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- Print your name in the above box and **check your section**.
- Do not detach pages or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- All functions are assumed to be smooth and nice unless stated otherwise.
- **Show your work**. Except for problems 1-3, we need to see details of your computation.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
14		10
Total:		150

Problem 1) True/False questions (20 points). No justifications are needed.

1) **T** F — The parametrization $\vec{r}(u,v) = \langle v^2, v^2\cos(u), v^2\sin(u)\rangle$ describes a cone.

Solution:
Indeed $y^2 + z^2 = x^2$

2) T **F** — If three vectors $\vec{u}, \vec{v}, \vec{w}$ satisfy $\vec{u} \cdot \vec{v} = 0$ and $\vec{v} \times \vec{w} = \vec{0}$, then $\vec{u} \cdot \vec{w} = 0$.

Solution:
One could think that if the vectors v, w are parallel and $u \cdot v = 0$, then $u \cdot w = 0$. The answer however is false because if $v = 0$ and $u = w = \langle 1, 0, 0\rangle$ then $u \cdot v = 0$ and $v \times w = 0$, but u and w are not orthogonal.

3) **T** F — Let S be a surface bounding a solid E and $\vec{F}$ is a vector field in space which is incompressible, $\text{div}(\vec{F}) = 0$, then $\iint_S \vec{F} \cdot d\vec{S} = 0$.

Solution:
By the divergence theorem.

4) T **F** — If $\text{curl}(\vec{F})(x,y,z) = 0$ for all (x,y,z) then $\iint_S \vec{F} \cdot d\vec{S} = 0$ for any closed surface S.

Solution:
The line integrals would all be zero

5) **T** F — If $\vec{F}$ is a conservative vector field in space, then $\vec{F}$ has zero curl everywhere.

Solution:
Yes, since $\text{curl}(\text{grad}(f)) = 0$

6) T **F** — If $\vec{F}, \vec{G}$ are two vector fields which have the same divergence then $\vec{F} - \vec{G}$ is constant.

Solution:
Take any two fields which are the curl of two fields.

7) T **F** The linearization of the constant function $f(x,y) = 3$ at $(x,y) = (1,1)$ is the function $L(x,y) = 0$.

Solution:
The linearization is 3.

8) T **F** The surface area of a surface S is $\int\int_S \langle x,y,z \rangle \cdot d\vec{S}$.

Solution:
This only would be true for the unit sphere

9) **T** F There is a non-constant vector field $\vec{F}(x,y,z)$ such that $\text{curl}(\vec{F}) = \text{curl}(\text{curl}(\vec{F}))$

Solution:
Take a gradient field or the ¡cos(y),0,cos(y)¿

10) **T** F If $\vec{F}$ is a vector field and E is the unit ball then $\iiint_E \text{div}(\text{curl}(\vec{F}))\, dV = 0$.

Solution:
Because $\text{div}(\text{curl}(\vec{F}) = \vec{0}$.

11) T **F** If the vector field $\vec{F}$ has constant divergence 1 everywhere, then the flux of $\vec{F}$ through any closed surface S is zero.

Solution:
The flux is the volume.

12) T **F** The equation $\text{grad}(\text{div}(\text{grad}(f))) = \vec{0}$ always holds.

Solution:
Take $f = x^3$, then the result is $\langle 1, 0, 0\rangle$.

13) | T | F | (T highlighted)

The vector $\vec{k} \times (\vec{j} \times \vec{i})$ is the zero vector, if $\vec{i} = \langle 1, 0, 0\rangle, \vec{j} = \langle 0, 1, 0\rangle$, and $\vec{k} = \langle 0, 0, 1\rangle$.

Solution:
$\vec{j} \times \vec{i} = -\vec{k}$ which is parallel to $\vec{k}$.

14) | T | F | (F highlighted)

If f is minimized at (a, b) under the constraint $g = c$, then $\nabla f(a, b)$ and $\nabla g(a, b)$ are perpendicular.

