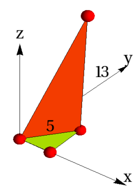
1: Geometry and Distance

A point in the **plane** has two **coordinates** $P = (x, y)$ like $P = (2, -3)$. A point in space is determined by three coordinates $P = (x, y, z)$ like $P = (1, 2, 4)$. The signs of the coordinates define 4 **quadrants** in the plane and 8 **octants** in space. The point $P = (1, 2, 4)$ is in the **first octant**. These regions by intersect at the **origin** $O = (0, 0)$ or $O = (0, 0, 0)$ and are separated by **coordinate axes** $\{y = 0\}$ and $\{x = 0\}$ or **coordinate planes** $\{x = 0\}, \{y = 0\}, \{z = 0\}$.

1 Describe the location of the points $P = (1, 2, 3), Q = (0, 0, -5), R = (1, 2, -3)$ in words. **Possible Answer:** $P = (1, 2, 3)$ is in the positive octant of space, where all coordinates are positive. The point $R = (0, 0, -5)$ is on the negative z axis. The point $S = (1, 2, -3)$ is below the xy-plane. When projected onto the xy-plane it is in the first quadrant.

2 **Problem.** Find the midpoint M of $P = (1, 2, 5)$ and $Q = (-3, 4, 9)$. **Answer.** The midpoint is obtained by taking the average of each coordinate $M = (P + Q)/2 = (-1, 3, 7)$.

The **Euclidean distance** between two points $P = (x, y, z)$ and $Q = (a, b, c)$ in space is defined as $d(P, Q) = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}$.



This definition of Euclidean distance is **motivated** by the **Pythagorean theorem**. [1]

3 Find the distance $d(P, Q)$ between the points $P = (1, 2, 5)$ and $Q = (-3, 4, 7)$ and verify that $d(P, M) + d(Q, M) = d(P, Q)$. **Answer:** The distance is $d(P, Q) = \sqrt{4^2 + 2^2 + 2^2} = \sqrt{24}$. The distance $d(P, M)$ is $\sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}$. The distance $d(Q, M)$ is $\sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}$. Indeed $d(P, M) + d(M, Q) = d(P, Q)$.

A **circle** of radius r centered at $P = (a, b)$ is the collection of points in the plane which have distance r from P.

A **sphere** of radius ρ centered at $P = (a, b, c)$ is the collection of points in space which have distance ρ from P. The equation of a sphere is $(x-a)^2 + (y-b)^2 + (z-c)^2 = \rho^2$.

4 Is the point $(3, 4, 5)$ outside or inside the sphere $(x-2)^2 + (y-6)^2 + (z-2)^2 = 16$? **Answer**: The distance of the point to the center of the sphere is $\sqrt{1 + 4 + 9}$ which is smaller than 4 the radius of the sphere. The point is inside.

[1] It appears in an appendix to "Geometry" of "Discours de la méthode" from 1637, **René Descartes** (1596-1650). More about Descartes in Aczel's book "Descartes Secret Notebook".

The **completion of the square** of an equation $x^2 + bx + c = 0$ is the idea to add $(b/2)^2 - c$ on both sides to get $(x + b/2)^2 = (b/2)^2 - c$. Solving for x gives the solution $x = -b/2 \pm \sqrt{(b/2)^2 - c}$.

[2]

5 Solve $2x^2 - 10x + 12 = 0$. **Answer.** The equation is equivalent to $x^2 + 5x = -6$. Adding $(5/2)^2$ on both sides gives $(x + 5/2)^2 = 1/4$ so that $x = 2$ or $x = 3$.

6 Find the center of the sphere $x^2 + 5x + y^2 - 2y + z^2 = -1$. **Answer:** Complete the square to get $(x+5/2)^2 - 25/4 + (y-1)^2 - 1 + z^2 = -1$ or $(x-5/2)^2 + (y-1)^2 + z^2 = (5/2)^2$. We see a sphere **center** $(5/2, 1, 0)$ and **radius** $5/2$.

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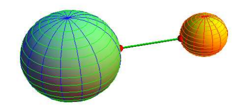

Distance between spheres

7 Find the set of points $P = (x, y, z)$ in space which satisfy $x^2 + y^2 = 9$. **Answer:** This is a cylinder of radius 3 around the z-axes parallel to the y axis.

8 What is $x^2 + y^2 = z^2$. **Answer**: this is the set of points for which the distance to the z axes is equal to the distance to the xy-plane. It must be a cone.

9 Find the distances of $P = (12, 5, 3)$ to the xy-plane. **Answer:** 3. Find the distance of $P = (12, 5, 0)$ to z axes. **Answer:** 13.

10 Describe $x^2 + 2x + y^2 - 16y + z^2 + 10z + 54 = 0$. **Answer:** Complete the square to get a sphere $(x+2)^2 + (y-8)^2 + (z+5)^2 = 36$ with center $(-2, 8, -5)$ and radius 6.

11 Describe the set $xz = x$. **Answer:** We either must have $x = 0$ or $z = 1$. The set is a union of two coordinate planes.

12 Find an equation for the set of points which have the same distance to $(1, 1, 1)$ and $(0, 0, 0)$. **Answer:** $(x-1)^2 + (y-1)^2 + (z-1)^2 = x^2 + y^2 + z^2$ gives $-2x + 1 - 2y + 1 - 2z + 1 = 0$ or $2x + 2y + 2z = 3$. This is the equation of a plane.

13 Find the distance between the spheres $x^2 + (y-12)^2 + z^2 = 1$ and $(x-3)^2 + y^2 + (z-4)^2 = 9$. **Answer:**The distance between the centers is $\sqrt{3^2 + 4^2 + 12^2} = 13$. The distance between the spheres is $13 - 3 - 1 = 9$.

[2] Due to **Al-Khwarizmi** (780-850) in *"Compendium on Calculation by Completion and Reduction"* The book "The mathematics of Egypt, Mesopotamia,China, India and Islam, a Source book, Ed Victor Katz, contains translations of some of this work.