# Homework 2: Vectors and Dot product

This homework is due Monday, 9/12 respectively Tuesday 9/13 at the beginning of class.

1 A kid drags a sled with a force $\vec{F} = \langle 3, 1, 4 \rangle$. The sled moves with velocity $\vec{v} = \langle 4, -1, 1 \rangle$. The dot product of $\vec{F}$ with $\vec{v}$ is called **power**.
a) Find the angle between the force and the velocity.
b) Find the **vector projection** of the force onto the velocity vector.

2 Light shines long the vector $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and reflects at the three coordinate planes where the angle of incidence equals the angle of reflection. Verify that the reflected ray is $-\vec{a}$.
**Hint**. Reflect first at the $xy$-plane and find the components of the ray after reflection. You can assume that in that case, the reflected vector is in the plane spanned by $\vec{k} = \langle 0, 0, 1 \rangle$ and $\vec{a}$.

3 Given the vectors $\vec{a} = \langle 1, 2, 1 \rangle, \vec{b} = \langle 1, -1, 1 \rangle, \vec{c} = \langle 0, 1, 1 \rangle, \vec{d} = \langle 2, 3, 4 \rangle$, compute all possible dot products and determine which pairs are perpendicular.

4 a) Find the angle between a diagonal of a cube and the diagonal in one of its faces.
b) The hypercube or tesseract has vertices $(\pm 1, \pm 1, \pm 1, \pm 1)$. Find the angle between the hyper diagonal connecting $(1, 1, 1, 1)$ with $(-1, -1, -1, -1)$ and the space diagonal connecting $(1, 1, 1, 1)$ with $(-1, -1, -1, 1)$.

5 a) Verify that if $\vec{a}, \vec{b}$ are nonzero, then $\vec{c} = |\vec{a}|\vec{b} + |\vec{b}|\vec{a}$ bisects the angle between $\vec{a}, \vec{b}$ if $\vec{c}$ is not zero.
b) Verify the parallelogram law $|\vec{a}+\vec{b}|^2 + |\vec{a}-\vec{b}|^2 = 2|\vec{a}|^2 + 2|\vec{b}|^2$.

## Main definitions

Two points $P = (a, b, c)$ and $Q = (x, y, z)$ in space define a **vector** $\vec{v} = \langle x - a, y - b, z - c \rangle$. As it connects $P$ with $Q$, we also write $\vec{v} = \vec{PQ}$. The real numbers $v_1, v_2, v_3$ in $\vec{v} = \langle v_1, v_2, v_3 \rangle$ are called the **components** of $\vec{v}$. The **length** $|\vec{v}|$ of a vector $\vec{v} = \vec{PQ}$ is defined as the distance $d(P, Q)$ from $P$ to $Q$. A vector of length 1 is called a **unit vector**. The **addition** of two vectors is $\vec{u} + \vec{v} = \langle u_1, u_2, u_3 \rangle + \langle v_1, v_2, v_3 \rangle = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$. The **scalar multiple** $\lambda \vec{u} = \lambda \langle u_1, u_2, u_3 \rangle = \langle \lambda u_1, \lambda u_2, \lambda u_3 \rangle$. The difference $\vec{u} - \vec{v}$ can best be seen as the addition of $\vec{u}$ and $(-1) \cdot \vec{v}$.

The **dot product** of two vectors $\vec{v} = \langle a, b, c \rangle$ and $\vec{w} = \langle p, q, r \rangle$ is defined as $\vec{v} \cdot \vec{w} = ap + bq + cr$. The **Cauchy-Schwarz inequality** tells $|\vec{v} \cdot \vec{w}| \leq |\vec{v}||\vec{w}|$.

The **angle** between two nonzero vectors is defined as the unique $\alpha \in [0, \pi]$ satisfying $\vec{v} \cdot \vec{w} = |\vec{v}| \cdot |\vec{w}| \cos(\alpha)$. Two vectors are called **orthogonal** or **perpendicular** if $\vec{v} \cdot \vec{w} = 0$. The zero vector $\vec{0}$ is orthogonal to any vector. For example, $\vec{v} = \langle 2, 3 \rangle$ is orthogonal to $\vec{w} = \langle -3, 2 \rangle$. The vector $\mathrm{P}(\vec{v}) = \frac{\vec{v} \cdot \vec{w}}{|\vec{w}|^2} \vec{w}$ is called the **projection** of $\vec{v}$ onto $\vec{w}$. The **scalar projection** $\frac{\vec{v} \cdot \vec{w}}{|\vec{w}|}$ is plus or minus the length of the projection of $\vec{v}$ onto $\vec{w}$. The vector $\vec{b} = \vec{v} - P(\vec{v})$ is a vector orthogonal to $\vec{w}$. **Pythagoras tells:** if $\vec{v}$ and $\vec{w}$ are orthogonal, then $|v - w|^2 = |v|^2 + |w|^2$.