Solution:
They are parallel

15) | T | F | (T highlighted)

If A, B, C, D are four points in space such that the line through A, B intersects the line through C, D, then A, B, C, D lie on some plane.

Solution:
Yes, it is the plane spanned by the two lines.

16) | T | F | (T highlighted)

The chain rule assures that for a vector field $\vec{F} = \langle P, Q, R\rangle$ the formula $\frac{\partial}{\partial u}\vec{F}(\vec{r}(u, v)) = \langle \nabla P \cdot \vec{r}_u, \nabla Q \cdot \vec{r}_u, \nabla R \cdot \vec{r}_u\rangle$ holds.

Solution:
Apply the chain rule to each component.

17) | T | F | (T highlighted)

The vector field $\vec{F} = \langle \cos(y), 0, \sin(y)\rangle$ satisfies $\text{curl}(\vec{F}) = \vec{F}$. By the way, it is called the Cheng-Chiang field.

Solution:
Compute the curl and see that the fixed point property holds. Cheng-Chiang is is a Harvard PhD who now is at MIT.

18) | T | F | (T highlighted)

The unit tangent vector $\vec{T}(t)$, the normal vector $\vec{N}(t)$ and the binormal vector $\vec{B}(t)$ for a given curve $\vec{r}(t)$ span a cube of volume 1 at $t = 1$.

Solution:
We have to normalize

19) T F The vector field $\vec{F}(x, y, z) = \langle z, z, z \rangle$ can not be the curl of a vector field.

Solution:
Its divergence is not zero

20) T F The expression curl(grad(div(grad(div(curl($\vec{F}$)))))) is a well defined vector field in three dimensional space.

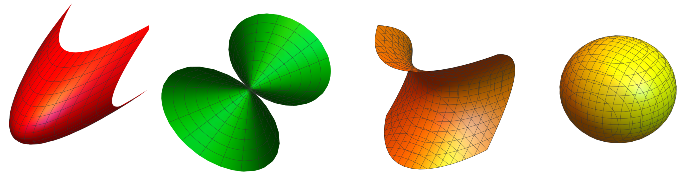
Solution:
This is the implicit equation.

Problem 2) (10 points) No justifications are necessary.

a) (2 points) Match the following surfaces. There is an exact match.

Surface	1-4
$\vec{r}(u,v) = \langle u, u^2 - v^2, v \rangle$	
$y - x^2 - z^2 = 0$	
$\vec{r}(u,v) = \langle u\cos(v), u, u\sin(v) \rangle$	
$x^2 + y^2 = 1 - z^2$	

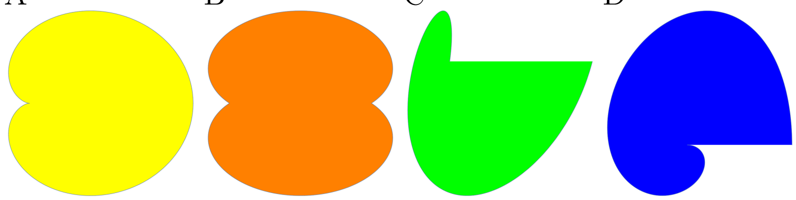
1 2 3 4



b) (2 points) Match the following regions given in polar coordinates (r, θ):

Region	A-D
$r < \theta^2$	
$r < 1 + \cos(\theta)$	
$r < 1 + \|\sin(\theta)\|$	
$r < (4\pi^2 - \theta^2)$	

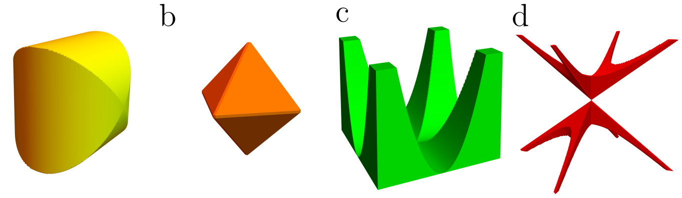
A B C D



c) (2 points) Match the regions. There is an exact match.

