11/2/2016 Second Hourly Practice VI **Math 21a, Fall 2016**

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TTH 11:30 Jake Marcinek
TTH 11:30 George Melvin

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3,8, we need to see **details** of your computation.
- All functions can be differentiated arbitrarily often unless otherwise specified.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) True/False questions (20 points), no justifications needed

1) T F

If $f(x,y) = 1$ is a curve, and near $(2,3)$ one can write y as a function of x, then $y' = -f_y(2,3)/f_x(2,3)$.

Solution:
The order is wrong: the f_y is in the denominiator.

2) T F

If $\iint_R f(x,y)\, dA = 0$, then the function $f(x,y)$ is everywhere zero on $R = \{x^2 + y^2 \leq 1\}$.

Solution:
If $f = xy$ and R is the disc, then the integral is zero but f is nonzero.

3) T F

The directional derivative in the direction of the gradient is $|\nabla f|$.

Solution:
Indeed, $D_{\nabla f/|\nabla f|} = \nabla f \cdot \nabla f/|\nabla f| = |\nabla f|$.

4) T F

The linearization of $f(x,y) = x^3 + y^3$ at $(1,1)$ is the quadratic function $L(x,y) = 3x^2 + 3y^2$.

Solution:
The linearization is a linear (affine) function and not quadratic.

5) T F

The function $f(x,y) = x^2 + y^2$ satisfies the partial differential equation $D = f_{xx}f_{yy} - f_{xy}^2 = 1$.

Solution:
Almost, a factor 4 is missing.

6) T F

The function x^2y^2 has no local minimum at $(0,0)$ because the discriminant function D is zero there.

Solution:
There can be a local minimum with $D = 0$.

7) T **F** The double integral $\int_0^{\pi/4} \int_0^2 r^3 \; dr d\theta$ is the volume of the part of a solid cylinder $x^2 + y^2 \leq 4$ which is below the paraboloid $z = x^2 + y^2$ and above the xy plane.

Solution:
We do not integrate from 0 to 2π.

8) **T** F The gradient of $f(x, y, z)$ at (x_0, y_0, z_0) is perpendicular to the level surface of f through (x_0, y_0, z_0).

Solution:
It indeed is! This is an important fact.

9) **T** F If $f(x, y, z) = 3x - 4z$, then the minimal possible directional derivative $D_{\vec{u}} f$ at any point in space is -5.

Solution:
The gradient has length 5. The directional derivative into the direction of the gradient is the length of the gradient.

10) **T** F If (x, y) is not a critical point, then the directional derivative $D_{\vec{v}} f$ can take both positive and negative values for different choices of $\vec{v}$.

Solution:
The directional derivative changes sign if $\vec{v}$ is replaced by $-\vec{v}$.

Solution:
Because of the symmetry $D_{-v} f = -D_v f$ the integral is zero.

11) **T** F Using linearization of $f(x, y) = x/y$ we can estimate $1.01/1.001 = f(1.01, 1.001) \sim 1 + 0.01 - 0.001 = 1.009$.

Solution:
$L(x, y) = 1 + 1 \cdot 0.01 - 1 \cdot 0.001$.

12) T **F** If $(0,0)$ is a critical point of $f(x,y)$ with nonzero discriminant $D = f_{xx}f_{yy} - f_{xy}^2$, we know that it is either a saddle, a global maximum or a global minimum.

Solution:
Local max or min, but not necessary global max or min.

13) T **F** For a rectangular region R, Fubini tells that $\int_0^2 \int_0^3 f(x,y) \; dxdy = \int_0^2 \int_0^3 f(x,y) \, dydx$ for any continuous function $f(x,y)$.

Solution:
We also have to switch the integration bounds.

14) T **F** If a function $f(x,y)$ has only one critical point $(0,0)$ in $G = \{x^2 + y^2 \leq 1\}$ which is a local maximum and $f(0,0) = 1$, then $\iint_G f(x,y) \, dxdy > 0$.

