

Name:

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| MWF 9 Jun-Hou Fung |
| MWF 9 Koji Shimizu |
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| MWF 10 Dusty Grundmeier |
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| MWF 11 Kate Penner |
| MWF 12 Yusheng Luo |
| MWF 12 YongSuk Moon |
| MWF 12 Will Boney |
| TTH 10 Peter Smillie |
| TTH 10 Chenglong Yu |
| TTH 11:30 Lukas Brantner |
| TTH 11:30 Yu-Wen Hsu |

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work.** Except for problems 1-3, we need to see **details** of your computation.
- All functions can be differentiated arbitrarily often unless otherwise specified.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

| | | |
|--------|--|-----|
| 1 | | 20 |
| 2 | | 10 |
| 3 | | 10 |
| 4 | | 10 |
| 5 | | 10 |
| 6 | | 10 |
| 7 | | 10 |
| 8 | | 10 |
| 9 | | 10 |
| 10 | | 10 |
| Total: | | 110 |

Problem 1) True/False (TF) questions (20 points)

Mark for each of the 20 questions the correct letter. No justifications are needed.

- 1) T F If two planes $ax + by + cz = d$ and $Ax + By + Cz = D$ are parallel then $a = A, b = B,$ and $c = C$.

Solution:

We can have $A = 2a, B = 2b, C = 2c, D = 2d$ for example.

- 2) T F The point $(x, y, z) = (1, 1, \sqrt{2})$ has the spherical coordinates $(\rho, \theta, \phi) = (2, \pi/4, \pi/4)$.

Solution:

Use the transformation formula.

- 3) T F Every point on the parametric curve $\vec{r}(t) = \langle t, t^2, -t \rangle$ lies on the surface $xz + y = 0$.

Solution:

Check with $x = t, y = t^2, z = -t$.

- 4) T F The two surfaces $f(x, y, z) = 3$ and $f(x, y, z) = 5$ of the function $f(x, y, z) = 2x^2 + y^3 + z^4$ do not intersect at any point in space.

Solution:

The function is continuous so that level surfaces to different values can not intersect. The function would take two values at such a point.

- 5) T F $\vec{u} \times \vec{i}$ and $\vec{u} \times \vec{j}$ are perpendicular for all vectors \vec{u} .

Solution:

Take $\vec{u} = \vec{i} + \vec{j}$.

- 6) T F If \vec{u} and \vec{v} are parallel (remember that this means $\vec{u} = \lambda\vec{v}$ for some real λ) then $\vec{u} \cdot \vec{v} \geq |\vec{u} \times \vec{v}|$.

Solution:

We can have $\vec{u} = -\vec{v}$ in which case the left hand side is negative if \vec{v} has positive length.

- 7)

| | |
|---|---|
| T | F |
|---|---|

 If a surface has the property that all intersections with the planes $y = \text{constant}$ are straight lines, then the surface is a plane.

Solution:

Take the function $y = x^2$ for example. Its graph is not a plane but $y = \text{constant}$ are lines.

- 8)

| | |
|---|---|
| T | F |
|---|---|

 For any non-zero vectors \vec{u} and \vec{w} , we must have $\text{Proj}_{\vec{u}}\vec{w} = -\text{Proj}_{\vec{w}}\vec{u}$.

Solution:

The projection onto \vec{u} is parallel to \vec{u} and the projection onto \vec{v} is parallel to \vec{v} .

- 9)

| | |
|---|---|
| T | F |
|---|---|

 In the parametric surface $\vec{r}(s, t) = \langle \sqrt{1+e^t} \cos(s), \sqrt{1+e^t} \sin(s), t \rangle$ the grid curves with constant s are ellipses.

Solution:

Take $s = 0$ to get the curve $\langle \sqrt{1+e^t}, 0, t \rangle$ which is the graph of a function of one variable in the xz plane.

- 10)

| | |
|---|---|
| T | F |
|---|---|

 There is a vector \vec{v} with the property that $\vec{v} \times \langle 1, 1, 1 \rangle = \langle 0, 0, 1 \rangle$.

