Name:

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TTH 11:30 Lukas Brantner
TTH 11:30 Yu-Wen Hsu

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3,we need to see **details** of your computation.
- All functions can be differentiated arbitrarily often unless otherwise specified.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) True/False questions (20 points), no justifications needed

1) T F — The length of the gradient $\nabla f(0,0)$ is the maximal directional derivative $|D_{\vec{v}} f(0,0)|$ among all unit vectors $\vec{v}$.

Solution:
By the cos-formula.

2) T F — The relation $f_{xxyyxx} = f_{xxyxyx}$ holds everywhere for $f(x,y) = \cos(\exp(x^{10}) + \sin(x-y))$.

Solution:
By Clairot

3) T F — $\int_0^4 \int_0^{4x} f(x,y)\, dydx = \int_0^{16} \int_{y/4}^{16} f(x,y)\, dxdy$.

Solution:
The inner integral on the right should be $\int_{y/4}^4$.

4) T F — $g(x,y) = \int_y^0 \int_0^x f(s,t)\, dsdt$ satisfies $g_{xy} = -f(x,y)$.

Solution:
By the fundamental theorem of calculus.

5) T F — If $\vec{r}(u,v)$ is a parametrization of the level surface $f(x,y,z) = c$, then $\nabla f(\vec{r}(u,v)) \cdot \vec{r}_v(u,v) = 0$.

Solution:
Because the vector r_v is tangent to the surface.

6) T F — If $D_{\langle 1/\sqrt{2}, 1/\sqrt{2}\rangle} f(a,b) = 3$ and $D_{\langle 1/\sqrt{2}, -1/\sqrt{2}\rangle} f(a,b) = 5$, then $D_{\vec{v}} f(a,b) \geq 0$ for all unit directions $\vec{v}$.

Solution:
If we change $\vec{v}$ to $-\vec{u}$ then the sign of the directional derivative changes.

7) **T** F

Given a parametrization $\vec{r}(t)$ of a curve and a function $f(x, y)$ we have $\frac{d}{dt} f(\vec{r}(2t)) = 2\nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$ at $t = 0$.

8) **T** F

If $u(t, x)$ solves both the heat and wave equation, then $u_t = c\, u_{tt}$ for some constant c.

Solution:
Equal to u_{xx}.

9) T **F**

If the Lagrange multiplier λ at a solution to a Lagrange problem is negative then this point is neither a maximum nor a minimum.

Solution:
The sign of λ has nothing to say about the nature of the critical point.

10) **T** F

The equation $f_x^2 + f_y^2 + f_z^2 = 1$ is an example of a partial differential equation.

Solution:
Write it out. It is an equation for a function f involving partial derivatives.

11) T **F**

If the discriminant D of $f(x, y)$ is zero at $(0, 0)$ then $\nabla f(0, 0) = \langle 0, 0 \rangle$.

Solution:
We can have $f(x, y) = x + y$ for example, which has zero discriminant but no critical point.

12) T **F**

If $f(x, y, z) = 0$ describes the unit sphere, then the gradient ∇f points outwards.

Solution:
the gradients of $x^2 = y^2 + z^2 - 1 = 0$ or $1 - x^2 - y^2 - z^2 = 0$ point in different directions.

13) T **F**

If $f(x, y)$ is a continuous function then $\int_0^2 \int_0^1 f(x, y) \quad dxdy = \int_0^2 \int_0^1 f(y, x)\, dxdy$.

Solution:
Take a simple example like $f(x,y) = x$, then $f(y,x) = y$, which gives an other result.

14) | T | **F** | The point $(5,5,5)$ is a critical point of $f(x,y,z) = x + y + z$.

Solution:
the function f has no critical point.

15) | T | **F** | Assume $\nabla f(0,0) = \langle 0,0 \rangle$ with discriminant $D > 0$, then $-f(x,y)$ has the same critical point $(0,0)$ with discriminant $D < 0$.

Solution:
We still have $D > 0$

16) | T | **F** | $\int\int_R |\nabla f|^2 \, dxdy$ is the surface area of the cubic paraboloid $z = f(x,y) = x^3 + y^3$ defined over the region R.

Solution:
It is not the formula.

17) | **T** | F | If $D(x,y)$ is the discriminant of f at (x,y) then the following poetic formula of the **d**irectional **d**erivative of the **d**iscriminant holds: $D_{\langle 1,0 \rangle} D = \partial_x D$.

