**11/2/2016 Second Hourly** **Math 21a, Fall 2016**

| Name: |
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|---|
| MWF 9 Koji Shimizu |
| MWF 10 Can Kozcaz |
| MWF 10 Yifei Zhao |
| MWF 11 Oliver Knill |
| MWF 11 Bena Tshishiku |
| MWF 12 Jun-Hou Fung |
| MWF 12 Chenglong Yu |
| TTH 10 Jameel Al-Aidroos |
| TTH 10 Ziliang Che |
| TTH 10 George Melvin |
| TTH 11:30 Jake Marcinek |
| TTH 11:30 George Melvin |

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3,we need to see **details** of your computation.
- All functions can be differentiated arbitrarily often unless otherwise specified.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

| | | |
|---|---|---|
| 1 | | 20 |
| 2 | | 10 |
| 3 | | 10 |
| 4 | | 10 |
| 5 | | 10 |
| 6 | | 10 |
| 7 | | 10 |
| 8 | | 10 |
| 9 | | 10 |
| Total: | | 100 |

Problem 1) True/False questions (20 points), no justifications needed

1) T F The function $f(x,y) = x^3y/(x^6+y^5)$ can be filled in at the origin with a value $f(0,0) = a$ so that $f$ is continuous everywhere.

2) T F The chain rule assures that $\int_0^1 (\nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t))\, dt = f(\vec{r}(1)) - f(\vec{r}(0))$.

3) T F The formula $\int_0^1 \int_0^1 f(x,y)\, dy\, dx = \int_0^1 \int_0^1 f(y,x)\, dy\, dx$ holds.

4) T F If $u(x,t)$ solves the partial differential equation $u_t = u_x$, then so does the function $u_x$.

5) T F There is a surface $S$ containing the curve $\vec{r}(t) = \langle t, t^2, t^3 \rangle$ for which the tangent plane to $S$ at $(0,0,0)$ is $x + 2y + 3z = 0$.

6) T F For any two unit vectors $\vec{u}$ and $\vec{v}$, and any $f$, we have $D_{\vec{u}} D_{\vec{v}} f = D_{\vec{v}} D_{\vec{u}} f$.

7) T F If the tangent plane to $z = f(x,y)$ at $(0,0,f(0,0))$ is $4 + 3x + 2y + z = 0$, then $L(x,y) = 4 + 3x + 2y$ is the linearization of $f(x,y)$ at $(0,0)$.

8) T F For $f(x,y) = x^3 e^{y^2 \cos y} - x^4 \cos y$ the function $f_{xyxyxyxyxy}$ is zero everywhere.

9) T F The point $(0,0)$ is a critical point of $f(x,y) = x^3y^2$.

10) T F The gradient of $f(x,y) = x^2 + y^2$ is a vector perpendicular to the surface $z = f(x,y)$.

11) T F If the function $f(x,y)$ attains an absolute maximum on the region $x^2+y^2 \leq 4$ at the point $(2,0)$, then we must have $f_{xx}(2,0) \leq 0$.

12) T F If $f(x,y) \leq 5$ for all values of $(x,y)$, then $\int_0^{2\pi} \int_0^7 f(r\cos\theta, r\sin\theta) r\, dr\, d\theta \leq 5\pi(7^2)$.

13) T F For any constant $a$, we have $\int_{-a}^a \int_0^a \left(e^{x^2} \sin y\right) dx\, dy = 0$.

14) T F The linearization of the function $f(x,y) = e^{x^2+y}$ at the point $(0,0)$ is the function $L(x,y) = 1 + 2x^2 e^{x^2+y} + y e^{x^2+y}$.

15) T F Let $\vec{u}$ be the unit vector in the direction $\langle 1,1 \rangle / \sqrt{2}$. Then $D_{\vec{u}} f = f_{xy}$.

16) T F The integral of $f(x,y) = \sqrt{x^2+y^2}$ over the unit disk is $\int_0^{2\pi} \int_0^1 r\, dr d\theta$.

17) T F There is a function $f(x,y)$ for which $D_{\vec{v}} f(0,0) = 1$ for all directions $\vec{v}$.