Solid	a-d
$\|x\| + \|y\| + \|z\| \leq 1$	
$x^2 + y^2 \leq 1, x^2 + z^2 \leq 1$	
$x \leq y^2, x \leq z^2$	
$0 \leq x^2 - y^2 \leq 4, 0 \leq y^2 - z^2 \leq 4$	

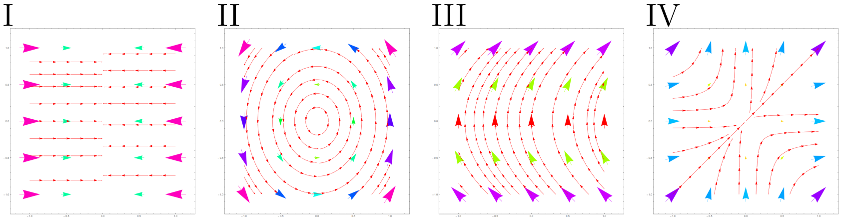
a b c d



d) (2 points) The figures display vector fields in the plane. There is an exact match.

Field	I-IV
$\vec{F}(x,y) = \langle y, 1 \rangle$	
$\vec{F}(x,y) = \langle -2y, 3x \rangle$	
$\vec{F}(x,y) = \langle -x, 0 \rangle$	
$\vec{F}(x,y) = \langle x^2, y^2 \rangle$	

I II III IV



e) (1 point) Find the linearization $L(x, y)$ of $f(x, y) = xy$ at $(2, 1)$.

f) (1 point) Write down the wave equation for a function $f(t, x)$:

Solution:

a) 3,1,2,4

b) C,A,B,D

c) b,a,c,d

d) III,II,I,IV

e) $2 + (x - 2) + 2(y - 1)$.

f) $f_{tt} = f_{xx}$.

Problem 3) (10 points)

a) (6 points) In the following, f is a function, $\vec{F}$ a vector field, I is an interval, G is a region in the two dimensional plane, S is a closed surface parametrized by $\vec{r}(u,v)$, C is a closed curve parametrized by $\vec{r}(t)$ and E is a solid. Fill the blanks using "volume", "area", "length" or "0", where a choice can appear a multiple times.

[ ] $= \int\int_G \text{curl}(\langle x, 0\rangle)\ dxdy$

[ ] $= \int\int_G \text{curl}(\langle -y, 0\rangle)\ dxdy$

[ ] $= \int\int\int_E \text{div}(\langle x, x, x\rangle)\ dxdydz$

[ ] $= \int\int_S |\vec{r}_u \times \vec{r}_v|\ dudv$

[ ] $= \int_C |\vec{r}\,'(t)|\ dt + \int_C \text{grad}(f) \cdot d\vec{r}$

[ ] $= \int\int_S \text{curl}(\vec{F}) \cdot d\vec{S}$

b) (4 points)

Complete the formulas of the following boxes

$\int_0^{2\pi} \int_0^{\pi} \int_0^1$ $\boxed{}$ $d\rho d\phi d\theta$ Integral for volume of unit ball $x^2 + y^2 + z^2 \leq 1$.

$\int_0^{2\pi} \int_0^1$ $\boxed{}$ $dr d\theta$ Integral for area of unit disc $x^2 + y^2 \leq 1$

$g_x(x, y) = \frac{\boxed{}}{f_z(x,y,z)}$ Implicit derivative of g satisfying $f(x, y, g(x, y)) = 0$

$|\vec{u} \cdot (\boxed{})|$ Volume of parallelepiped spanned by $\vec{u}, \vec{v}, \vec{w}$.

