region and so zero. The reverse is easier to see: if a potential f satisfying $\nabla f = \langle f_x, f_y \rangle = \langle P, Q \rangle$ exists, then $Q_x - P_y = f_{yx} - f_{xy} = 0$ (Clairot).

Problem 2) (10 points)

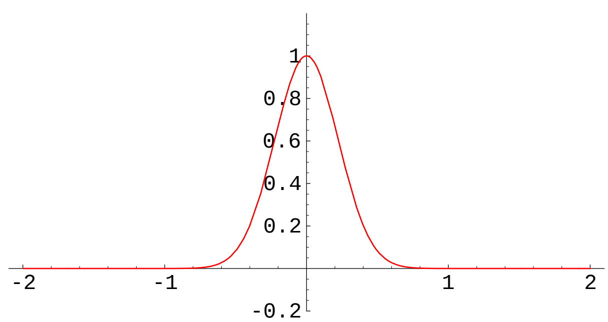
2 a) (5 points) Fill in names of the mathematicians: Green, Stokes, Gauss, Fubini, Clairot. If there is no name associated to the theorem, write the name of the theorem.

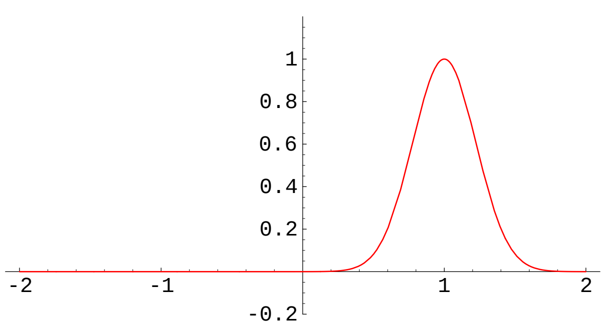
Formula	Name of the theorem
$\int_C \vec{F} \cdot d\vec{r} = \int \int_S \text{curl}(\vec{F}) \cdot dS$	
$f_{xy}(x, y) = f_{yx}(x, y)$	
$\int_C \vec{F} \cdot dr = \int \int_R \text{curl}(\vec{F})\ dxdy$	
$\int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t))\ dt = f(\vec{r}(b)) - f(\vec{r}(a))$	
$\int \int_S F \cdot dS = \int \int \int_E \text{div}(F)\ dV$	
$\int_a^b \int_c^d f(x, y)\ dxdy = \int_c^d \int_a^b f(x, y)\ dydx$	

Solution:

Formula	Name of the theorem
$\int_C \vec{F} \cdot d\vec{r} = \int \int_S \text{curl}(\vec{F}) \cdot dS$	Stokes
$f_{xy}(x, y) = f_{yx}(x, y)$	Clairot
$\int_C \vec{F} \cdot dr = \int \int_R \text{curl}(\vec{F})\ dxdy$	Green
$\int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t))\ dt = f(\vec{r}(b)) - f(\vec{r}(a))$	Fundamental theorem of line integrals
$\int \int_S F \cdot dS = \int \int \int_E \text{div}(F)\ dV$	Gauss
$\int_a^b \int_c^d f(x, y)\ dxdy = \int_c^d \int_a^b f(x, y)\ dydx$	Fubini

2 b) (5 points) We have a function $u(t, x)$ which is a solution to a partial differential equation. In all cases, we have $u(0, x) = e^{-x^2}$. The picture to the right shows this function $u(0, x)$. Which partial differential equation is involved, when you see the function $u(1, x)$ as a graph?