Solution:

Whatever the vector is, the right hand side would be perpendicular to $\langle 1, 1, 1 \rangle$.

- 11)

| | |
|---|---|
| T | F |
|---|---|

 We can assign a value $f(0, 0)$ such that the function $f(x, y) = (x^3 + y^3)/(x^2 + y^2)$ is continuous at $(0, 0)$.

Solution:

Use polar coordinates to get $f = r^3(\cos^3(\theta) + \sin^3(\theta))/r^2 = r(\cos^3(\theta) + \sin^3(\theta))/r$.

- 12)

| | |
|---|---|
| T | F |
|---|---|

 The curvature of a curves $\vec{r}(t) = \langle t, t^2, t^3 \rangle$ and $\vec{R}(t) = \langle t^2, t^4, t^6 \rangle$ are the same at $t = 1$.

Solution:

Curvature is independent of the parametrization.

- 13) T F The curve given in spherical coordinates as $\phi = \pi/2, \rho = \pi/2$ is a circle.

Solution:

The result is zero

- 14) T F Two nonparallel planes with normal vectors \vec{n}, \vec{m} intersect in a line parallel to $\vec{n} \times \vec{m}$.

Solution:

Make a picture. This is a way to compute the intersection line.

- 15) T F If $f(x, y) = x^3/3 - y^2$, then the graph of the function $f(x, y)$ is called an elliptic paraboloid.

Solution:

It is a hyperbolic paraboloid

- 16) T F The equation $\rho \cos(\theta) \sin(\phi) = 2$ in spherical coordinates defines a plane.

Solution:

In spherical coordinates, we have $x = \rho \cos(\theta) \sin(\phi)$.

- 17) T F The vector $\langle 3, -2 \rangle$ in the two dimensional plane is perpendicular to the line $3x - 2y = 7$.

Solution:

It is the gradient $\langle 1, 2 \rangle$.

- 18) T F The volume of the parallelepiped spanned by the vectors $\langle 1, 0, 0 \rangle, \langle 0, 2, 0 \rangle$ and $\langle 1, 1, 1 \rangle$ is 2.

Solution:

Compute the triple scalar product which is 2.

- 19) T F If $\vec{r}(t)$ is a curve and $|\vec{r}'(t)| > 0$ and $|\vec{T}'| > 0$, we have $\vec{T}(t) \cdot (\vec{N}(t) \times \vec{B}(t)) = 1$.

Solution:

The three vectors are defined and are all perpendicular to each other and have length 1. They span a cube of volume 1.

- 20) T F The arc lengths of $\vec{r}(t) = \langle t, t^2, t^3 \rangle$ and $\vec{R}(t) = \langle t^2, t^4, t^6 \rangle$ are the same for $0 \leq t \leq 1$.

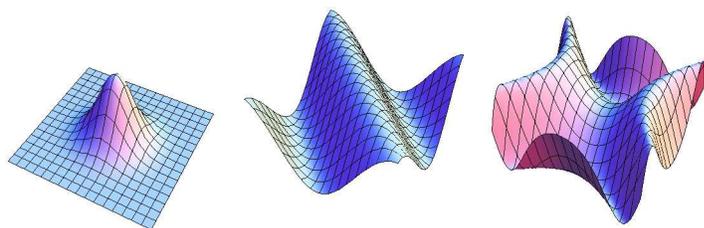
Solution:

This is an important property of arc length.

Total

Problem 2) (10 points)

a) (2 points) Match the graphs $z = f(x, y)$ with the functions. Enter O, if there is no match. In each of the problems a) - d), each entry O,I,II,III appears exactly once.