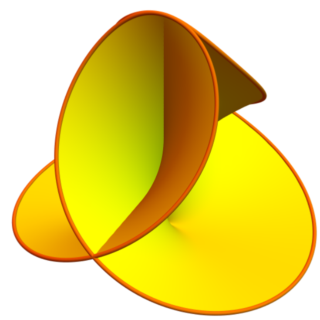
Solution:
a) 0, area, volume, area, length, 0
b) $\rho^2 \sin(\phi)$, r, $-f_x(x, y, z)$, and $\vec{v} \times \vec{w}$

Problem 4) (10 points)

Find the surface area of the surface parametrized by

$$\vec{r}(u, v) = \langle u \cos(v), u \sin(v), \frac{v^2}{2} \rangle \ ,$$

where (u, v) are in the domain $u^2 + v^2 \leq 9$.



P.S. You do not have to worry that this cool surface has self-intersections.

Solution:

$$\vec{r}_u = \langle \cos(v), \sin(v), 0 \rangle$$
$$\vec{r}_v = \langle -u\sin(v), u\cos(v), v \rangle \ .$$

The cross product

$$\vec{r}_u \times \vec{r}_v = \langle v\sin(v), -v\cos(v), u \rangle$$

has length $\sqrt{u^2+v^2}$. Now we can compute the integral

$$\int \int_R |\vec{r}_u \times \vec{r}_v| \; dudv = \int_0^{2\pi} \int_0^3 r \; r \; drd\theta = 18\pi \ .$$

Problem 5) (10 points)

On planet **Tatooine**, Luke Skywalker travels along a path C parametrized by

$$\vec{r}(t) = \langle t\cos(t), t\sin(t), 0 \rangle$$

from $t = 0$ to $t = 2\pi$. What is the work done

$$\int_C \vec{F} \cdot d\vec{r}$$

by the "force"

$$\vec{F} = \langle x^2 + y + z, y^3 + x, z^5 + x \rangle \ .$$

Solution:
This is a problem for the fundamental theorem of line integrals. It is applicable because the curl of $\vec{F}$ is zero. The potential f can be obtained by integration. We have

$$f(x,y,z) = x^3/3 + y^4/4 + z^6/6 + xy + xz \ .$$

Now, we just have to plug in the end point $\vec{r}(2\pi) = \langle 2\pi, 0, 0 \rangle$ and initial point $\langle 0,0,0 \rangle$ and get

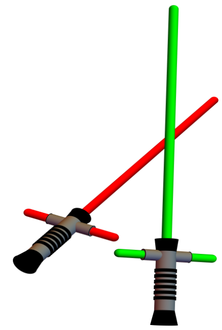
$$\int_0^{2\pi} \vec{F}(\vec{r}(t)) \; \vec{r}\,''(t) \; dt = f(\vec{r}(2\pi)) - f(\vec{r}(0)) = 8\pi^3/3 \ .$$

Problem 6) (10 points)

Tomorrow, on December 18, the "force awakens". There will be light sabre battles, without doubt.

a) (7 points) What is the distance between two light sabres given by cylinders of radius 1 around the line $\vec{r}(t) = \langle t, -t, t \rangle$ and the line connecting $(0, 14, 0)$ with $(3, 5, 6)$.

b) (3 points) A spark connects the two points of the sabre which are closest to each other. Find a vector in that direction.



Sabre by Connor (mathematica project).

Solution:
a) First compute the distance between the central lines. They contain the vectors $\langle 1, -1, 1 \rangle$ and $\langle 1, -3, 2 \rangle$. The cross product is $\vec{n} = \langle 1 - 1, -2 \rangle$. Pick a point $Q = (0, 14, 0)$ on one curve and $P = (0, 0, 0)$. Now

$$d = |\vec{PQ} \cdot \langle 1, -1, -2 \rangle| / |\langle 1, -1, -2 \rangle| = 7\sqrt{6}/3$$