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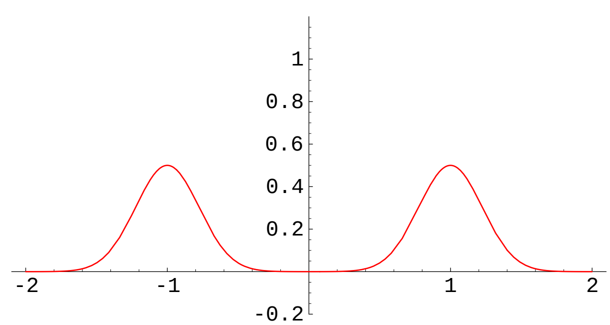

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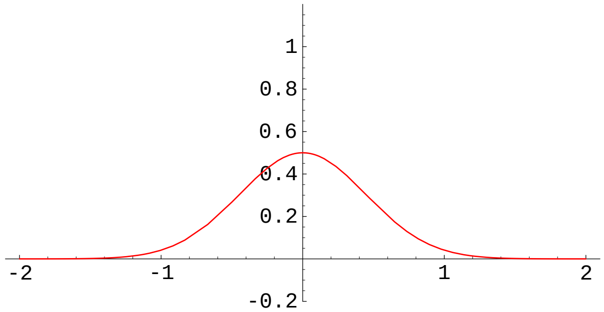

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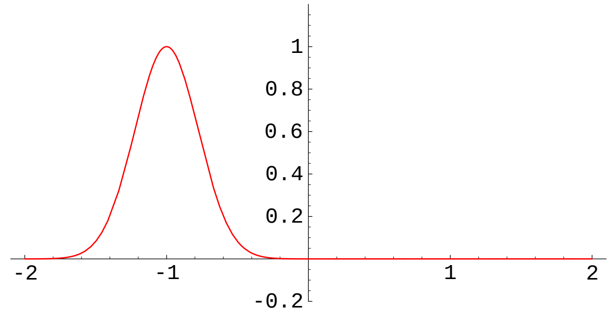


IV

Enter I,II,III,IV here	Equation
	$u_t(x,t) = u_x(x,t)$
	$u_t(x,t) = u_{xx}(x,t)$
	$u_{tt}(x,t) = u_{xx}(x,t)$
	$u_t(x,t) = -u_x(x,t)$

Solution:

Enter I,II,III,IV here	Equation
IV	$u_t(x,t) = u_x(x,t)$
III	$u_t(x,t) = u_{xx}(x,t)$
II	$u_{tt}(x,t) = u_{xx}(x,t)$
I	$u_t(x,t) = -u(x,t)$

Some more explanation:
Case $u_t = u_x$: If you look at the traces where t is fixed. This gives functions of one variable. $u_t = u_x$ means that the function will increase, where the slope u_x is positive and decrease, where the slope u_x is negative. This makes the graph move to the left. With $g(x) = u(0,x)$, the general solution is $u(t,x) = g(x+t)$ as you can check.
Case: $u_t = -u_x$: Dito, the graph moves to the right. With $g(x) = u(0,x)$, the general solution is $u(t,x) = g(x-t)$ as you can check.
Case: $u_t = u_{xx}$: The function will decrease where the second derivative u_{xx} is positive. So, the bump will become smaller. The tails of the bump have $u_x x < 0$, there, the function will increase.
Case: $u_{tt} = u_{xx}$: This is the wave equation. You best look at it physically. Pluck an infinite string in the middle. The disturbance will travel to both sides. You might also recall that with $g(x) = u(0,x)$ the function $u(t,x) = (g(x-t) + g(x+t))/2$ is the general solution.

Problem 3) (10 points)

a) Find an equation for the plane Σ passing through the points $P = (1, 0, 1)$, $Q = (2, 1, 3)$ and $R = (0, 1, 5)$.

b) Find the distance from the origin $O = (0, 0, 0)$ to Σ.

c) Find the distance from the point P to the line through Q, R.

d) Find the volume of the parallelepiped with vertices O, P, Q, R.

Solution:
a) Take the cross product between two vectors in the plane to get a normal vector $n = (a, b, c) = (1, 1, 2) \times (-1, 1, 4) = (2, -6, 2)$. The plane has the equation $ax + by + cz = d$, where d is obtained from plugging in one of the points. The answer is $\boxed{x - 3y + z = 2.}$.

b) The distance is $|(1, 0, 1) \cdot \vec{n}|/|n| = \boxed{2/\sqrt{11}}$.

c) The distance is $|\vec{PQ} \times \vec{QR}|/|\vec{QR}| = \sqrt{11/2}$.

d) The volume is $\det \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 0 & 1 & 5 \end{bmatrix} = 4$.