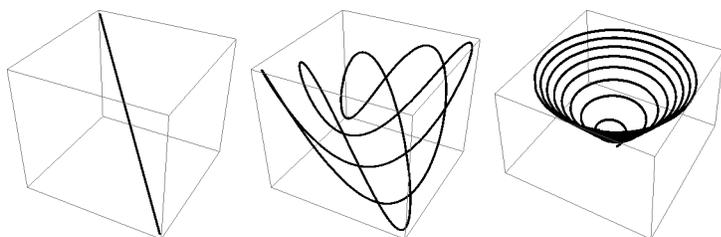
I

II

III

| Function $f(x, y) =$ | O,I,II or III |
|----------------------|---------------|
| $e^{-x^2-y^2}$ | |
| $\cos(x + y)$ | |
| $\sin(x^2 - y^2)$ | |
| $x^4 + y^4$ | |

b) (3 points) Match the space curves with their parametrizations $\vec{r}(t)$. Enter O, if there is no match.



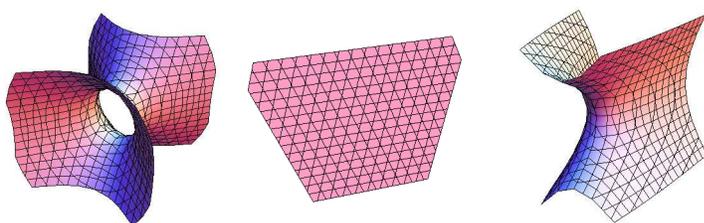
I

II

III

| Parametrization $\vec{r}(t) =$ | O, I,II,III |
|--|-------------|
| $\langle 1 + t, 1 - t, t \rangle$ | |
| $\langle t \cos(t^2), t \sin(t^2), t \rangle$ | |
| $\langle t, t, \sin(t^3) \rangle$ | |
| $\langle \cos(3t), \sin(2t), \sin(5t) \rangle$ | |

c) (2 points) Match the functions g with the level surface $g(x, y, z) = 1$. Enter O, where no match.



I

II

III

| $g(x, y, z) =$ | O, I,II,III |
|-----------------------------|-------------|
| $(x - 1)^2 - y^2 + z^2 = 1$ | |
| $(x - 1)^2 + y + z^2 = 1$ | |
| $(x - 1) + y + z = 1$ | |
| $(x - 1)^2 - y - z^2 = 1$ | |

d) (3 points) Match the surface with the parametrization. Enter O, where no match.



I

II

III

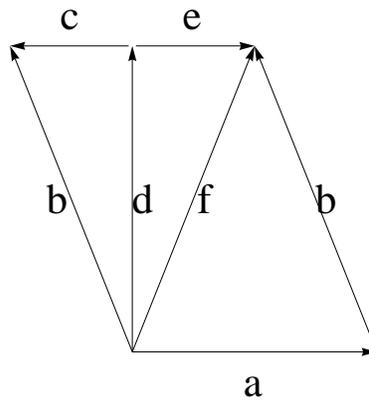
| Parametrization $\vec{r}(s, t) =$ | O,I,II,III |
|---|------------|
| $\langle s \cos(t), s \sin(t), s^2 \rangle$ | |
| $\langle t - 1, s, s + t \rangle$ | |
| $\langle \cos(t), \sin(t), s \rangle$ | |
| $\langle s \cos(t), s \sin(t), s^2 \sin(t) \rangle$ | |

Solution:

- a) I,II,III,O
- b) I,III,O,II
- c) I,O,II,III
- d) I,II,III,O

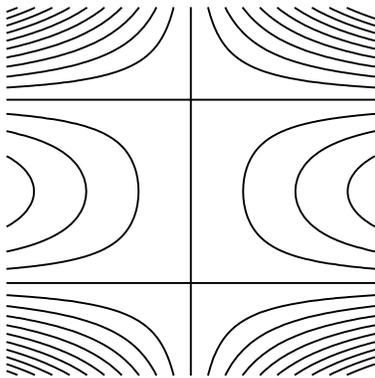
Problem 3) (10 points)

a) (7 points) Each of the vectors $a, b, c, d, e, f, 0$ (written without arrows for clarity) will appear in the blanks exactly once. As the picture indicates, you know $d \cdot e = d \cdot c = 0$.

