Homework 11: Partial differential equations

This homework is due Wednesday, 10/7 rsp Friday 10/9.

1 a) Which of the following functions solves the **Laplace's equation** $u_{xx} + u_{yy} = 0$.
a) $u = 5x^2 + 5y^2$ b) $u = 7x^2 - 7y^2$ c) $u = x^3 + 3xy^2$
d) $u = \log\sqrt{x^2 + y^2}$ e) $u = e^{-x}\cos y - e^{-y}\cos x$
As usual, $\log = \ln$ denotes the natural log.
b) Verify that each of the following solves the **wave equation** $u_{tt} = u_{xx}$.
a) $u = \sin(kx)\sin(kt)$ b) $u = \left(\frac{t}{t^2 - x^2}\right)$
c) $u = (x-t)^6 + (x+t)^6$ d) $u = \sin(x-t) + \log(x+t)$

2 The differential equation

$$f_t = f - xf_x - x^2 f_{xx}$$

is an example of the **infamous Black-Scholes equation**. Here $f(x,t)$ is the prize of a call option and x the stock prize ad t is time. Verify that $f(x,t) = e^t \log(x)$ solves this equation. (We know also that $x, e^t, e^{-3t}x^2, (1+x^2)/(2x)$ solves this equation. If you should find an other one, mail it in.)

3 Show that the **Cobb-Douglas** production function $P = L^\alpha K^\beta$ satisfies the equation

$$L\frac{\partial P}{\partial L} + K\frac{\partial P}{\partial K} = (\alpha + \beta)P \ .$$

The constants α and β are fixed. L is labor and K is capital.

4 a) Run Mathematica code for the one-dimensional wave equation. The example code is given on the website and below. We want to

know the value of $u(t,x)$ at $t = 0.6$ and $x = 0.7$.
b) Run Mathematica code for the two dimensional wave equation. The example code is given on the website and below. Plot the graph of the function $u(t, 0.3, 0.4)$ from $t = 0$ to $t = 1$. You can either print out your output or copy what you see on the screen.

5 The partial differential equation

$$f_t + f f_x = f_{xx}$$

called **Burgers equation** describes waves at the beach. In higher dimensions, it leads to the **Navier-Stokes** equation which are used to describe the weather. Use Mathematica to verify that the function

$$f(t,x) = \frac{\left(\frac{1}{t}\right)^{3/2} x e^{-\frac{x^2}{4t}}}{\sqrt{\frac{1}{t}} e^{-\frac{x^2}{4t}} + 1}$$

solves the Burgers equation.

Use of Mathematica

Part of the homework is also to install and run Mathematica. You can use this software to make tough computations. You can use the software to do the problems above. Here is an example. After entering, type return while holding down the shift key:

```
f[t_,x_]:=(1/Sqrt[t])*Exp[-x^2/(4t)];
Simplify[ D[f[t,x],t] == D[f[t,x],{x,2}]]
```

You have verified that the function

$$\frac{1}{\sqrt{t}} e^{-x^2/(4t)}$$

satisfies the heat equation. As any real programming language, Mathematica is particular about syntax. Watch brackets, capitalization, double equal signs ==!

> An equation for an unknown function $f(x, y)$ which involves partial derivatives with respect to at least two different variables is called a **partial differential equation**.

Here is Mathematica code for an example of the two dimensional wave equation. You can copy paste from the website:

```
A=Rectangle[{0,0},{1,1}]; Clear[t,x,y];
f[x_,y_]:=Sin[2 Pi x] Abs[Sin[3 Pi y]];
g[x_,y_]:=3 Sin[Pi x] Sin[Pi y];
U=NDSolveValue[{D[u[t,x,y],{t,2}]
 -Inactive[Laplacian][u[t,x,y],{x,y}]==0,
   u[0,x,y] == f[x,y],
   Derivative[1,0,0][u][0,x,y]==g[x,y],
   DirichletCondition[u[t,x,y] ==0,True]},
   u,{t, 0, 2 Pi}, {x,y} \[Element] A];
Plot3D[U[4,x,y],{x,0,1},{y,0,1}]
Animate[ContourPlot[U[t,x,y],
     {x,0,1},{y,0,1}],{t,0,2Pi}]
```

Here is an example for Mathematica code for the one dimensional wave equation. You can copy paste from the website:

```
f[x_]:=Sin[Pi 7x];
g[x_]:=5 Sin[5 Pi x];
U = NDSolveValue[
   {D[u[t,x],{t,2}]-D[u[t,x],{x,2}]==0,
   u[0,x]==f[x],
   Derivative[1,0][u][0,x] == g[x],
   DirichletCondition[u[t,x]==f[0],x==0],
   DirichletCondition[u[t,x]==f[1],x==1]},
   u,{t,0,1},{x,0,1}];
Animate[Plot[U[t,x],{x,0,1},
   PlotRange->{-2,2}],{t,0,1,0.01}]
Plot[U[t,0.5], {t, 0, 1}]
```