Problem 4) (10 points)

The equation $f(x, y, z) = e^{xyz} + z = 1 + e$ implicitly defines z as a function $z = g(x, y)$ of x and y.

a) Find formulas (in terms of x,y and z) for $g_x(x, y)$ and $g_y(x, y)$.

b) Estimate $g(1.01, 0.99)$ using linear approximation.

Solution:
a) By the chain rule $g_x = -f_x/f_z = -yze^{xyz}/(1+xye^{xyz})$ and $g_y = -f_y/f_z = -xze^{xyz}/(1+xye^{xyz})$.
b) Note that $f(1, 1, 1) = 1 + e$ so that $(1, 1, 1)$ is on the surface and $g(1, 1) = 1$. From $a)$ we know $g_x(1, 1) = -e/(1+e)$ and $g_y(1, 1) = -e/(1+e)$. By linearization, $g(1+a, 1+b) = g(1, 1)+g_x(1, 1)a+g_y(1, 1)b = 1-(a+b)e/(1+e)$. In our case, where $a = 0.01, b = -0.01$ we estimate $g(1.01, 0.99) = 1$.

Problem 5) (10 points)

Find the surface area of the surface S parametrized by $\vec{r}(u,v) = \langle u, v, 2 + \frac{u^2}{2} + \frac{v^2}{2} \rangle$ for (u,v) in the disc $D = \{u^2 + v^2 \leq 1\}$.

Solution:
$r_u = (1,0,u), r_v = (0,1,v)$ and $r_u \times r_v = (-u,-v,1)$.
The surface area is $\int\int_D \sqrt{1+u^2+v^2}\ dudv = \int_0^1 \int_0^{2\pi} \sqrt{1+r^2} r\ d\theta dr = 2\pi \int_0^1 \sqrt{1+r^2} r\ dr =$ $2\pi(1+r^2)^{3/2}/3|_0^1 = \boxed{2\pi(\sqrt{8}-1)/3}$.

Problem 6) (10 points)

Find the local and global extrema of the function $f(x,y)$ which is the curl of $\vec{F}(x,y) = \langle -y^4/12 + y^3/6 - y, x^4/12 - x^3/6 \rangle$ on the disc $\{x^2 + y^2 \leq 4\}$.

a) Classify every critical point inside the disc $x^2 + y^2 < 4$.

b) Find the extrema on the boundary $\{x^2 + y^2 = 4\}$ using the method of Lagrange multipliers.

c) Determine the global maxima and minima on all of D.

Solution:
a) The function is $f(x,y) = x^3/3 + y^3/3 - x^2/2 - y^2/2 + 1$. We have $\nabla f = (x^2 - x, y^2 - y)$. The critical points of f are $(0,0), (0,1), (1,0), (1,1)$.
The Hessian determinant (=discriminant) is $D = (2x-1)(2y-1)$, which is $1, -1, -1, 1$. The point $(0,0)$ is a local maximum, the point $(1,1)$ is a local minimum and $(0,1), (1,0)$ are saddle points.
b) $g(x,y) = x^2 + y^2 - 4$. The Lagrange equations are
$x^2 - x = \lambda 2x$
$y^2 - y = \lambda 2y$
$x^2 + y^2 - 4 = 0$
which have solutions $\boxed{(-2,0), (0,-2), (0,2), (2,0), (-\sqrt{2}, -\sqrt{2}), (\sqrt{2}, \sqrt{2})}$. At these points, the function f takes values $-14/3, -14/3, 2/3, 2/3, -\sqrt{2}, -2-4\sqrt{2}/3, -2+4\sqrt{2}/3$.
c) To find the maximum and minimum, just compare the values at all the candidates obtained in a) and b):
$\boxed{\text{the minimum is at the two points } (-2,0), (0,-2)}$
where f takes the value $-14/3$.
$\boxed{\text{The global maximum is at } (0,0 \text{ because the value is there larger than } f(2,0) = f(0,2) = 2/3}$.