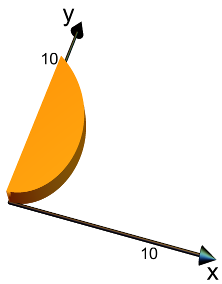
18) T F Given $f(x, y(x)) = 0$, then $f_x + f_y \frac{dy}{dx} = 0$.

19) T F Any function on a closed and bounded region must have a critical point.

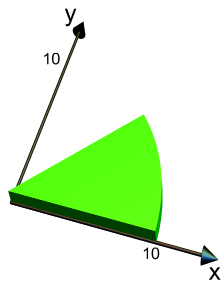
20) T F The integral $\iint_{x^2+y^2 \leq 1} |f(x,y)|\, dxdy$ computes the surface area of the surface $z = f(x,y), x^2+y^2 \leq 1$.

Problem 2) (10 points) No justifications needed

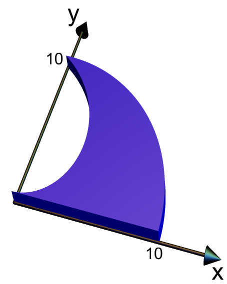
a) (6 points) Double integrals like $\iint_R 1\, dxdy$ or $\iint_R r\, drd\theta$ can be interpreted both as the **area** of the region $R$ as well as the **volume** of the solid under the graph of the constant function $f(x,y) = 1$ or $f(\theta, r) = 1$. Match the regions with the integrals:



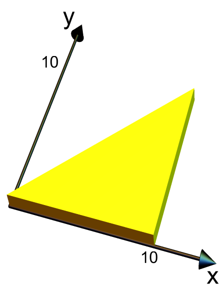

A



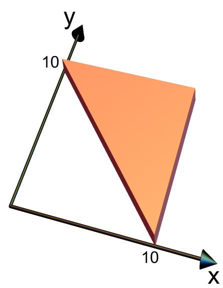

B



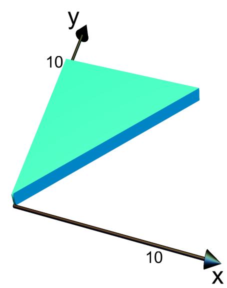

C




D




E

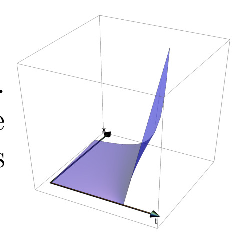



F

| Enter A-F | Integral |
|---|---|
| | $\int_0^{10} \int_0^{x} 1\ dydx$ |
| | $\int_0^{10} \int_0^{\pi/4} r\ d\theta dr$ |
| | $\int_0^{\pi/2} \int_0^{20\theta/\pi} r\ drd\theta$ |
| | $\int_0^{\pi/2} \int_{20\theta/\pi}^{10} r\ drd\theta$ |
| | $\int_0^{10} \int_0^{y} 1\ dxdy$ |
| | $\int_0^{10} \int_{10-x}^{10} 1\ dydx$ |

b) (4 points)

You know the **Transport**, **Wave**, **Heat**, or **Burgers** equation. Given in a possibly different order, these differential equations are $u_t = u_{xx}, u_t = u_x, u_{tt} = u_{xx}, u_t + uu_x = u_{xx}$. Check all the boxes where the given function solves the given PDE.



| $f(x,t) = x/(1+t)$ solves | Name |
|---|---|
| | Burgers |
| | Transport |

| $f(x,t) = xt$ solves | Name |
|---|---|
| | Heat |
| | Wave |

Problem 3) (10 points)

3a) (7 points) All the parts of this problem refer to the labeled points and the differentiable function $f(x, y)$ whose level curves are shown in the following plot:

