

Homework 25: Line integrals

This homework is due Friday, 11/13 resp Tuesday 11/17.

- 1 a) Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$ if $\vec{F}(x, y, z) = \langle x + y, y - z, z^2 \rangle$ and $\vec{r}(t) = \langle t^2, t^3, t^2 \rangle$ with $0 \leq t \leq 2$.
b) Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$ if $\vec{F}(x, y, z) = \langle 3z, 3y, -3x \rangle$ and $\vec{r}(t) = \langle t, \sin(t), \cos(t) \rangle$, $0 \leq t \leq \pi$.
- 2 An electric current I produces a magnetic field \vec{B} whose flow lines are circles circling the wire. Let $C : \langle r \cos(t), r \sin(t), 0 \rangle$. Ampère's law is $\int_C \vec{B} \cdot d\vec{r} = \mu_0 I$, where μ_0 is a constant called permeability. Show that the magnitude $B(r) = |\vec{B}|$ of the magnetic field at a distance r from the center of the wire is $B = \frac{\mu_0 I}{2\pi r}$. Note that B is a scalar function and \vec{B} is a vector field. Use first the information provided to find the vector field \vec{B} .
- 3 Determine from each of the following cases, whether or not \vec{F} is conservative or not. If it is, find a function f such that $\vec{F} = \nabla f$.
 - a) $\vec{F}(x, y) = \langle 7e^x \sin(y), 7e^x \cos(y) \rangle$
 - b) $\vec{F}(x, y) = \langle 3x^2 - 2y^2, 4xy + 3 \rangle$
 - c) $\vec{F}(x, y, z) = \langle x + y, y + x, z^5 - \sin(z) \rangle$
 - d) $\vec{F}(x, y, z) = \langle x, z, y \rangle$
- 4 Evaluate the line integral $\int_C \langle 1 - ye^{-x}, e^{-x} \rangle \cdot d\vec{r}$, where C is the path $\vec{r}(t) = \langle t, 1 + t + \sin(\sin(t)) \rangle$ and t is from 0 to π . You probably will have difficulty. A future "you" (who has a time machine) tells you that you can compute the integral also in a different way: find a function f which is a potential to the vector field, then evaluate $f(\vec{r}(\pi)) - f(\vec{r}(0))$. You can use this without justification for now. We will learn about this "warp" feature in the next lecture.

- 5 The topological notions appearing next are cool but not very essential for the course. Determine whether or not the given set is (a) open, (b) connected, and (c) simply-connected.
- a) $\{(x, y) \mid 0 < y < 3\}$, b) $\{(x, y) \mid 1 < |x| < 2\}$
c) $\{(x, y) \mid 1 \leq x^2 + y^2 \leq 4, y \geq 0\}$ d) $\{(x, y) \mid (x, y) \neq (1, 2)\}$
e) $\{(x, y, z) \mid (x, y, z) \neq (1, 2, 3)\}$

Main definitions

If \vec{F} is a vector field and $C : t \mapsto \vec{r}(t)$ is a curve defined on the interval $[a, b]$ then $\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt$ is called the **line integral** of \vec{F} along the curve C .

A vector field is **conservative** in a region R if the line integral from A to B is path independent. It has the **closed loop property** if the line integral along any closed loop is zero. It is **irrotational** if $\text{curl}(F) = Q_x - P_y$ is zero everywhere in R .

A subset G of the plane is **open** if every point (x, y) in G is contained in a small disc centered at (x, y) which is also in G . A subset is **connected**, if one can connect any two points in G with a curve inside G . A subset G is **simply connected** if it is connected and every closed curve in G can be pulled together to a point within G .

Clairaut test: Zero curl is necessary for a gradient field.