Problem 7) (10 points)

a) Given two nonzero vectors $\vec{u} = \langle a, b, c \rangle$ and $\vec{v} = \langle d, e, f \rangle$ in space, write down a formula for the cosine of the angle between them. Find a nonzero vector $\vec{v}$ that is perpendicular to $\vec{u} = \langle 3, 2, 1 \rangle$. Describe geometrically the set of all $\vec{v}$, including zero, that are perpendicular to this vector $\vec{u}$.

b) Consider a function f of three variables. Explain with a picture and a sentence what it means geometrically that $\nabla f(P)$ is perpendicular to the level set of f through P.

c) Assume the gradient of f at P is nonzero. Write a few sentences that would convince a skeptic that $\nabla f(P)$ is perpendicular to the level set of f at the point P.

d) Assume the level set of f is the graph of a function $g(x,y)$. Explain the relation between the gradient of g and the gradient of f. Especially, how do you relate the orthogonality of ∇f to the level set of f with the orthogonality of ∇g to the level set of g?

Solution:
a) $\cos(\alpha) = ad + be + cf/(\sqrt{a^2+b^2+c^2}\sqrt{d^2+e^3+f^2})$. The vector $(1,-1,-1)$ is perpendicular to $(3,2,1)$. The set of points which are perpendicular to $(3,2,1)$ satisfies the equation $3x+2y+z=0$.
b) ∇f is orthogonal to every tangent vector to the level surface. ∇f is orthogonal to the velocity vector of a curve $r(t)$ on the level surface.
c) Take two curves on the level surface which cross transversally. Because f is constant on the level surface, $f(r(t)) = c$ is constant or by the chain rule, $d/dt f(r(t)) = \nabla f(r(t)) \cdot r'(t) = 0$. This means that ∇f is orthogonal to the velocity vector $r'(t)$. By b), ∇f is orthogonal to the level surface.
d) In the special case $f(x,y,z) = g(x,y) - z$, we have $\nabla f = (\nabla g, -1)$. We see that projecting the vector ∇f onto the xy plane gives the vector ∇g.

Problem 8) (10 points)

Let R be the region inside the circle $x^2+y^2=4$ and above the line $y=\sqrt{3}$. Evaluate

$$\int\int_R \frac{y}{x^2+y^2}\, dA\ .$$

Solution:
The key is to set up the integral in polar coordinates:

$$\int_{\pi/3}^{2\pi/3} \int_{\sqrt{3}/\sin(\theta)}^{2} \frac{r\sin(\theta)}{r^2}\, r dr d\theta\ .$$

This gives $\int_{\pi/3}^{2\pi/3} (2\sin(\theta) - \sqrt{3})\, d\theta = (-2\cos(\theta) - \sqrt{3}\theta)|_{\pi/3}^{2\pi/3} = -2(-1/2-1/2) - \sqrt{3}\pi/3 =$ $\boxed{2-\sqrt{3}\pi/3}$.

Problem 9) (10 points)

A region W in $\mathbf{R}^3$ is given by the relations

$$x^2+y^2 \le z^2 \le 3(x^2+y^2)$$
$$1 \le x^2+y^2+z^2 \le 4$$
$$x \ge 0$$

1. Sketch the region W.

2. Find the volume of the region W.

Solution:

1. First inequalities: the region is sandwiched between two cones. Second inequalities: the region is sandwiched between two spheres.
2. Use spherical coordinates:

$$\int_1^2 \int_{\pi/6}^{\pi/4} \int_{-\pi/2}^{\pi/2} \rho^2 \sin(\phi)\; d\theta d\phi d\rho + \int_1^2 \int_{3\pi/4}^{5\pi/6} \int_{-\pi/2}^{\pi/2} \rho^2 \sin(\phi)\; d\theta d\phi d\rho$$

Both integrals are the same so that we have to compute one and multiply in the end by 2. The answer is the product of the integrals $\int_1^2 \rho^2\; d\rho = 7/3$, $\int_{-\pi/2}^{\pi/2}\; d\theta = \pi$ and $\int_{\pi/6}^{\pi/4} \sin(\phi)\; d\phi = (\sqrt{3} - \sqrt{2})/2$ which is $(7/3)\pi(\sqrt{3} - \sqrt{2})/2$. The final answer is $\boxed{(7/3)\pi(\sqrt{3} - \sqrt{2})}$.