a) At the point ☐, the gradient $\nabla f$ has maximal length

b) At the point ☐, $f_x > 0$ and $f_y = 0$

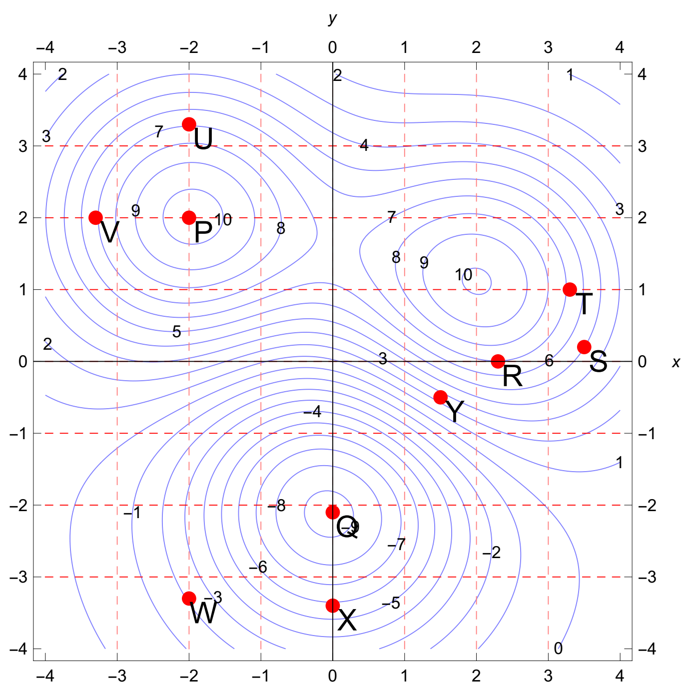
c) At the point ☐, $f_x < 0$ and $f_y < 0$

d) At the point ☐, $D_{\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle} f = 0$ and $f_x \neq 0$

e) At the point ☐, $f$ achieves a global min on $-4 \leq x \leq 4$ and $-4 \leq y \leq 4$

f) At the point ☐, $\nabla f = \vec{0}$ and $f_{xx} < 0$

g) At the point ☐, $\nabla f$ points straight toward the top of the page.




3b) (3 points) Check the cases where the maximum, minimum or saddle point of the function can be established **conclusively** using the second derivative test. Don't check the box if the test does not apply, (even if it might be a sort of minimum, maximum or saddle).

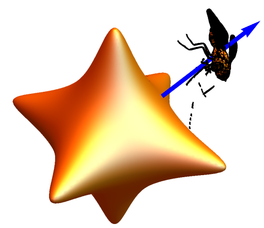
| Critical Point | $x^4 + y^2$ | $xy$ | $x^2 - y^4$ |
|---|---|---|---|
| Maximum | | | |
| Minimum | | | |
| Saddle point | | | |

Problem 4) (10 points)

a) (4 points) A **math candy** of the form

$$f(x,y,z) = 3x^2y^2 + 3y^2z^2 + 3x^2z^2 + x^2 + y^2 + z^2 = 12$$

is leaning at $(1,1,1)$ at the plane tangent to it. Find that plane.



b) (3 points) Estimate $f(1.1, 1.01, 0.98)$ using linearization.

c) (3 points) A fruit fly just dipped some sugar from the candy at $(1,1,1)$ and moves along a path $\vec{r}(t)$ with constant speed 1 perpendicularly away from the candy. What is $\frac{d}{dt} f(\vec{r}(t))$ at the moment of take-off?

| Problem 5) (10 points) |
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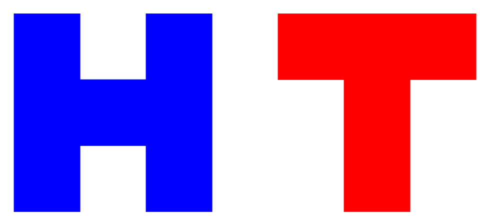
In order to figure out the **Egos** $x$ and $y$ of the US presidential candidates, we want to minimize the sum of the perimeter of the letters $H$ and $T$ written in units $x$ and $y$ if the total area is fixed. The letter $H$ has area $7x^2$ and perimeter $16x$, the letter $T$ has area $5y^2$ and perimeter $12y$. Minimize




$$f(x,y) = 16x + 12y \ .$$

under the constraint

$$g(x,y) = 7x^2 + 5y^2 = 2016 \ .$$

We don't actually need to know $x$ and $y$. **As political pundits, we are only interested in the ratio $y/x$ at the minimum.** Find this ratio!