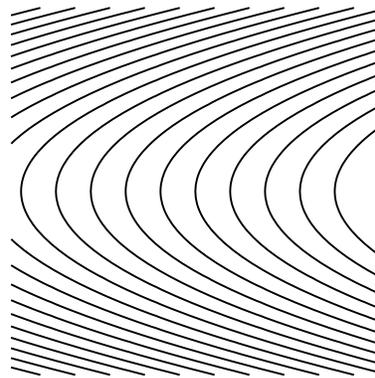


| the vector | is equal to |
|-------------------|-------------|
| $\text{Proj}_d f$ | |
| $f - d$ | |
| $-2c$ | |
| $d - c$ | |
| $-e$ | |
| $\text{Proj}_d e$ | |
| $d + c$ | |

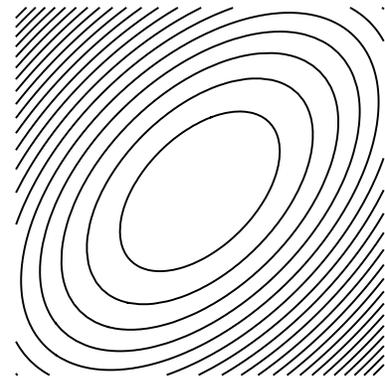
b) (3 points) Match the contour maps with the functions



I



II



III

| Function $f(x, y) =$ | Enter O,I,II or III |
|----------------------|---------------------|
| $y - x$ | |
| $(y^2 - 1)x$ | |
| $y^2 + x^2 - xy$ | |
| $y^2 - x$ | |

Solution:

Its obviously a "deaf cob":

| the vector | is equal to |
|---------------------------------|-------------|
| $\text{Proj}_{\vec{d}} \vec{f}$ | \vec{d} |
| $\vec{f} - \vec{d}$ | \vec{e} |
| $-2\vec{e}$ | \vec{a} |
| $\vec{d} - \vec{e}$ | \vec{f} |
| $-\vec{e}$ | \vec{c} |
| $\text{Proj}_{\vec{d}} \vec{e}$ | \vec{o} |
| $\vec{d} + \vec{e}$ | \vec{b} |

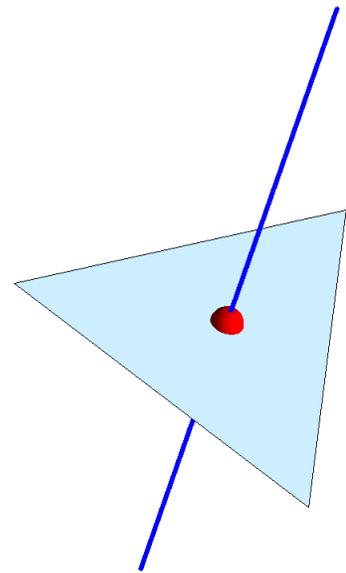
b) O,I,III,II

| |
|------------------------|
| Problem 4) (10 points) |
|------------------------|

a) (4 points) The **center** of the triangle $A = (3, 2, 1), B = (1, 1, 1), C = (2, 0, 4)$ is the point $P = (A + B + C)/3 = (2, 1, 2)$. Find the line L perpendicular to the plane which contains A, B, C and which goes through P .

b) (3 points) Find the equation of the plane through A, B, C .

c) (3 points) Find the area of the triangle ABC .



Solution:

The vectors $\vec{BA} = \langle 2, 1, 0 \rangle, \vec{BC} = \langle 1, -1, 3 \rangle$ are in the plane. Their cross product $\vec{n} = \langle 3, -6, -3 \rangle$ gives the direction normal to the plane as well as the direction of the line.

a) The equation of the line is $\vec{OP} + t\vec{n} = \langle 2, 1, 2 \rangle + t\langle 3, -6, -3 \rangle$.

b) Since the plane contains the point $(1, 1, 1)$, the equation of the plane is $3x - 6y - 3z = -6$.

c) The area of the triangle is half the area of the parallelogram which is the length of the cross product divided by 2. This is $3\sqrt{6}/2$.