Problem 10) (10 points)

Consider the vector field

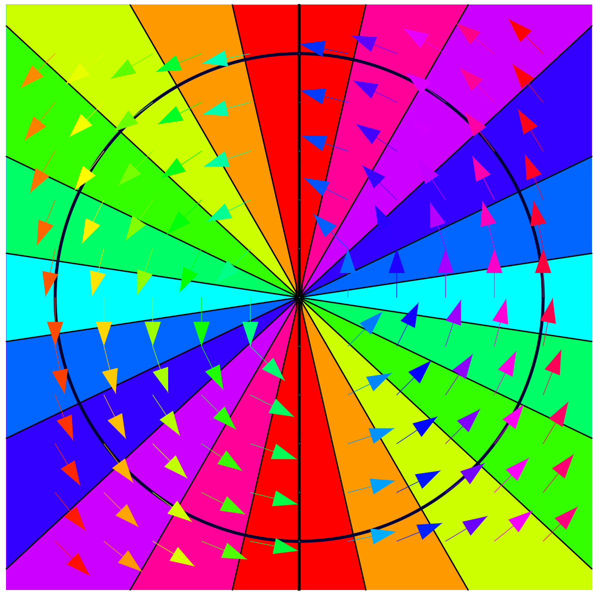
$$\vec{F}(x, y) = \langle \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \rangle$$

defined everywhere in the plane $\mathbf{R}^2$ except at the origin.

a) Let C be any closed curve which bounds a region D. Assume that $(0, 0)$ is not contained in D and does not lie on C. Explain why

$$\int_C \vec{F} \cdot d\vec{r} = 0 \; .$$

b) Let C be the unit circle oriented counterclockwise. What is $\int_C \vec{F} \cdot d\vec{r}$? Explain why your answer shows that there is no function f for which $\vec{F}(x, y) = \nabla f(x, y)$ everywhere.



Solution:
a) F is conservative in the region $\mathbf{R}^2 \setminus \{(0,0)\}$ because with $\vec{F} = \langle P, Q \rangle$ we have $Q_x - P_y = 0$ and Greens theorem assures that the line integral along any closed curve is zero, provided the region does not contain 0.
b) A direct calculation with $r(t) = (x(t), y(t)) = (\cos(t), \sin(t))$ gives $\int_0^{2\pi} 1 \; dt = 2\pi$. If there would exist a potential function f defined all over the plane, then the line integral along any closed curve would be zero. This would contradict the result for the curve along the unit circle.

Problem 11) (10 points)

First use rectangular, then cylindrical and finally spherical coordinates to integrate the function $f(x, y, z) = xyz$ over the solid in space described by the inequalities $0 \leq z \leq \sqrt{(1 - x^2 - y^2)}, x^2 + y^2 \leq 1, x - y \geq 0, y \geq 0$.

Solution:
Euclidean: $\int_0^{1/\sqrt{2}} \int_y^{\sqrt{1-y^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz \; dz \; dx \; dy$.
Cylindrical: $\int_0^1 \int_0^{\pi/4} \int_0^{\sqrt{1-r^2}} r^3 \cos(\theta) \sin(\theta) z \; dz \; d\theta \; dr$.
Spherical: $\int_0^1 \int_0^{\pi/4} \int_0^{\pi/2} \rho^3 \cos(\phi) \sin^2(\phi) \cos(\theta) \sin(\theta) \rho^2 \sin(\phi) \; d\phi d\theta \; d\rho$.

Problem 12) (10 points)

Let $\vec{F}(x,y)$ be a vector field in the plane given by the formula

$$\vec{F}(x,y) = \langle x^2 - 2xye^{-x^2} + 2y, e^{-x^2} + \frac{1}{\sqrt{y^4+1}} \rangle .$$

If C is the path which goes from from $(-1,0)$ to $(1,0)$ along the semi circle $x^2+y^2=1$, $y \geq 0$, evaluate $\int_C \vec{F} \cdot d\vec{r}$.