Solution:
This is true for all functions f, not only for the D. The directional derivative in the x direction is the partial derivative f_x.

18) | **T** | F | Assume $f(x,y) = -x^2 + y^4$ and a curve $\vec{r}(t)$ satisfying $\vec{r}\,'(t) = \nabla f(\vec{r}(t))$, then $\frac{d}{dt} f(\vec{r}(t)) \geq 0$ for all t.

Solution:
The assumption tells that the gradient is perpendicular to the velocity. By the chain rule, the expression is zero.

19) | T | F |

The Lagrange equations for extremizing $f(x, y)$ under the constraint $g(x, y) = c$ have the same solutions as the Lagrange equations for extremizing $F = f + g$ under the constraint $g = c$.

Solution:
Write down the Lagrange equations.

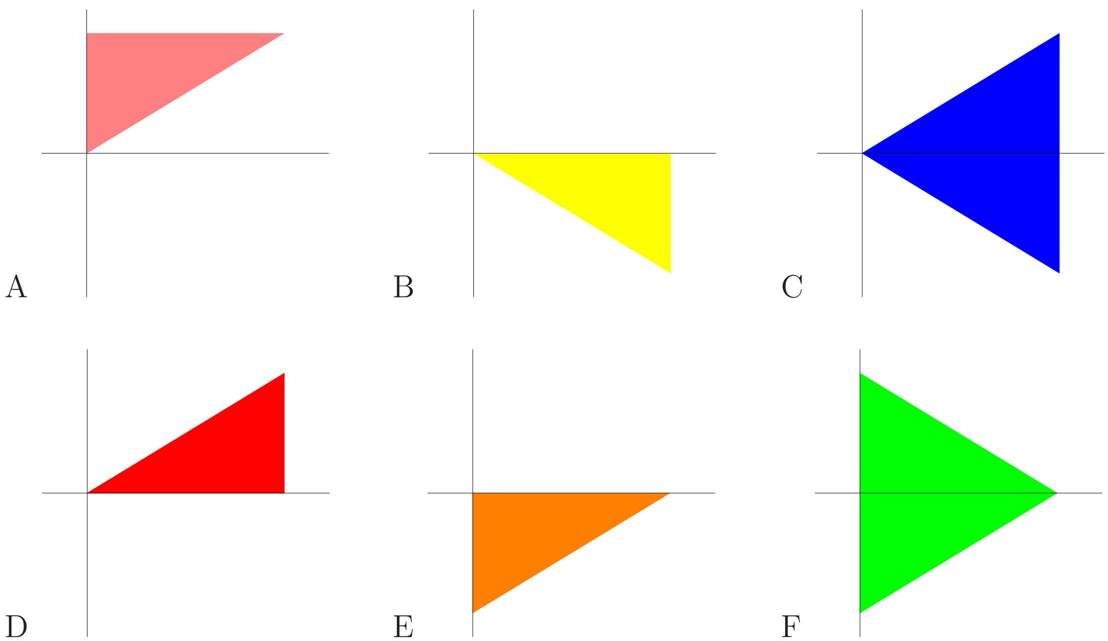
20) | T | F |

If f is a maximum under the constraint $g = 1$ at $(0, 0)$, and $(0, 0)$ is not a critical point for both f and g, then the level curves of f and g have the same tangent line at $(0, 0)$.

Solution:
The tangent line is determined by the point and the gradient. The tangent planes are the same if the gradients are parallel as long as they are not zero.

Problem 2) (10 points) No justifications needed

a) (6 points) Match the regions with the integrals. Each integral matches one region $A-F$.



Enter A-F	Integral
	$\int_{-1}^{1} \int_{\|y\|}^{1} f(x,y)\ dxdy$
	$\int_0^1 \int_0^x f(x,y)\ dydx$
	$\int_{-1}^{0} \int_{-y}^{1} f(x,y)\ dxdy$
	$\int_0^1 \int_{x-1}^{0} f(x,y)\ dydx$
	$\int_0^1 \int_{x-1}^{1-x} f(x,y)\ dydx$
	$\int_0^1 \int_x^1 f(x,y)\ dydx$

b) (4 points) Name the partial differential equations correctly. Each equation matches one name.