Solution:
The critical point can be surrounded by a small region only, where f is positive.

15) **T** F If $\vec{r}(t)$ is a curve in space for which the speed is 1 at all times and $f(x,y,z)$ is a function of three variables, then $d/dt f(\vec{r}(t)) = D_{\vec{r}\,'(t)}(f)$.

Solution:
Yes, this is the chain rule.

16) **T** F $\int_0^1 \int_0^1 f_{xy}(x,y) \, dydx = f(1,1) - f(1,0) - f(0,1) + f(0,0)$.

Solution:
This is a consequence of the fundamental theorem of calculus

17) **T** F If $f_{yy}(x,y) > 0$ everywhere, then f can not have any local maximum.

Solution:
We would have $f_{yy} < 0$ at a local maximum.

18) T **F**

The double integral $\int_0^1 \int_0^1 x^2 - y^2 \, dxdy$ is the volume of the solid below the graph of $f(x,y) = x^2 - y^2$ and above the square $0 \leq x \leq 1, 0 \leq y \leq 1$ in the xy-plane.

Solution:
It is a signed volume. There can be part below.

19) **T** F

For any unit vector $\vec{v}$ and any differentiable function f, one has $D_{\vec{v}}(f) + D_{-\vec{v}}(f) = 0$.

Solution:
Write down the definition. The sum is $\nabla f \cdot (v - v) = 0$.

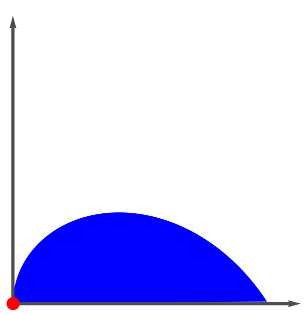
20) **T** F

The surfaces $x + y + z = 0$ and $x^2 + y^2 + z^2 + x + y + z = 0$ have the same tangent plane at $(0, 0, 0)$.

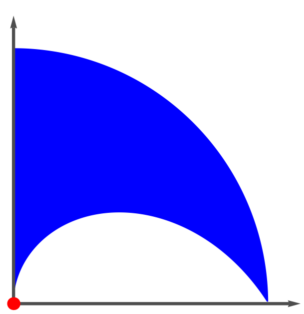
Solution:
They have the same gradient at $(0, 0, 0)$.

Problem 2) (10 points)

a) (6 points) Match the regions with the corresponding polar double integrals



A



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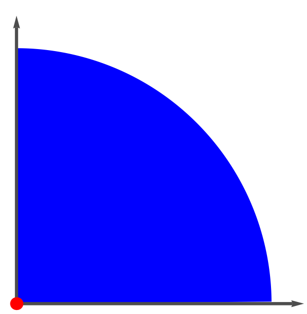

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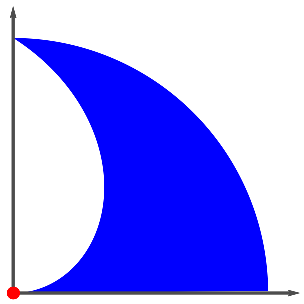

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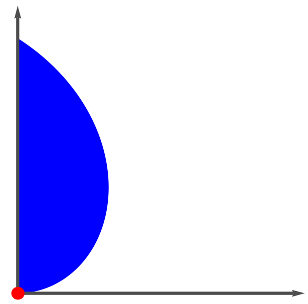

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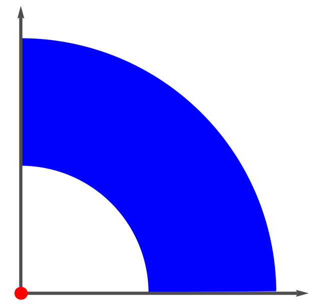