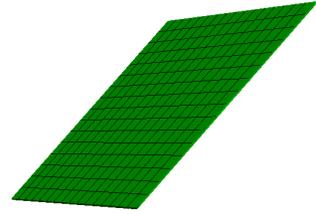
| |
|------------------------|
| Problem 5) (10 points) |
|------------------------|

Complete the parametrizations:

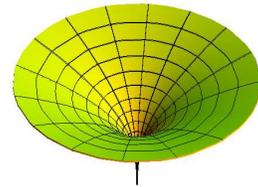
a) (3 points) $\vec{r}(u, v) = \langle 2 + 3 \cos(u) \sin(v), 3 + \sin(u) \sin(v), \boxed{} \rangle$ parametrizes the ellipsoid $(x - 2)^2/9 + (y - 3)^2 + (z - 5)^2/16 = 1$.



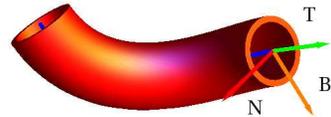
b) (2 points) $\vec{r}(u, v) = \langle u, v, \boxed{} \rangle$ parametrizes the plane $x + y + z = 1$.



c) (3 points) $\vec{r}(u, v) = \langle v^3 \cos(u), \boxed{}, v \rangle$ parametrizes the surface of revolution $x^2 + y^2 = z^6$.



d) (2 points) $\vec{r}(u, v) = \vec{r}(v) + \cos(u)\vec{N}(v) + \sin(u)\boxed{}$ parametrizes a tube around a curve $\vec{r}(v)$ which has unit tangent vector $\vec{T}(v)$, normal vector $\vec{N}(v)$ and binormal vector $\vec{B}(v)$.



Solution:

a) This is an ellipsoid centered at $(2, 3, 5)$ which is deformed. The last entry is $\boxed{5 + 4 \cos(v)}$.

b) This is a plane. Solve for $z = 1 - x - y$ and use u, v to get $\boxed{1 - u - v}$.

c) This is a surface of revolution and we have $\boxed{v^3 \sin(u)}$.

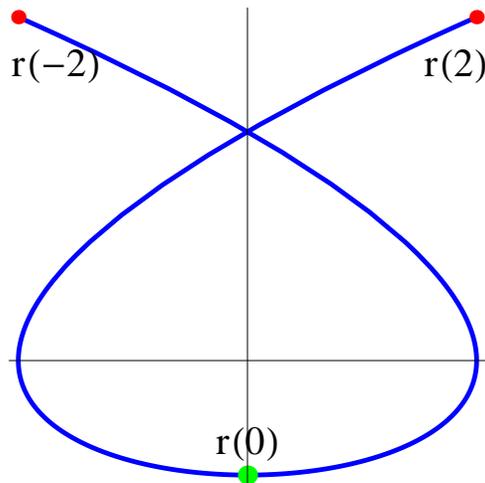
d) Since the tube circular and the grid curves with constant v are circles, we can use N, B to draw the circle. This is almost completed. We only have to fill in $\boxed{\vec{B}(v)}$. This example is actually one of the main motivations to use the TNB frame. It allows to draw beautiful tubes around a given curve.

Problem 6) (10 points)

We look at the parametrized curve

$$\vec{r}(t) = \left\langle \frac{t^3}{3} - t, t^2 - 1, 0 \right\rangle$$

whose image you see in the picture showing it in the xy plane for $-2 \leq t \leq 2$.



a) (3 points) Find the velocity $\vec{r}'(t)$, the acceleration $\vec{r}''(t)$ and speed $|\vec{r}'(t)|$.

b) (2 points) Evaluate this at $t = 0$ to get $\vec{r}'(0)$, $\vec{r}''(0)$ and $|\vec{r}'(0)|$.

c) (2 points) Find the curvature $|\vec{r}'(0) \times \vec{r}''(0)|/|\vec{r}'(0)|^3$ at $(0, -1, 0)$.

d) (3 points) Find the arc length of the curve $\vec{r}(t)$ from $-2 \leq t \leq 2$.