Fill in 1-4	Name
	Laplace
	Wave
	Transport
	Heat

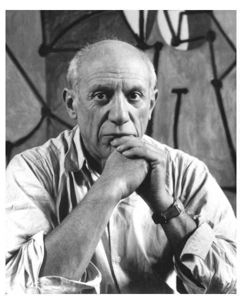
Equation number	Formula for PDE
1	$\frac{\partial}{\partial_t} u - \frac{\partial}{\partial_y} u = 0$
2	$\frac{\partial}{\partial_t} u - \frac{\partial^2}{\partial_x^2} u = 0$
3	$\frac{\partial^2}{\partial_t^2} u - \frac{\partial^2}{\partial_x^2} u = 0$
4	$\frac{\partial^2}{\partial_x^2} u + \frac{\partial^2}{\partial_y^2} u = 0$

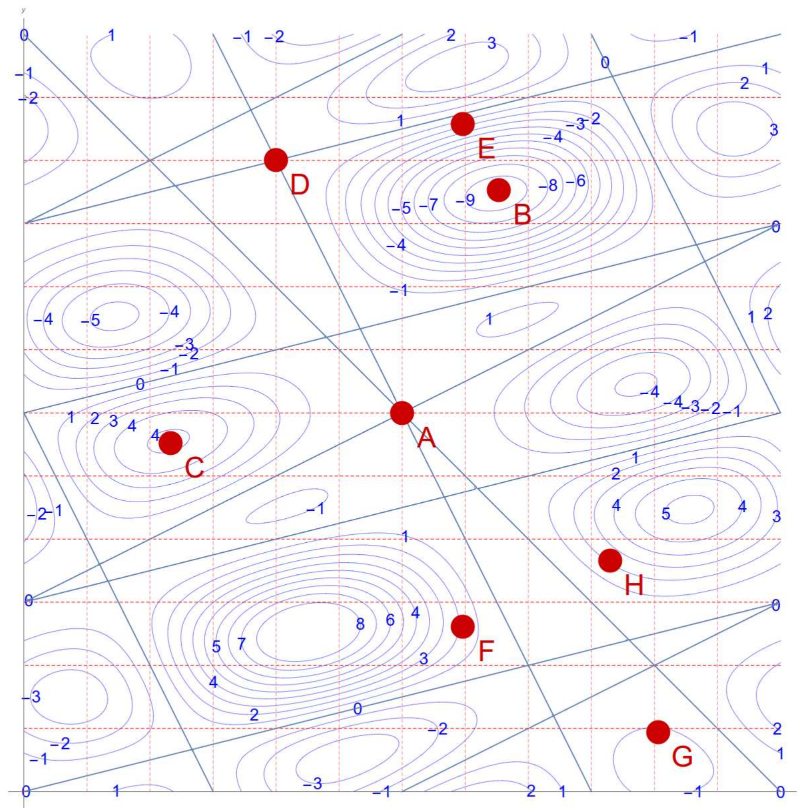
Solution:
CDBEFA
4,3,1,2

Problem 3) (10 points)

a) (7 points) The following contour map is inspired by a cubistic style of Picasso. Each of the points A-H fit exactly once

A point where $f_x \neq 0, f_y = 0$	
A saddle point of f	
A local maximum of f	
A critical point with $D = 0$	
A local minimum	
A point with $f_y \neq 0, f_x = 0$	
$\lvert\nabla f\rvert$ maximal among A-H	
Point where $D_{\langle -1,1\rangle/\sqrt{2}} f = 0$	





The painting "Pigeon with Green Peas" by Pablo Picasso was stolen in 2010. The thief got scared and disposed it to trash shortly after the theft. The garbage was emptied and taken away, the painting lost for ever. Or the thief had been clever ...