| Problem 6) (10 points) |
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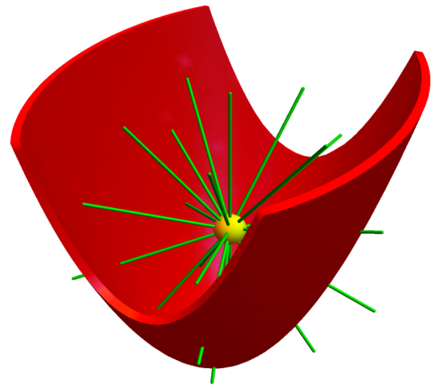
With $F(x,y,z) = 2x^2 + y^2 + z^2$ and the surface $S$ parametrized by $\vec{r}(x,y) = \langle 2x, y, 2x^2+y^2-1 \rangle$, the function $f(x,y) = F(\vec{r}(x,y))$ giving the value $F$ on $S$ is

$$f(x,y) = 4x^4 + 4x^2y^2 + 4x^2 + y^4 - y^2 + 1 \ .$$



a) (8 points) Find all the critical points of $f$ and classify them with the second derivative test. Please organize your work carefully so that we can see your method and your conclusions easily.

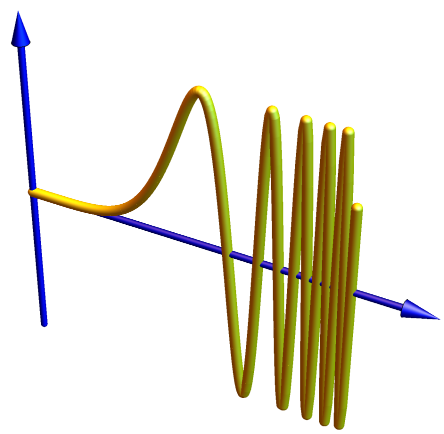
b) (2 points) The minimum could be obtained by minimizing $F(x,y,z)$ on the surface $G(x,y,z) = x^2/2 + y^2 - 1 - z = 0$. We would then use a method found by some mathematician. Which one? Just check the name. No additional work is needed in b).

| Fubini | Burgers | Laplace | Lagrange | Bolzano | Clairaut |
|---|---|---|---|---|---|
| | | | | | |

Problem 7) (10 points)

Integrate

$$\int_0^1 \int_{(1-y)^{1/4}}^1 \sin(x^5) \, dxdy \ .$$



The figure just shows a fancy plot of the function $\sin(x^5)$.

Problem 8) (10 points)

Integrate the double integral

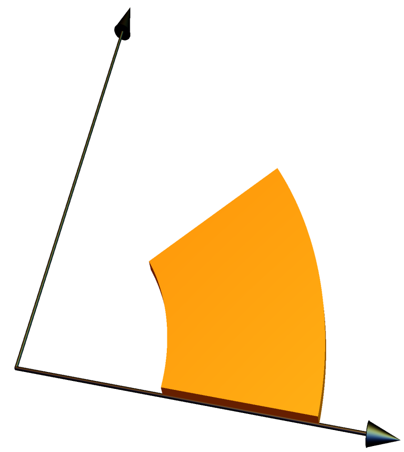
$$\int\int_R x^2 \; dxdy \; ,$$

where $R$ is the region

$$1 \leq x^2 + y^2 \leq 4$$

and

$$x \geq 0, y \geq 0, y \leq x \; .$$



| Problem 9) (10 points) |
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a) (7 points) Compute $A = |\vec{r}_\theta \times \vec{r}_\phi|$ for the half cylinder parametrized by

$$\vec{r}(\theta, \phi) = \langle \cos(\theta), \sin(\theta), \cos(\phi) \rangle \; .$$

with $0 \leq \phi \leq \pi/2$ and $0 \leq \theta \leq \pi$ and use this to find the surface area of the half cylinder

b) (3 points) Compute $B = |\vec{r}_\theta \times \vec{r}_\phi|$ for the quarter sphere parametrized by

$$\vec{r}(\theta, \phi) = \langle \sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi) \rangle$$

with $0 \leq \phi \leq \pi/2$ and $0 \leq \theta \leq \pi$ to show that (remarkably!) it is the same factor than in part a).

**Remark:** The fact that the surface area elements $A$ and $B$ are the same has been realized by Archimedes already. It allowed him to compute the surface area of the sphere in terms of the surface area of the cylinder.