Homework 18: Global extrema

This homework is due Monday, 10/24 rsp Tuesday 10/25.

1 a) We suppose that the Cobb Douglas production formula $Q(L,K) = L^{1/4}K^{3/4} = 100$, which tells that the quantity Q is constant. What values of L and K minimizes the cost function $C(L,K) = 4L + 5K$ under the constraint $Q(L,K) = 100$?
b) Is there a global maximum or minimum for $C(L,K)$ on the region $L \geq 0, K \geq 0$ without the constraint $Q = 100$? If yes, what is the maximum, or what is the minimum?

2 Find the extreme values of f on the region described by the inequality. $f(x,y) = 2x^2 + 3y^2 - 4x - 5, \quad x^2 + y^2 \leq 16$.

3 Find the absolute maximum and minimum values of

$$f(x,y) = e^{-x^2-y^2}(x^2 + 2y^2) \quad ,$$

on the disk $D = \{x^2 + y^2 \leq 4\}$.

4 a) Use Lagrange multipliers to find the maximum and minimum values of f subject to the given constraint

$$f(x,y) = \frac{1}{x} + \frac{1}{y}; \; g(x,y) = \frac{1}{x^2} + \frac{1}{y^2} = 1 \; .$$

b) Is there a global maximum or global minimum of f on $g = 1$? Is this a case for the Bolzano theorem?

5 A package in the shape of a rectangular box can be mailed by the **US Postal Service** if the sum of its length and girth (the perimeter of a cross-section perpendicular to the length) is at most 108. Find the dimensions of the package with largest volume $V(x,y,z) = xyz$ that can be mailed under the constraint $x + 2y + 2z \leq 108$.

Main definitions:

Standard assumption is still that all functions have continuous first and second derivatives. A point (x_0, y_0) is an **absolute maximum** = **global maximum** on a domain R, if $f(x, y) \leq f(x_0, y_0)$ for all (x, y) in R.

To find a global maximum, we look at the local maxima and minima as well as the maxima and minima on the boundary. The later is a Lagrange problem. If the domain is unbounded, we also have to look at the behavior of the function when $x, y \to \infty$.

Example.
$f(x, y) = x^2 + y^2 - x^4 - y^4$ has a local minimum at $(0, 0)$ but this is not a global minimum because $f(1000, 1000)$ for example is smaller than $f(0, 0) = 0$.
If $f(x, y)$ is considered on the domain $R = \{x^2 + y^2 \leq 1\}$ then the situation has changed and we need to look at extrema on the boundary too and $(0, 0), (\pm 1, 0), (0, \pm 1)$ are all global minima.

Bolzano theorem: A curve C or region R is bounded if there is r such that R is contained in a disc of radius r. A curve or region is closed if it contains all boundary points. A continuous function bounded and closed curve or bounded and closed region always has a global maximum and a global minimum on R. (You know this already in single variable with a continuous function on a closed interval).