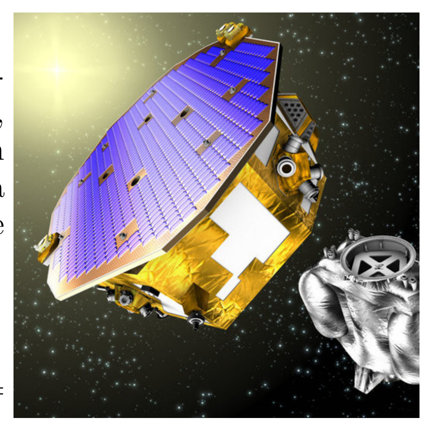
The distance between the cylinders is $7\sqrt{6}/3 - 2$.
b) The vector $\langle 1, -1, -2 \rangle$ found in a) is perpendicular to both lines. It already gives the connection between the two points!

Problem 7) (10 points)

100 years ago, Einstein proposed gravitational waves. To measure them, the LISA pathfinder was launched on December 3, 2015. It carries two cubes to the Lagrangian point between Earth and Moon aiming to measure the waves. Assume a gravitational wave from a black hole merger produces a force leading to an acceleration

$$\vec{r}\,''(t) = \langle \sin(t), \cos(t), \sin(t) \rangle .$$

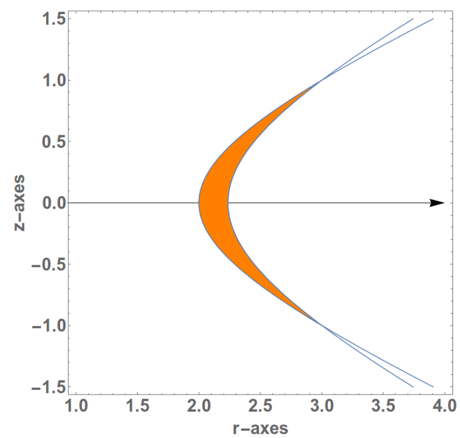
What is $\vec{r}(t)$ at time $t = \pi$ if $\vec{r}\,'(0) = \langle 2, 0, 0 \rangle$ and $\vec{r}(0) = \langle 1, 0, 3 \rangle$.



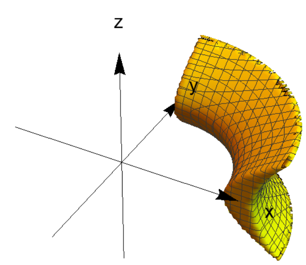
LISA proof of concept will be followed by eLISA in 2034.

Solution:
Integrate
$$\vec{r}\,''(t) = \langle \sin(t), \cos(t), \sin(t) \rangle$$
to get
$$\vec{r}\,'(t) = \langle 3 - \cos(t), \sin(t), 1 - \cos(t) \rangle \ .$$
Now integrate again to get
$$\vec{r}\,(t) = \langle 1 + 3t - \sin(t), 1 - \cos(t), 3 + t - \sin(t) \rangle \ .$$
We have $\vec{r}(\pi) = \langle 1 + 3\pi, 2, 3 + \pi \rangle$.

Problem 8) (10 points)



a) (5 points) Find the integral $\int\int_G r \ drdz$, where G is the region enclosed by the curves $r^2 - 4z^2 = 5$ and $r^2 - 5z^2 = 4$ and contained in $r \geq 0$.



b) (5 points) The Galactic Empire builds a space craft E given as a solid in $x \geq 0, y \geq 0$, enclosed by
$$x^2 + y^2 - 4z^2 = 5$$
and
$$x^2 + y^2 - 5z^2 = 4 \ .$$
Find its volume. P.S. You can make use of problem a) to solve part b) as the problems are related.

