Homework 14: Tangent lines and planes

This homework is due Friday, 10/14 rsp Thursday 10/13.

1 a) Find the tangent plane to the surface

$$x^2 + y^2 - x^2y^2 - z^2 = 0$$

at the point $(x, y, z) = (1, 2, 1)$.
b) Find the tangent line to the curve

$$x^2 + y^2 - x^2y^2 = -23$$

at the point $(x, y) = (3, 2)$.

2 a) Find an equation of the tangent plane to the surface $z = 3(x - 1)^2 + 2(y + 3)^2 + 7$ at the point $(2, -2, 12)$.
b) Find an equation of the tangent plane to the surface

$$z = \log(x - 2y)$$

(with $\log = \ln$ as natural log as usual) at the point $(3, 1, 0)$.

3 a) Find an equation of the tangent plane to the parametric surface

$$\vec{r}(u, v) = \langle u^2, v^2, uv \rangle$$

at the point $(u, v) = (1, 1)$.
b) The surface satisfies the equation $xy - z^2 = 0$. Find the tangent plane to this surface at the same point $(x, y, z) = (1, 1, 1)$ by computing the gradient.

4 Find an equation of the tangent plane and the normal line to the surface $x - z - 4\arctan(yz) = 0$ through the point $(1 + \pi, 1, 1)$.

5 a) Show that the ellipsoid $6x^2 + 4y^2 + 2z^2 = 18$ and the sphere $x^2 + y^2 + z^2 - 8x - 6y - 8z + 24 = 0$ are tangent to each other at the point $(1, 1, 2)$, meaning that they have the same tangent plane at that point.
b) Find a surface different from a plane for which $x + y + 2z = 4$ is the tangent plane at the point $(1, 1, 1)$.

Main definitions

The **gradient** in two dimensions is $\nabla f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle$. In three dimensions, it is $\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z)\rangle$.

From the chain rule, we can deduce:

Theorem: Gradients are orthogonal to level curves and level surfaces.

The tangent line through (x_0, y_0) to a level curve $f(x, y) = c$ is $ax + by = d$, where $\nabla f(x_0, y_0) = \langle a, b\rangle$ and d is obtained by plugging in the point. The tangent plane through (x_0, y_0, z_0) to a level surface $f(x, y, z) = C$ is $ax + by + cz = d$, where $\nabla f(x_0, y_0, z_0) = \langle a, b, c\rangle$ and d is obtained by plugging in the point.

For parametrized surfaces $\vec{r}(u, v)$, the tangent plane is computed using the vectors $\vec{r}_u, \vec{r}_v$ are velocity vectors of grid curves and so tangent to the surface. The normal is $\vec{n} = \vec{r}_u \times \vec{r}_v = \langle a, b, c\rangle$ and then get $ax + by + cz = d$, where d is obtained by plugging in the point $\vec{r}(u_0, v_0)$.