EXACT CURLS

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1. The question

Let's treat $\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial x}, \frac{\partial}{\partial x} \rangle$ as a vector of differential operators. Then all 3-dimensional vector operations can be extended to the algebra including ∇ and all vector fields. $\mathbb{R}^3$ is flat so ∇ even commutes with itself, however ∇ does not commute with ordinary functions and vector fields. We can ask which vector algebra identities still hold.

For example, $u \cdot (v \times w) = [u, v, w]$, so if $u = v$, the expression should evaluate to 0. On the other hand $F \cdot (\nabla \times F)$ is NOT always 0, take for instance $F = \langle -y, x, 1 \rangle$. The curl factor is not affected by the addition of a gradient field since the curl of a gradient is 0, so perhaps the equation is true up to the addition of a gradient field:

Question 1.1. *Given any vector field F in $\mathbb{R}^3$, is there a function f such that*

$$(\nabla \times F) \cdot F = (\nabla \times F) \cdot \nabla f? \tag{1}$$

Answer 1.2. Our reasoning shows that such an f can exist ONLY when $(\nabla \times F) \cdot F$ already vanishes.

2. Quaternions

We can do all 3d vector algebra using the algebra of quaternions. This is the set of all $q = a + ib + jc + kd$ with $a, b, c, d \in \mathbb{R}$ and noncommutative multiplication rule $i^2 = j^2 = -1$ and $ij = k$. Our usual notion of a scalar a will by represented by $\Re q = a$ and our usual notion of a vector $\langle b, c, d \rangle$ will be represented $\Im q = ib + jc + kd$. In this setting, real multiplication and scalar multiplication behave as usual while vector multiplication behaves as $vw = v \times w - v \cdot w$.

Note 2.1. (1) $\Im v^2 = 0$ since $v \times v = 0$.
(2) Quaternion multiplication is associative $u(vw) = (uv)w$, however...

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(3) The vector operator $\nabla = i\partial_x + j\partial_y + k\partial_z$ satisfies the Leibniz rule

$$\nabla(xy) = (\nabla x)y + x(\nabla y)$$

for any quaternions x, y.

Now let F, G be vector fields. $(G \times F) \cdot F = -\Re(GF)F = -\Re G(F^2) = 0$ by the note. However, if we try the same trick replacing G with ∇...

$$(\nabla \times F) \cdot F = -\Re(\nabla F)F = -\Re\nabla(F^2) + \Re F(\nabla F) = \Re F(\nabla F)$$

which is not obviously zero.

The next two sections give counterexamples: F such that there is no function f with $(\nabla \times F) \cdot F = (\nabla \times F) \cdot \nabla f$.

3. Flow lines

Thinking of the right hand side of (1) as the directional derivative $D_{\nabla \times F} f$, my first thought was to find a vector field F so that $\nabla \times F$ has a closed flow line over which the line integral F is nonzero. This would contradict the fundamental theorem of line integrals implying no such f can exist.

Let

$$F = \langle -2y + \frac{4}{3}y^3, 0, -\frac{1}{2}(x^2 + y^2) \rangle.$$

In this case, $\nabla \times F = \langle -y, x, 2 - 4y^2 \rangle$ has the closed flow line $r(t) = \langle \cos t, \sin t, \sin 2t \rangle$ which projects onto the unit circle in the $xy - plane$. Lastly, $F \cdot (\nabla \times F) = y^2 + \frac{2}{3}y^4 + 2y^2x^2 - x^2$ which integrates positively over any closed loop that projects onto the unit circle.

4. Contact forms

Another approach is motivated by contact structures.

Definition 4.1. A *contact form* on $\mathbb{R}^3$ is a vector field F such that $(\nabla \times F) \cdot F > 0$.

Example 4.2. One contact form is $F = \langle 0, x, 1 \rangle$. Here, $\nabla \times F = \langle 0, 0, 1 \rangle$ so $(\nabla \times F) \cdot F = 1$.

Let F be any *contact form* on $\mathbb{R}^3$ and h a function which is positive inside the unit ball and outside. Suppose $(\nabla \times hF) \cdot hF = (\nabla \times hF) \cdot \nabla f$. By the divergence theorem, (check the vector identity here)

$$0 = \int_{S^2} f\nabla \times hF = \int_{B^3} \nabla \cdot (f\nabla \times hf) = \int_{B^3} h^2 \nabla f \cdot (\nabla \times F) dV > 0.$$

To obtain Answer 1.2, apply this cutoff function argument to any open set on which the left hand side of (1) is nonzero.