Solution:
a) $\int_{-1}^{1} \int_{\sqrt{4+5z^2}}^{\sqrt{5+4x^2}} r dr dz = \int_{-1}^{1} (5 + 4z^2 - 4 - 5z^2)/2 \, dz = 2/3$.
b) E is described in terms of cylindrical coordinates by

$$-1 \leq z \leq 1, \qquad \sqrt{4z^2+5} \leq r \leq \sqrt{5z^2+4}, \qquad 0 \leq \theta \leq 2\pi .$$

Hence, the volume of E is

$$\begin{aligned}
&= \iiint_E 1 \, dV \\
&= \int_{-1}^{1} \int_0^{\pi/2} \int_{\sqrt{4z^2+5}}^{\sqrt{5z^2+4}} r \, dr \, d\theta \, dz \\
&= \int_{-1}^{1} \int_0^{\pi/2} \frac{r^2}{2} \Big|_{\sqrt{4z^2+5}}^{\sqrt{5z^2+4}} dr \, d\theta \, dz \\
&= \pi/3 \, .
\end{aligned}$$

Since a) and b) are related, one an also use the result in a) directly. The volume is $\int_0^{\pi/2} \int_{-1}^{1} \frac{r^2}{2} \Big|_{\sqrt{4z^2+5}}^{\sqrt{5z^2+4}} dr \, dr \, d\theta$. The inner integral is 2/3 from a). So, the result is $\pi/2$ times 2/3 which is $\pi/3$.

Problem 9) (10 points)

In September 2015, the west side of the Harvard Science center honored the concept of curl by displaying paddle wheels. One of the wheel tips moves on an oriented curve $\vec{r}(t) = \langle \cos(t), 0, \sin(t) \rangle$ bounding the disc parametrized by $\vec{r}(u,v) = \langle u\cos(v), 0, u\sin(v) \rangle$ with $0 \leq u \leq 1, 0 \leq v \leq 2\pi$. Let $\vec{F}$ be the wind vector field

$$\vec{F}(x,y,z) = \langle x^9 + y^7 + 3z, x^9 + y^9 + \sin(z), z^5 e^z \rangle \, .$$

Find the line integral $\int_C \vec{F} \cdot d\vec{r}$ measuring the energy gain during one rotation along the curve C parametrized by $\vec{r}(t)$.

Solution:
We use Stokes theorem with a disc surface bounded by the curve C. This surface can be parametrized by

$$\vec{r}(u,v) = \langle u,0,v\rangle, \vec{r}_u \times \vec{r}_v = \langle 0,-1,0\rangle \ ,$$

and where the parameter domain is $R : u^2+v^2 \leq 1$. The curl is

$$\mathrm{curl}(\vec{F})(x,y,z) = \langle -\cos(z), 3, 9x^8 - 5y^4\rangle \ .$$

The flux of the curl through the disc R

$$\int\int_R \langle -\cos(v), 3, 9u^8\rangle \cdot \langle 0,-1,0\rangle \ dudv \ ,$$

which we can compute by using polar coordinates

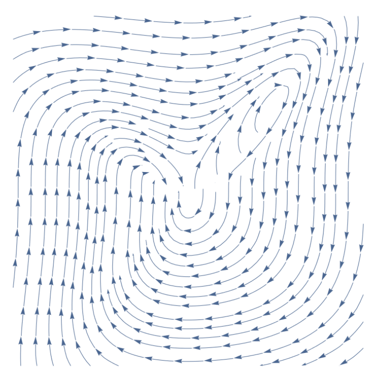
$$\int\int_R (-3) dudv = \int_0^{2\pi}\int_0^1 (-3r) \ drd\theta = -3\pi \ .$$

This is already the right sign as the parametrization of the surface and the line was already given in a compatible form.