F

Enter A-F	Integral of $f(r,\theta)$
	$\int_0^{\pi/2} \int_0^{\pi/2} f(r,\theta) r \; dr d\theta$
	$\int_0^{\pi/2} \int_0^{\theta} f(r,\theta) r \; dr d\theta$
	$\int_0^{\pi/2} \int_0^{\pi/2-\theta} f(r,\theta) r \; dr d\theta$

Enter A-F	Integral of $f(r,\theta)$
	$\int_0^{\pi/2} \int_{\theta}^{\pi/2} f(r,\theta) r \; dr d\theta$
	$\int_0^{\pi/2} \int_{\pi/2-\theta}^{\pi/2} f(r,\theta) r \; dr d\theta$
	$\int_0^{\pi/2} \int_{\pi/4}^{\pi/2} f(r,\theta) r \; dr d\theta$

b) (4 points) Match the partial differential equations (PDE's) for the functions $u(t,s)$ with their names. No justifications are needed.

Enter A,B,C,D here	PDE
	$u_t + u u_s - u_{ss} = 0$
	$u_{tt} + u_{ss} = 0$

Enter A,B,C,D here	PDE
	$u_{tt} - u_{ss} = 0$
	$u_t - u_{ss} = 0$

A) Wave equation	B) Heat equation	C) Burgers equation	D) Laplace equation

Solution:

	C	D
a)	E	B
	A	F
b)	C	A
	D	B

Problem 3) (10 points)

a) (7 points) Find and classify all the critical points of the function

$$f(x,y) = 5 + 3x^2 + 3y^2 + y^3 + x^3 \ .$$

b) (3 points) Is there a global maximum or a global minimum for $f(x,y)$?

Solution:
The gradient of f is

$$\langle 6x + 3x^2, 6y + 3y^2 \rangle = \langle 3x(2+x), 3y(2+y) \rangle \ .$$

There are 5 critical points:

x	y	D	f_{xx}	Type	f value
-2	-2	36	-6	maximum	13
-2	0	-36	-6	saddle	9
0	-2	-36	6	saddle	9
0	0	36	6	minimum	5

b) There is no global maximum (nor global minimum). For $y = 0$, have $5 + 3x^2 + x^3$ which grows like x^3 for $x \to \infty$.

Problem 4) (10 points)

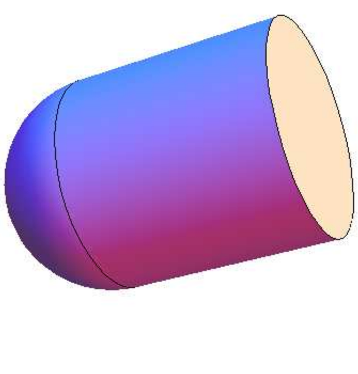
A **solid bullet** made of a half sphere and a cylinder has the volume $V = 2\pi r^3/3 + \pi r^2 h$ and surface area $A = 2\pi r^2 + 2\pi r h + \pi r^2$. Doctor Manhatten designs a bullet with fixed volume and minimal area. With $g = 3V/\pi = 1$ and $f = A/\pi$ he therefore minimizes

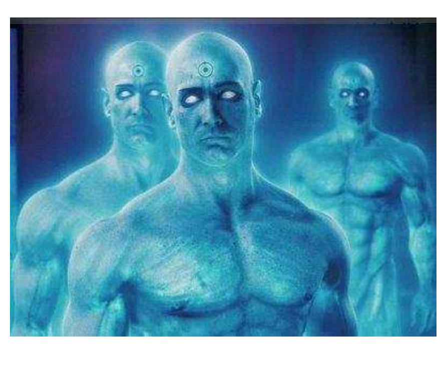
$$f(h,r) = 3r^2 + 2rh$$

under the constraint

$$g(h,r) = 2r^3 + 3r^2 h = 1 \ .$$

Use the Lagrange method to find a local minimum of f under the constraint $g = 1$.