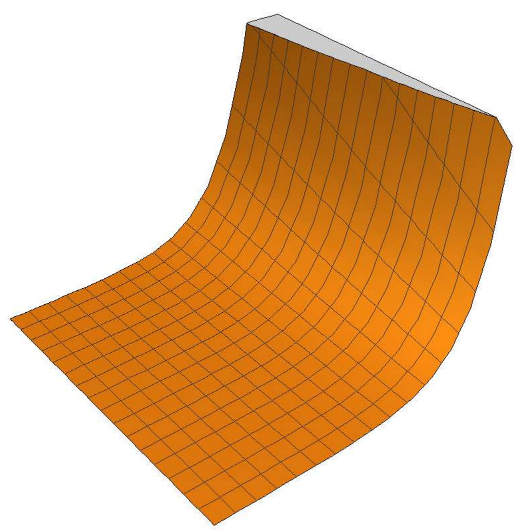
Solution:

b) (3 points) Given a function $f(x, y)$ and a curve $\vec{r}(t)$. Let L be the linearization of f at $\vec{r}(0)$. Each of the following 3 vectors $\vec{a}, \vec{b}, \vec{c}$ is placed exactly twice in the puzzle to the right below.

$\vec{a} = \nabla f(\vec{r}(0))$	$\frac{d}{dt} f(\vec{r}(t))_{t=0} = \boxed{} \cdot \boxed{}$
$\vec{b} = \vec{r}\,'(0)$	$L(\vec{r}(t)) = \quad f(\vec{r}(0)) + \boxed{} \cdot \boxed{}$
$\vec{c} = \vec{r}(t) - \vec{r}(0)$	$\boxed{} = \lim_{t \to 0} \frac{1}{t} \boxed{}$

Solution:
FDCABGEH
ab,ac,bc

Problem 4) (10 points)



a) (5 points) Find the tangent plane to the **skate board ramp**

$$z - f(x, y) = z - \sqrt{x^{30}y^3 + x} = 0$$

at the point $(1, 2, 3)$.

b) (5 points) Estimate

$$f(1.006, 1.98) = \sqrt{1.006^{30} \cdot 1.98^3 + 1.006}$$

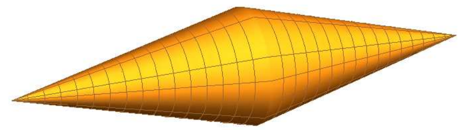
by linearizing the function $f(x, y)$ at $(1, 2)$.

Solution:
a) The gradient is $\langle (-30x^{29}y^3 + 1)/(2\sqrt{x^{30}y^3 + x}), (-3x^{30}y^2/(2\sqrt{x^{30}y^3 + x})), 1 \rangle$. At $(1, 2, 3)$ this is $\langle -241/6, -2, 1 \rangle$. The equation of the plane is $241x + 12y - 6z = d$, where d is the constant obtained by plugging in the point. It is $d = 220$.
b) We have already computed the gradient $\langle 40, 8 \rangle$ in a). The linearization is $L(x, y) = 3 + \langle 241/6, 2 \rangle \cdot \langle (x - 1), (y - 2) \rangle$. We have $L(1.006, 1.98) = 3 + 241/6 \cdot 0.006 + 2(-0.02)$

Problem 5) (10 points)

A **croissant** of length $2h$ and radius r in the shape of two cones has fixed volume

$$V(r,h) = \frac{2\pi r^2 h}{3} = 18 \; .$$



Use Lagrange to find the values r and h for which the surface area

$$A(r,h) = 2\pi r\sqrt{r^2+h^2}$$

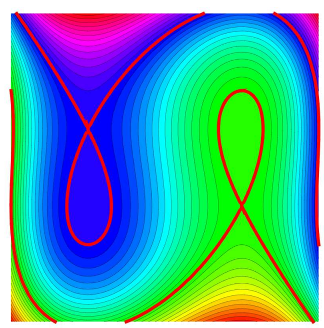
is minimal. **Hint:** as you have seen in homework, it is much more convenient to minimize $f(r,h) = A(r,h)^2$ instead.

Solution:
The Lagrange equations are
$2x^3 + 2x(x^2+y^2) = 2Lxy/3$
$2x^2y = \lambda x^2/3$. $y = \sqrt{2}x$. Plug that into the constraint to get $x = 3/2^{1/6}$, $y = 3 * 2^{1/3}$.

Problem 6) (10 points)

Find the local maxima, minima and saddle points of the **tadpole function**

$$f(x,y) = 3y^2 + 4x^3 + 2y^3 - 12x \; .$$



First tadpole: "What is your favorite book?" Second tadpole: "Metamorphosis by Kafka. What is your favorite year?" First tadpole: "Leap year!" Both croak with laughter. Then, sick of frog jokes, they turn green.