Problem 10) (10 points)

The value of the line integral of the vector field $\vec{F}(u,v) = (2/\pi)\langle -uv^2+v^3, uv-u^3\rangle$ along a curve $\vec{r}(t) = \langle x+\cos(t), y+\sin(t)\rangle$ depends only on the center point (x,y) and is given by

$$f(x,y) = -3 - 6x^2 + 2y + 4xy - 6y^2 \ .$$



a) (7 points) Find all critical points (x,y) for the function $f(x,y)$ and analyze them using the second derivative test.
b) (3 points) Given that

$$f(x,y) = -3 - (x-2y)^2 - 5x^2 - 2y^2 + 2y \ ,$$

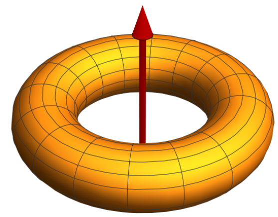
decide whether there is a global maximum for f.

Solution:
a) The gradient is $\langle -12x+4y, 2+4x-12y\rangle$ which when put to zero $\langle 0,0\rangle$ gives $y = 3/16$ and $x = 1/16$. The discriminant is 128 and $f_{xx} = -12$. The function f has therefore a local maximum at $(1/16, 3/16)$. The value is $-(45/16)$.
b) Completing the square gives $f(x,y) = -2 - (x-2y)^2 - 5x^2 - y^2 - (y-1)^2$. Since this is a sum of negative squares, this goes to $-\infty$ in any direction. The maximum found in a) is therefore a global maximum. (P.S. there are functions of two variables with one critical point which is a local maximum without that this is a global maximum. So, the fact alone that we have only one critical point is not enough.)

Problem 11) (10 points)

The moment of inertia $f(x, y)$ of a torus of mass 4 with smaller tube radius x and bigger center curve radius y is

$$f(x, y) = 3x^2 + 4y^2 \ .$$



a) (7 points) Find the parameters (x_0, y_0) for the torus which have minimal moment of inertia under the constraint that

$$g(x, y) = x + 4y = 13 \ .$$

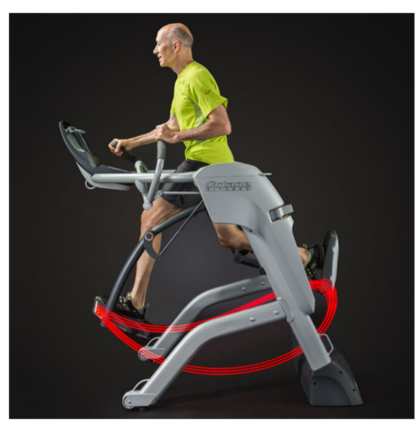
b) (3 points) Write down the equation of the tangent line to the level curve of f which passes through (x_0, y_0).

Solution:
a) This is a Lagrange problem. We have

$$\begin{aligned} 6x &= \lambda\, 1 \\ 8y &= \lambda\, 4 \\ x + 4y &= 13 \end{aligned}$$

Elminating λ gives $24x = 8y$ meaning $y = 3x$. Plugging into the constraint gives $x = 1, y = 3$.
b) Since the gradient of f is parallel to the gradient of g which we know and the tangent line through $(1, 3)$ is the constraint $x + 4y = 13$, the later equation is already the equation of the constraint.

Problem 12) (10 points)



A new **elliptical machine** has been designed to simulate running better. The leg of a runner moves on the curve parametrized by

$$\vec{r}(t) = \langle 8\cos(t), 2\sin(t) + \sin(2t) + \cos(2t) \rangle$$

with $0 \leq t \leq 2\pi$. Find the area of the region enclosed by the curve.

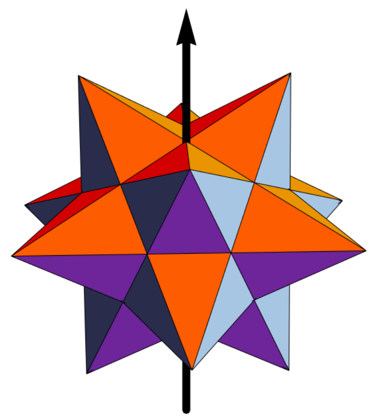
Solution:
We use Green's theorem with the vector field $\vec{F} = \langle 0, x \rangle$ which has curl 1. The line integral is

$$\int_0^{2\pi} \langle 0, 8\cos(t) \rangle \cdot \langle -8\sin(t), 2\cos(t) + 2\cos(2t) - 2\sin(2t) \rangle \, dt$$

This is $\int_0^{2\pi} 8\cos(t)(2\cos(t) + 2\cos(2t) - 2\sin(2t)) \, dt = 16\pi$.