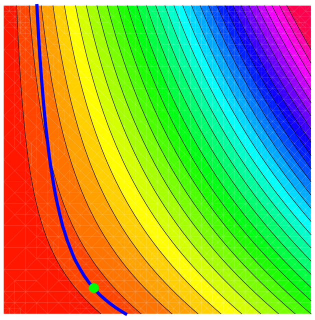




Solution:
The Lagrange equations are

$$\begin{aligned} 2h + 6r &= \lambda(6hr + 6r^2) \\ 2r &= 3\lambda r^2 \\ 3hr^2 + 2r^3 &= 1 \ . \end{aligned}$$

Because $r = 0$ is incompatible with the third equation, we can divide the second equation by r. This allows to eliminate λ and $2h+6r = 4h+4r$ which is $h = r$. The third equation gives us $h = r = 1/5^{1/3}$. The point where the minimum occurs is $(1/5^{1/3}, 1/5^{1/3})$. The minimal value of f is $5/5^{2/3}$.

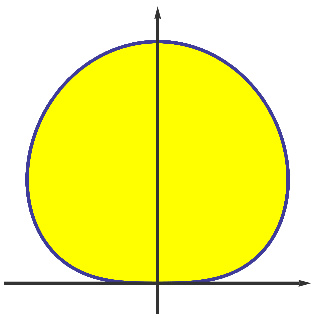


Problem 5) (10 points)

A region R in the plane shown to the right is called the **"blob of nothingness"**. It does not have any purpose nor meaning. It just sits there. The region is given in polar coordinates as $0 \leq r \leq \theta(\pi - \theta)$ for $0 \leq \theta \leq \pi$. Find the area

$$\int \int_R 1 \; dxdy$$

of this nihilistic object.



Solution:

$$\int_0^\pi \int_0^{\theta(\pi-\theta)} r \; dr d\theta = \int_0^\pi \theta^2(\pi-\theta)^2/2 \; d\theta = \pi^5/60 \; .$$

Problem 6) (10 points)

a) (4 points) If

$$f(x,y) = y\cos(x-y) \; ,$$

find equation of plane tangent to $z = f(x,y)$ at the point $(2,2,2)$.

b) (3 points) Find the equation of the tangent line to $f(x,y) = 2$ at $(2,2)$.

c) (3 points) Estimate $f(2.1, 1.9)$ using linear approximation.

Solution:
a) Define $g(x,y,z) = f(x,y) - z$. It is important to deal with a function of three variables when looking at planes. The graph of g is equal to the level surface $z - f(x,y) = 0$. Then $\nabla g(2,2,2) = \langle 0,1,-1 \rangle$. The tangent plane is of the form $y - z = d$ where d is a constant. Plugging in $(2,2,2)$ gives $y - z = 0$.
b) $\nabla f(x,y) = \langle -y\sin(x-y), \cos(x-y) + \sin(x-y)$ and $\nabla f(2,2) = \langle 0,2 \rangle$ so that the equation of the tangent line is $y = d$ for a constant d. Plugging in the point $(2,2)$ gives $y = 2$.
c) $L(2.1, 1.9) = 2 + 0(0.1) + 1(-0.1) = 1.9$.

Problem 7) (10 points)

A **Harvard robot bee** flies along the curve

$$\vec{r}(t) = \langle t - t^3, 3t^2 - 3t \rangle$$

and measures the temperature $f(x, y)$. It flies over the target point $(0, 0)$ at time $t = 0$ and time $t = 1$. At each time, its sensor measures the temperature change $g'(t)$ where $g(t) = f(\vec{r}(t))$.

Image source: Harvard Press release on robobees.seas.harvard.edu

a) (5 points) Assume you knew that the gradient of f at $(0, 0)$ is $\langle a, b \rangle$. What are the values of $g'(t) = d/dt f(\vec{r}(t))$ at $t = 0$ and $t = 1$ in terms of a and b?