Solution:
The curl of F is -2. The negative of line integral along the curve C_1 from $(-1,0)$ to $(1,0)$ plus the line integral along the curve $C_2 : r(t) = (t,0)$ for $t \in [-1,1]$ is by Green's theorem $\int \int_D (-2)\, dxdy = -\pi$.

$$-\int_{C_1} \vec{F}\, dr + \int_{C_2} \vec{F}\, dr = \int \int_G \text{curl}(F)\, dxdy = -\pi$$

The line integral along the line segment C_2 is $\int_{-1}^{1} t^2\, dt = \boxed{2/3}$.

$$-\int_{C_1} \vec{F}\, dr + (2/3) = -\pi$$

Therefore, the final result for $\int_{C_1} \vec{F}\, dr$ is $\boxed{2/3 + \pi}$.

Problem 13) (10 points)

In appropriate units, the charge density $\sigma(x,y,z)$ in a region in space is given by $\sigma = \nabla \cdot \vec{E} = \text{div}(\vec{E})$, where $\vec{E}$ is the electric field. Consider the cube of side lengths 1 given by $0 \leq x,y,z \leq 1$. What is the total charge in this cube if

$$\vec{E} = \langle x(1-x)\log(1+xyz), y(1-y)\tan(x^3+y^3+z^3), z(1-z)e^{\sqrt{x+y}} \rangle .$$

(The total charge is the integral of the charge density over the cube.)

Solution:
On the x-faces, we know that $\vec{F} = \langle 0, Q, R \rangle$. The flux through the x-faces (normal to the x-axes) is 0. Similarly, the fluxes through the other sides is zero. By the divergence theorem, the triple integral on the unit cube is $\boxed{0}$.

Problem 14) (10 points)

a) By calculating the integral $\int\int_S \vec{F} \cdot \vec{dS}$ directly, find the flux of the vector field $\vec{F}(x,y,z) = \langle 0,0,x+z \rangle$ through the sphere $x^2+y^2+z^2 = 9$, where the sphere is oriented with the normal pointing outward.

b) Find the flux of the vector field $\vec{F}(x,y,z) = \langle 0,0,x+z \rangle$ through the sphere $x^2+y^2+z^2 = 9$ using the divergence theorem.

c) Explain in words without invoking any integral theorem, why the flux integral of the vector field $\vec{F}(x,y,z) = \langle 0,0,x+z \rangle$ through any sphere with positive radius centered at $(0,0,0)$ is positive. A one or two sentence explanation is sufficient, but it should be formulated so that it makes sense to somebody who does not know calculus.

Solution:

a) We parametrize the sphere as usual and get $\vec{r}_\phi(\phi,\theta) \times \vec{r}_\theta(\theta,\phi) = \rho^2 \sin(\phi)\vec{r}(\theta,\phi)$ so that $\vec{F}(\vec{r}(\theta,\phi)) \cdot \vec{r}_\theta(\theta,\phi) \times \vec{r}_\phi(\theta,\phi) = \rho^3 \cos^2(\phi)\sin(\phi) + \rho^3 \cos(\theta)\sin^2(\phi)$. When we integrate the second summand over θ, we get zero. We are left with

$$\int_0^{2\pi} \int_0^{\pi} 27\cos^2(\phi)\sin(\phi)\ d\phi d\theta == 27 \cdot 2\pi \cdot (-\frac{\cos^3(\phi)}{3}|_0^\pi) =$$

$\boxed{36\pi}$. b) The divergence of $\vec{F}$ is 1. The integral of the divergence over the sphere of radius R is $4\pi R^3/3 = \boxed{36\pi}$.

c) The flux integral of $\vec{F}(x,y,z) = \langle 0,0,x \rangle$ is zero by symmetry: there is the same flux on upper and lower hemisphere with opposite sign. The flux integral of $\vec{F}(x,y,z) = \langle 0,0,z \rangle$ is positive because everywhere on the sphere, the normal vector and the field form an acute angle.