Solution:
Write down the gradient $\nabla f(x,y) = \langle 12x^2 - 12, 6y + 6y^2 \rangle$. This shows that $x = \pm 1$ and $y = 0$ or -1. There are 4 solutions.

x	y	D	f_{xx}	Type f
-1	-1	144	-24	maximum 9
-1	0	-144	-24	saddle 8
1	-1	-144	24	saddle -7
1	0	144	24	minimum -8

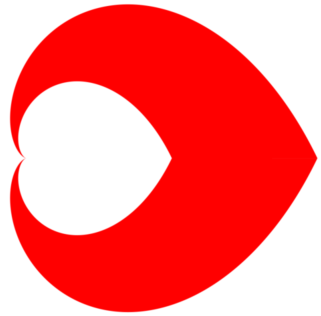
Problem 7) (10 points)

Find the area of the heart shaped polar region

$$(\theta - \pi)^2 \leq r \leq 2(\theta - \pi)^2$$

with $0 \leq \theta \leq 2\pi$.

Warning: Valentine cards displaying **"You are my** $r < (\theta - \pi)^2$**!"** *do not always work.*



Solution:
$(3/5)\pi^5$.

Problem 8) (10 points)

Mathematica 10 does not give an elementary expression for the integral

$$\int_0^1 \int_{\exp(y)}^e \frac{1}{\log(x)} \, dx \, dy \, ,$$

where log is the natural log. You can! "Humans are awesome 2014"! https://www.youtube.com/watch?v=ZBCOMG2F2Zk

The logarithmic integral $\mathrm{Li}(x) = \int_0^x dt/\log(t)$ is important in number theory. It was Gauss who proposed first that the number $\pi(x)$ of primes smaller than x is about $\mathrm{Li}(x)$. It is now known that $0.89\,\mathrm{Li}(x) \leq \pi(x) \leq 1.11\,\mathrm{Li}(x)$ for all large enough x. P.S. Mathematica can solve the double integral of course, but only if told to "FullSimplify".

Solution:
Switch the order of integration. It is pivotal to make a picture.

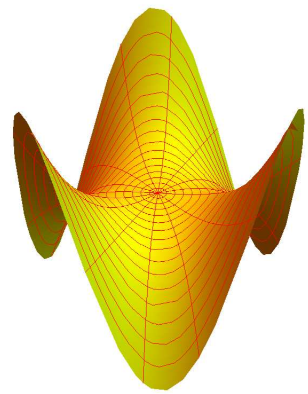
$$\int_1^e \int_0^{\log(x)} \frac{1}{\log(x)} \, dy \, dx = e - 1 \, .$$

Problem 9) (10 points)

Compute the weighted surface area

$$\int\int_R (u^2+v^2)|\vec{r}_u \times \vec{r}_v| \, dudv$$

of the monkey saddle parametrized by $\vec{r}(u,v) = \langle u, v, u^3 - 3uv^2 \rangle$ over the domain $R : u^2+v^2 \leq 1$. This quantity is also known as the moment of inertia of the surface. Spin that monkey!



Solution:
$\vec{r}_u = \langle 1, 0, 3u^2 - 3v^2 \rangle$ and $\vec{r}_v = \langle 0, 1, -6uv \rangle$ so that we get $2\pi \int_0^1 r^3 \sqrt{1+9r^4} \, dr$. This can be integrated with substitution $u = r^4$. The answer is $\boxed{10\sqrt{10} - 1)\pi/27}$.

Problem 10) (10 points)

The following two integrals are called "Mad Max" integrals because they were written while watching that movie:

a) (5 points) Integrate

$$\int_0^1 \int_{\arcsin(y)}^{\pi/2} \frac{xy}{\sin(x)} \, dxdy \, .$$

b) (5 points) Integrate the double integral

$$\int\int_R \sin(x^2+y^2) \, dxdy$$

where R is the disk of radius $\sqrt{\pi/2}$.

Solution:

a) Make a picture! Change the order of integration to get

$$\int_0^{\pi/2} \int_0^{\sin(x)} \frac{xy}{\sin(x)} \, dydx \; .$$

After solving the first integral, we get

$$\int_0^{\pi/2} x \sin(x) \; dx = 1/2 \; .$$

1/2.

b)

$$2\pi \int_0^{\sqrt{\pi/2}} \sin(r^2) r \; dr = -\pi \cos(r^2)|_0^{\sqrt{\pi}} = \pi \; .$$