Homework 17: Lagrange multipliers

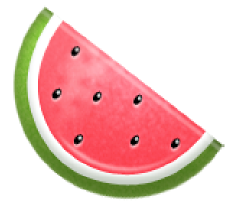
This homework is due Friday, 10/21 rsp Tuesday 10/25.

1 We look at a melon shaped candy. The outer radius is x, the inner is y. Assume we want to maximize the **sweetness function**

$$f(x, y) = x^2 - 2y^2$$

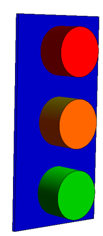
under the constraint that

$$g(x, y) = x - y = 2 \ .$$



Since this problem is so tasty, we require you to use the most yummy method known to mankind: the **Lagrange** method!

2 The material to build a traffic light is $g(x, y) = 6 + 6\pi xy + 3\pi x^2 = 12$ is fixed (the radius of each cylinder is x and the height is y and the constant 6 is the material for the back plate). We want to build a light for which the shaded region with volume $f(x, y) = 3\pi x^2 y$ is maximal. Use the Lagrange method.



3 The situation is the same, but we have two Lagrange multipliers (see box to the right). Use Lagrange multipliers to find the maximum and minimum f under the two constraints:

$$f(x, y, z) = 3x - y - 3z;$$

$$g(x, y, z) = x + y - z = 0$$

$$h(x, y, z) = x^2 + 2z^2 = 1 \ .$$

4 a) Use Lagrange multipliers to prove that the triangle with maximum area that has a given perimeter 2 is equilateral.
b) What are the minima? Why does the Lagrange method not establish them?
Hint: Use Heron's formula for the area:

$$A = \sqrt{s(s-x)(s-y)(s-z)}$$

where $s = 1$ and x, y, z are the lengths of the sides.

5 Which pyramid of height h over a square $[-a, a] \times [-a, a]$ with surface area is $4a\sqrt{h^2 + a^2} + 4a^2 = 4$ has maximal volume $V(h, a) = 4ha^2/3$? By using new variables (x, y) and multiplying V with a constant, we get to the equivalent problem to maximize $f(x, y) = yx^2$ over the constraint $g(x, y) = x\sqrt{y^2 + x^2} + x^2 = 1$. Use the later variables.

Main definitions

The system of equations $\nabla f(x, y) = \lambda \nabla g(x, y), g(x, y) = 0$ for the three unknowns x, y, λ are called **Lagrange equations**. λ is a **Lagrange multiplier**. The **two constraint case** appears only here in homework and is not covered in section $\nabla f(x, y, z) = \lambda \nabla g(x, y, z), g(x, y, z) = 0$ are the **Lagrange equations** $\nabla f(x, y, z) = \lambda \nabla g(x, y, z) + \mu \nabla h(x, y, z)$, $g(x, y, z) = 0$, $h(x, y, z) = 0$ are the **Lagrange equations** with two constraints.

Lagrange theorem: Maxima or minima of f on the constraint $g = c$ are either solutions of the Lagrange equations or critical points of g.