

Homework 27: Greens theorem

This homework is due Wednesday, 11/18 resp Thursday 11/19.

- 1 Evaluate the line integral

$$\int_C \vec{F} \cdot d\vec{r} ,$$

where $\vec{F}(x, y) = \langle 6y + x^5 - 1, \sin(y)y^{100} \rangle$ and C consists of the line segments from $(0, 1)$ to $(0, 0)$ and from $(0,0)$ to $(1,0)$ and the parabola $y = 1 - x^2$ from $(1, 0)$ to $(0, 1)$.

- 2 a) Find the line integral $\int_C \vec{F} \cdot d\vec{r}$.

$$\vec{F}(x, y) = \langle y^2 \cos x + \sin(\sin(x)), 15x + x^2 + 2y \sin x + \sin(\sin(y)) \rangle$$

where C is the triangular path from $(0,0)$ to $(2,6)$ to $(2,0)$ to $(0,0)$. Watch the orientation of the curve!

- b) Evaluate $\int_C \vec{F} \cdot d\vec{r}$

$$\vec{F}(x, y) = \langle y - \ln(x^2 + y^2), 2 \tan^{-1}(y/x) \rangle$$

C is the circle $(x - 2)^2 + (y - 3)^2 = 16$ oriented counterclockwise. Don't worry here about the singularity or the fact that arctan is not continuous.

- 3 A classical problem asks to compute the area of the region bounded by the **hypocycloid**

$$\vec{r}(t) = \langle 4 \cos^3(t), 4 \sin^3(t) \rangle, 0 \leq t \leq 2\pi .$$

We can not do that directly. Guess which theorem to use!

- 4 Calculate $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F}(x, y) = \langle x^2 + y, 3x - y^2 \rangle$ and C is the ellipse $x^2/100 + y^2/16 = 1$ oriented clockwise. (You might have seen in class the computation of the area of an ellipse using Green, you can use that).

5 Use Green's Theorem to evaluate

$$\int_C \langle \sin(\sqrt{1+x^3}), 7x \rangle d\vec{r},$$

where C is the boundary of the region $K(4)$. You see in the picture $K(0), K(1), K(2), K(3), K(4)$. The first $K(0)$ is an equilateral triangle of length 1. The second $K(1)$ is $K(0)$ with 3 equilateral triangles of length $1/3$ added. $K(2)$ is $K(1)$ with $3 * 4^1$ equilateral triangles of length $1/9$ added. $K(3)$ is $K(2)$ with $3 * 4^2$ of length $1/27$ added and $K(4)$ is $K(3)$ with $3 * 4^3$ triangles of length $1/81$ added. Remark. We could now find the line integral in the limit $K = K(\infty)$, a **fractal** called the **Koch snowflake** It has dimension $\log(4)/\log(3) = 1.26\dots$ which is between 1 and 2.



Main points

The **curl** of a vector field $\vec{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$ is the scalar field

$$\text{curl}(F)(x, y) = Q_x(x, y) - P_y(x, y).$$

Green's theorem: If $\vec{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$ is a vector field and G is a region for which the boundary C is parametrized so that R is "to the left", then

$$\int_C \vec{F} \cdot d\vec{r} = \iint_G \text{curl}(F) dx dy.$$