Problem 13) (10 points)

The polyhedron E in the figure is called **small stellated Dodecahedron**. The solid E has volume 10. Its moment of inertia $\iiint_E x^2 + y^2 \, dxdydz$ around the z-axis is known to be 1. Let S be the boundary surface of the polyhedron solid E oriented outwards.



a) (5 points) What is the flux of the vector field

$$\vec{F}(x, y, z) = \langle y^5 + x, z^5 + y, x^5 + z \rangle$$

through S?

b) (5 points) What is the flux of the vector field

$$\vec{G}(x, y, z) = \langle x^3/3, y^3/3, 0 \rangle$$

through S?

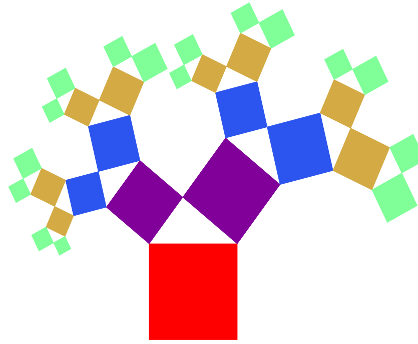
Solution:
a) We use the divergence theorem. The divergence of the vector field is 3. The flux therefore is 3 times the volume of the solid which is 30.
b) Again, we use the divergence theorem. The divergence is x^2+y^2. The integral $\int\int\int_E x^2 + y^2 \, dxdydz$ is 1.

Problem 14) (10 points)

Find the line integral of the vector field

$$\vec{F}(x,y) = \langle -y + x^8, x - y^9 \rangle$$

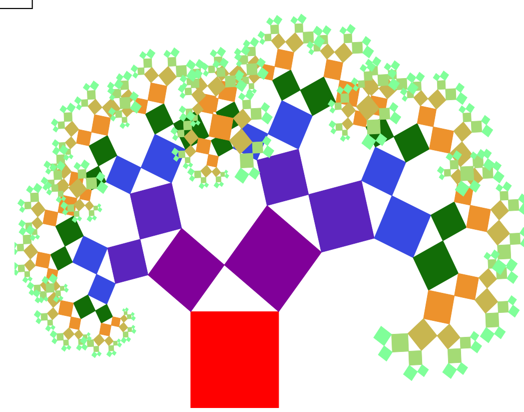
along the boundary C of the generation 4 **Pythagoras tree** shown in the picture. The curve C traces each of the 31 square boundaries counter clockwise. You can use the Pythagoras tree theorem mentioned below. We also included the proof of that theorem evenso you do not need to read the proof in order to solve the problem.



Pythagoras tree theorem:

The generation n Pythagorean tree has area $n+1$.

Proof: in each generation, new squares are added along a right angle triangle. The 0'th generation is a square of area $c^2 = 1$. The first generation tree got two new squares of side length a, b which by **Pythagoras** together have area $a^2 + b^2 = c^2 = 1$. Now repeat the construction. In generation 2, we have added 4 new squares which together have area 1 so that the tree now has area 3. In generation 3, we have added 8 squares of total area 1 so that the generation tree has area 4. Etc. Etc. The picture to the right shows generation 7. Its area of all its (partly overlapping) leafs is 8.



Solution:

We use the Green theorem. The curl of $\vec{F}$ is constant 2. The integral $\int\int_R \text{curl}(\vec{F})\, dxdy$ is therefore 2 times the area of R which is $2 \times 5 = \boxed{10}$.