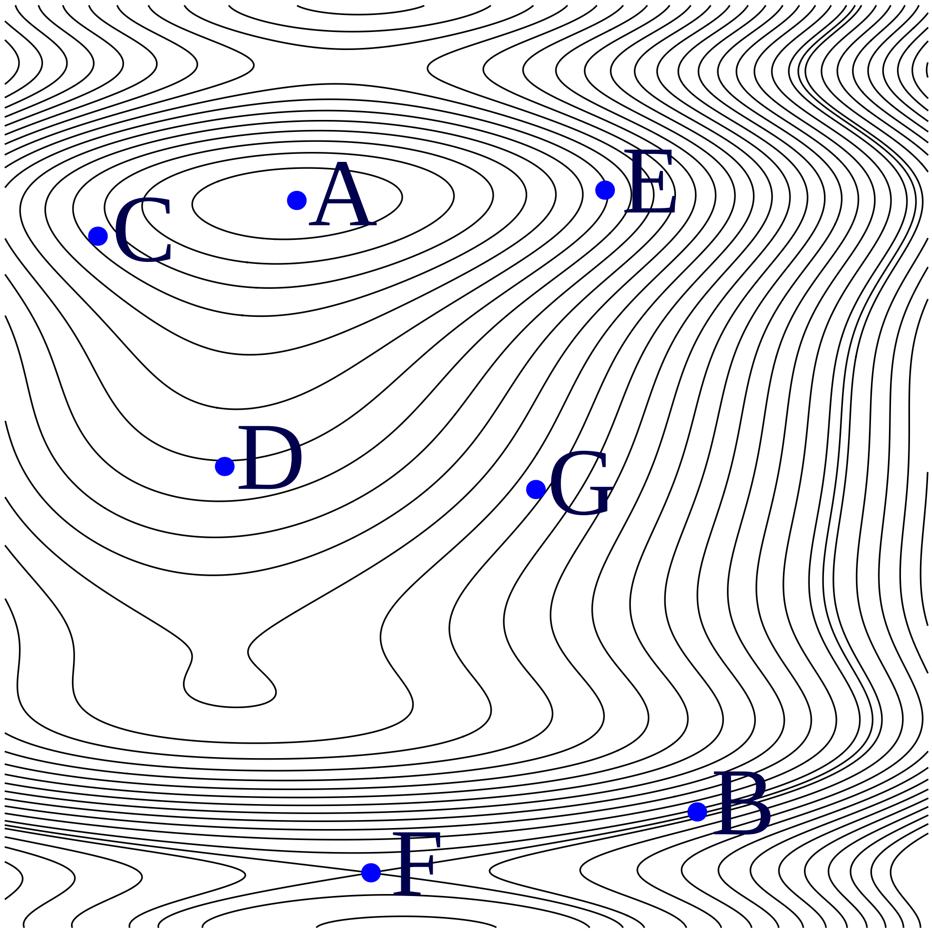
b) (5 points) The bee measures $g'(0) = 3$ and $g'(1) = 3$. What is the gradient $\nabla f(0, 0) = \langle a, b \rangle$ of f at $(0, 0)$?

Solution:
a) $\vec{r}'(t) = \langle 1 - 3t^2, 6t - 3 \rangle$. So that $\vec{r}'(0) = \langle 1, -3 \rangle, \vec{r}'(1) = \langle -2, 3 \rangle$. If the gradient of f at $(0, 0)$ is $\langle a, b \rangle$ we get by the chain rule $d/dt f(\vec{r}(t)) = \langle a, b \rangle \cdot \vec{r}\,'(t)$ which is either $\langle a, b \rangle \cdot \langle 1, -3 \rangle = a - 3b$ or $\langle a, b \rangle \cdot \langle -2, 3 \rangle = -2a + 3b$.
b) We know that $a - 3b = 3$ and $-2a + 3b = 3$. This is a system of linear equations which has the solution $\langle a, b \rangle = \langle -6, -3 \rangle$.

