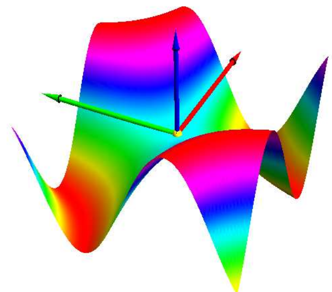
Lecture 5: Functions

A **function of two variables** $f(x,y)$ is a rule which assigns to two numbers x, y a third number $f(x,y)$. For example, the function $f(x,y) = x^2y+2x$ assigns to $(3,2)$ the number $3^2 2 + 6 = 24$. The **domain** D of a function is set of points where f is defined, the range is $\{f(x,y) \mid (x,y) \in D\}$. The **graph** of $f(x,y)$ is the surface $\{(x,y,f(x,y)) \mid (x,y) \in D\}$ in space Graphs allow to visualize functions.

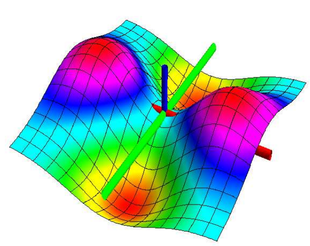


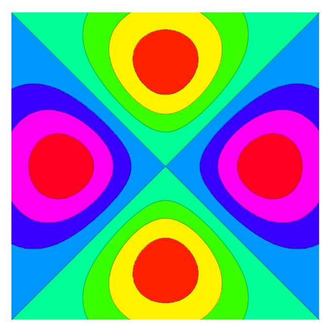
1 The graph of $f(x,y) = \sqrt{1-(x^2+y^2)}$ on the domain $D = \{x^2+y^2 < 1\}$ is a half sphere. The range is the interval $[0,1]$.

The set $f(x,y) = c =$ const is called a **contour curve** or **level curve** of f. For example, for $f(x,y) = 4x^2+3y^2$, the level curves $f = c$ are ellipses if $c > 0$. The collection of all contour curves $\{f(x,y) = c\}$ is called the **contour map** of f.

2 For $f(x,y) = x^2 - y^2$, the set $x^2 - y^2 = 0$ is the union of the lines $x = y$ and $x = -y$. The curve $x^2 - y^2 = 1$ is made of two hyperbola with with their "noses" at the point $(-1,0)$ and $(1,0)$. The curve $x^2 - y^2 = -1$ consists of two hyperbola with their noses at $(0,1)$ and $(0,-1)$.

3 For $f(x,y) = (x^2-y^2)e^{-x^2-y^2}$, we can not find explicit expressions for the contour curves $(x^2-y^2)e^{-x^2-y^2} = c$. but we can draw the curves with the computer:



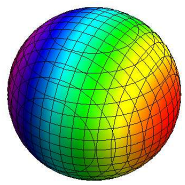


A function of three variables $g(x,y,z)$ assigns to three variables x, y, z a real number $g(x,y,z)$. We can visualize it by **contour surfaces** $g(x,y,z) = c$, where c is constant. It is helpful to look at the **traces**, the intersections of the surfaces with the coordinate planes $x = 0, y = 0$ or $z = 0$.

4 For $g(x,y,z) = z - f(x,y)$, the level surface $g = 0$ which is the graph $z = f(x,y)$ of a function of two variables. For example, for $g(x,y,z) = z - x^2 - y^2 = 0$, we have the graph $z = x^2 + y^2$ of the function $f(x,y) = x^2 + y^2$ which is a paraboloid. Most surfaces $g(x,y,z) = c$ are not graphs.

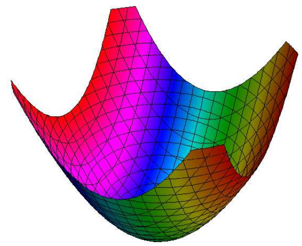
5 If $f(x,y,z)$ is a polynomial and $f(x,x,x)$ is quadratic in x, then $\{f=c\}$ is a **quadric**.

Sphere



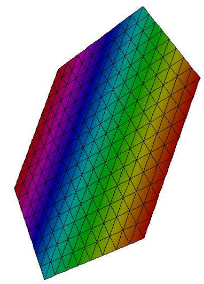
$x^2+y^2+z^2=1$

Paraboloid



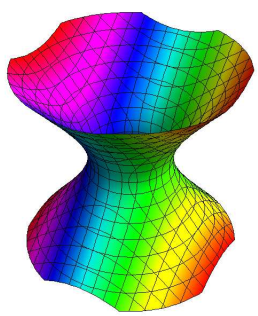
$x^2+y^2-c=z$

Plane



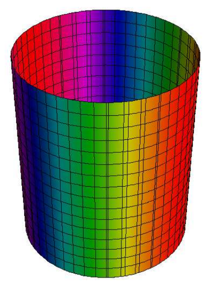
$ax+by+cz=d$

One sheeted Hyperboloid



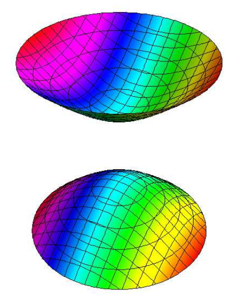
$x^2+y^2-z^2=1$

Cylinder



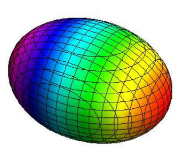
$x^2+y^2=r^2$

Two sheeted Hyperboloid



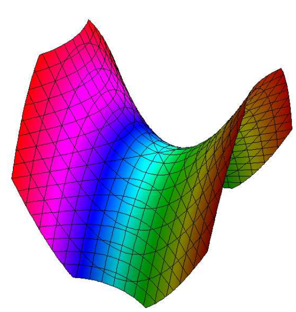
$x^2+y^2-z^2=-1$

Ellipsoid



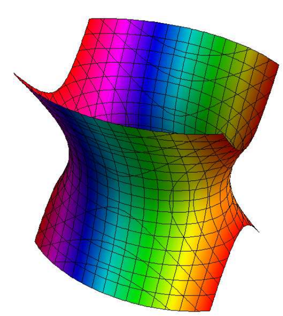
$x^2/a^2+y^2/b^2+z^2/c^2=1$

Hyperbolic paraboloid



$x^2-y^2+z=1$

Elliptic hyperboloid



$x^2/a^2+y^2/b^2-z^2/c^2=1$