Homework 11: Partial differential equations

This homework is due Wednesday, 10/5 rsp Thursday 10/7.

1 a) Identify the functions which solve the **Laplace equation** $u_{xx} + u_{yy} = 0$.

a) $u = 5x^2 - 5y^2$ b) $u = 7x - 7y$ c) $u = x^3 + 3xy^2$

d) $u = \log\sqrt{x^2 + y^2}$ e) $u = e^{-x}\cos y - e^{-y}\cos x$

As usual, $\log = \ln$ denotes the natural log.

b) Verify that each of the following solves the **wave equation** $u_{tt} = u_{xx}$.

a) $u = \sin(5x)\sin(5t)$ b) $u = \left(\frac{t}{t^2 - x^2}\right)$

c) $u = (x - t)^6 + (x + t)^6$ d) $u = \sin(x - t) + \log(x + t)$

2 The differential equation

$$f_t = f - xf_x - x^2 f_{xx}$$

is an example of the **infamous Black-Scholes equation**. Here $f(x, t)$ is the prize of a call option and x the stock prize and t is time. Verify that $f(x, t) = 3x + e^t \log(x)$ solves this equation. (We know also that $x, e^t, e^{-3t}x^2, (1 + x^2)/(2x)$ solves this equation. If you should find an other one, mail it in.)

3 a) Show that the **Cobb-Douglas** production function $P = L^\alpha K^\beta$ satisfies the equation

$$L\frac{\partial P}{\partial L} + K\frac{\partial P}{\partial K} = (\alpha + \beta)P \ .$$

The constants α and β are fixed. L is labor and K is capital.

b) Verify that $f(x, y) = \sqrt{x^2 + y^2}$ satisfies the **Eiconal equation** $f_x^2 + f_y^2 = 1$. This partial differential equation is important in optics.

4 Run Mathematica code for the two dimensional wave equation. The example code is given on the website and below. Plot the graph of the function $u(t, 0.3, 0.4)$ from $t = 0$ to $t = 1$. You can either print out your output or copy what you see on the screen.

5 The partial differential equation

$$f_t + f f_x = f_{xx}$$

called **Burgers equation** describes waves at the beach. In higher dimensions, it leads to the **Navier-Stokes** equation which is used to describe the weather. Use Mathematica to verify that the function

$$f(t, x) = \frac{\left(\frac{1}{t}\right)^{3/2} x e^{-\frac{x^2}{4t}}}{\sqrt{\frac{1}{t}} e^{-\frac{x^2}{4t}} + 1}$$

solves the Burgers equation.

An equation for an unknown function $f(x, y)$ which involves partial derivatives with respect to at least two different variables is called a **partial differential equation**. You have to be able to verify whether some function solves a partial differential equation. You will also have to know a 5 basic differential equations: the Laplace, the heat, the wave and the transport and the Burger equation. In lecture we provide some background on these equations and meaning which helps to know them without blindly memorizing them.

Use of Mathematica

Most of you have already installed Mathematica. If you have not done so, part of the homework is also to finally install and run Mathematica. You can use this software to make tough computations. You can use the software to do the problems above. Here is an example. After entering, type return while holding down the shift key:

```
f [ t_ , x_ ]:=( 1 / Sqrt [ t ] )∗Exp[ −x ˆ2/( 4 t ) ];
Simplify [ D[ f [ t , x ] , t ] == D[ f [ t , x ] ,{ x , 2 }]]
```

You have verified that the function

$$\frac{1}{\sqrt{t}}e^{-x^2/(4t)}$$

satisfies the heat equation. As any real programming language, Mathematica is particular about syntax. Watch brackets, capitalization, double equal signs ==!

Here is Mathematica code for an example of the two dimensional wave equation. You can copy paste from the website:

```
A=Rectangle[{0,0},{1,1}]; Clear[t,x,y];
f[x_,y_]:=Sin[2 Pi x] Abs[Sin[3 Pi y]];
g[x_,y_]:=3 Sin[Pi x] Sin[Pi y];
U=NDSolveValue[{D[u[t,x,y],{t,2}]
 -Inactive[Laplacian][u[t,x,y],{x,y}]==0,
   u[0,x,y] == f[x,y],
   Derivative[1,0,0][u][0,x,y]==g[x,y],
   DirichletCondition[u[t,x,y] ==0,True]},
   u,{t, 0, 2 Pi}, {x,y} \[Element] A];
Plot3D[U[4,x,y],{x,0,1},{y,0,1}]
Animate[ContourPlot[U[t,x,y],
   {x,0,1},{y,0,1}],{t,0,2Pi}]
```