Solution:

a) $\vec{r}'(t) = \langle t^2 - 1, 2t, 0 \rangle$. $\vec{r}''(t) = \langle 2t, 2, 0 \rangle$ and $|\langle t^2 - 1, 2t, 0 \rangle| = \sqrt{t^4 + 2t + 1} = \sqrt{(t^2 + 1)^2} = t^2 + 1$.

b) $\vec{r}'(0) = \langle -1, 0, 0 \rangle$, $\vec{r}''(0) = \langle 0, 2, 0 \rangle$ and $|\vec{r}'(0)| = 1$.

c) Since the speed is 1, the curvature is

$$|\langle -1, 0, 0 \rangle \times \langle 0, 2, 0 \rangle| = |\langle 0, 0, -2 \rangle| = 2$$

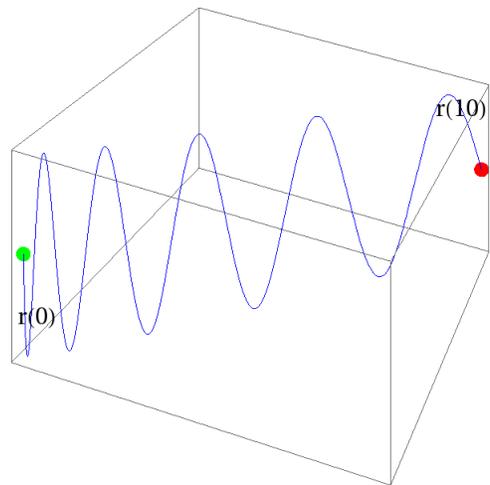
d) The arc length is $\int_{-2}^2 |\vec{r}'(t)| dt = \int_{-2}^2 t^2 + 1 dt = t^3/3 + t \Big|_{-2}^2 = 28/3$.

Problem 7) (10 points)

a) (4 points) We know $\vec{r}''(t) = \langle 1, 2, \pi^2 \sin(\pi t) \rangle$ and the initial velocity $\vec{r}'(0) = \langle 1, 0, -\pi \rangle$. Find $\vec{r}'(t)$.

b) (3 points) Assume we know also $\vec{r}(0) = \langle 0, 0, 10 \rangle$. Find $\vec{r}(10)$.

c) (3 points) What is the projection of $\vec{r}'(10)$ onto $\langle 1, 1, 0 \rangle$?



Solution:

a) Integrate to get

$$\vec{r}'(t) = \langle t, 2t, -\pi \cos(\pi t) \rangle + \langle c_1, c_2, c_3 \rangle .$$

Comparing the initial velocity gives the constants and so

$$\vec{r}'(t) = \langle 1 + t, 2t, -(\pi \cos(\pi t)) \rangle .$$

b) Integrate again and compare coefficients to get

$$\vec{r}(t) = \langle t + t^2/2, t^2, 10 - \sin(\pi t) \rangle .$$

we have $r(10) = \langle 60, 100, 10 \rangle$.

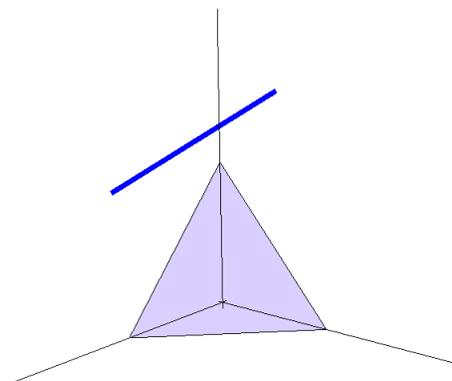
c) We have $\vec{r}'(10) = \langle 10, 20, -\pi \rangle$. The vector projection is $\langle 1, 1, 0 \rangle = \vec{r}'(10) \cdot \langle 1, 1, 0 \rangle / 2 = 15 \langle 1, 1, 0 \rangle$

Problem 8) (10 points)