Problem 8) (10 points)

A function $f(x, y)$ of two variables has level curves as shown in the picture. The function values at neighboring level curves differ by 1. [No justifications are needed in this problem. Naturally, since there are less points then boxes, some of the points A-G will appear more than once, but each box will only be filled with one letter.]

Enter A-G	is a point, where ...
	$f_x(x, y) = 0$ and $f_y(x, y) \neq 0$.
	$f_y(x, y) = 0$ and $f_x(x, y) \neq 0$.
	$f(x, y)$ has either a max or a min.
	$f(x, y)$ has a saddle point.
	$f(x, y)$ has no max nor min but is extremal under a constraint $y = c$ for some c.
	$f(x, y)$ has no max nor min but is extremal under a constraint $x = c$ for some c.
	the length of the gradient vector of f is largest among all points A-G.
	$D_{\langle 1/\sqrt{2}, 1/\sqrt{2} \rangle} f(x, y) = 0$ and $D_{\langle 1/\sqrt{2}, -1/\sqrt{2} \rangle} f(x, y) \neq 0$.
	$D_{\langle 1/\sqrt{2}, -1/\sqrt{2} \rangle} f(x, y) = 0$ and $D_{\langle 1/\sqrt{2}, 1/\sqrt{2} \rangle} f(x, y) \neq 0$.
	the tangent line to the curve is $x + y = d$ for some constant d.



Solution:
D,E,A,F,(D or F), (E or F), B,G,C,C.
To the tangent line (last choice): The equation $x+y=d$ means that the gradient vector is $\langle 1,1\rangle$ at the point. Use now that the gradient vector is perpendicular to the level curve.

Problem 9) (10 points)

Evaluate the following double integral

$$\int_0^1 \int_0^{(1-x)^2} \frac{x^3}{(1-\sqrt{y})^4}\, dy dx \ .$$

Solution:
We have to change the order of integration

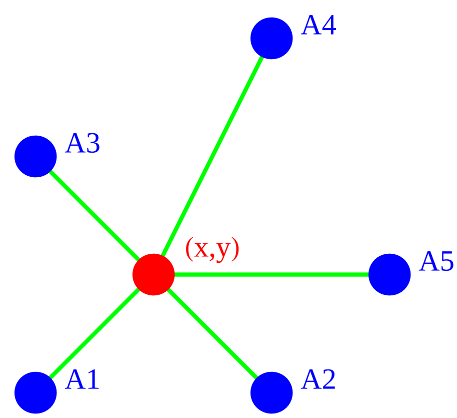
$$\int_0^1 \int_0^{1-\sqrt{y}} x^3/(1-\sqrt{y})^4 dxdy = \int_0^1 (1/4)\, dy = 1/4 \,.$$

Problem 10) (10 points)

A mass point with position (x, y) is attached by springs to the points $A_1 = (0, 0), A_2 = (2, 0), A_3 = (0, 2), A_4 = (2, 3), A_5 = (3, 1)$. It has the potential energy

$$f(x, y) = 31 - 14x + 5x^2 - 12y + 5y^2$$

which is the sum of the squares of the distances from (x, y) to the 5 points. Find all extrema of f using the second derivative test. The minimum of f is the position, where the mass point has the lowest energy.



Solution:
The gradient of f is

$$\nabla f(x, y) = \langle -14 + 10x, -12 + 10y \rangle \,.$$

It leads to the solution $(x, y) = (7, 6)/5 = (1.4, 1.2)$.
(Side remark: In general the average $\sum_{i=1}^n A_i/n$ is the only critical point because the function $f(X) = \sum_{i=1}^n |x - A_i|^2$ has the gradient $\sum_{i=1}^n 2(X - A_i) = 0$ showing $nX = \sum_i A_i$. This is true in any dimension and any number of mass points.)
The Hessian matrix is

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} \,.$$

From this we can read $D = 100$ and $f_{xx} = 10$. The second derivative test shows that the point is a minimum. We have $f(1.4, 1.2) = 14$.