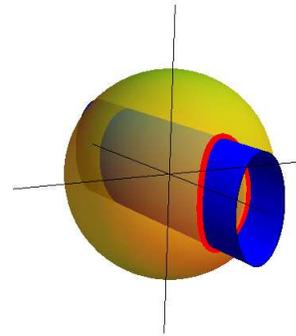
a) (5 points) Find the distance between the plane $x + y + z = 1$ and the line

$$x - 1 = \frac{(y - 1)}{-2} = z - 1$$

which is parallel to the plane.
(You do not have to check that it is parallel).



b) (5 points) The intersection of the cylinder $4x^2 + z^2 = 1$ with the sphere centered at $(0, 0, 0)$ with radius $\rho = \sqrt{2}$ cuts out two curves. Parametrize the curve which contains the point $(0, 1, 1)$.



Solution:

a) Choose a point on the plane $P = (1, 0, 0)$ and a point $Q = (1, 1, 1)$ on the line and compute the distance from P to the plane. We have $\vec{PQ} = \langle 0, 1, 1 \rangle$ and

$$d = \frac{|\langle 0, 1, 1 \rangle \cdot \langle 1, 1, 1 \rangle|}{|\langle 1, 1, 1 \rangle|} = 2/\sqrt{3}.$$

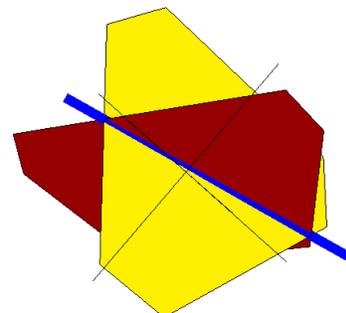
b) First parametrize the ellipse in the xz -plane with $\vec{r}(t) = \langle \cos(t)/2, \dots, \sin(t) \rangle$ where we do not know y yet. The sphere has the equation $x^2 + y^2 + z^2 = 2$ and we can solve for y and get the parametrization

$$\vec{r}(t) = \langle \cos(t)/2, \sqrt{2 - \cos^2(t)/4 - \sin^2(t)}, \sin(t) \rangle.$$

Problem 9) (10 points)

a) (5 points) Find a parametrization of the intersection line L of the two planes

$$\begin{aligned} 2x - 2y + z &= 1, \\ x + y + z &= 1. \end{aligned}$$



b) (5 points) Find a parametrization for the line M parallel to the line L computed in a) which passes through $(1, 2, 3)$.

Solution:

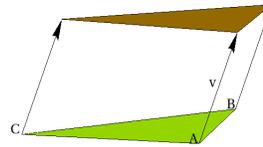
a) A point in the intersection of the plane is $P = (0, 0, 1)$. The cross product between the normal vectors is $\langle -3, -1, 4 \rangle$. The parametrization of the line is $\vec{r}(t) = \langle 0, 0, 1 \rangle + t\langle -3, -1, 4 \rangle$.

b) Now translate the line through $(1, 2, 3)$ to get $\vec{r}(t) = \langle 1, 2, 3 \rangle + t\langle -3, -1, 4 \rangle$.

Problem 10) (10 points)

a) (5 points) What is the area of the triangle through the points $A = (1, 1, 1)$ and $B = (0, 1, 0)$ and $C = (1, 2, 4)$.

b) (5 points) Find the volume of the prism which has the triangle T as base as well as a by $\vec{v} = \langle 0, 1, 1 \rangle$ translated triangle as roof.



Solution:

a) It is half the length of

$$\vec{AB} \times \vec{AC} = \langle -1, 0, -1 \rangle \times \langle 0, 1, 3 \rangle = \langle 1, 3, -1 \rangle$$

which has length $\sqrt{11}$ so that the answer is $\boxed{\sqrt{11}/2}$.

b) The volume is half of the volume of the parallel epiped and therefore half of the absolute value of the triple scalar product

$$\vec{v} \cdot (\vec{AB} \times \vec{AC}) = \langle 0, 1, 1 \rangle \cdot \langle 1, 3, -1 \rangle = 2$$

so that the answer is $\boxed{1}$.