Homework 23: Cylindrical/Spherical integration

This homework is due Monday, 11/7 rsp Tuesday 11/8.

1 Evaluate the following integrals.
a) $\int_0^{\pi} \int_0^2 \int_0^{9-r^2} r\,dz\,dr\,d\theta$
b) $\int_0^{2\pi} \int_{\pi/2}^{\pi} \int_1^2 \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta$

2 Use cylindrical coordinates to find the volume of the solid that lies within both the cylinder $x^2+y^2=1$ and the sphere $x^2+y^2+z^2=4$.

3 Use spherical coordinates to evaluate

$$\int\int\int_H (16 - x^2 - y^2)\,dV \ ,$$

where H is the solid hemisphere $x^2+y^2+z^2 \leq 16$, $z \geq 0$.

4 Evaluate the integral by changing to spherical coordinates.

$$\int_{-a}^{a} \int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} \int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} (x^2z + y^2z + z^3)\,dz\,dx\,dy$$

5 Show that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sqrt{x^2+y^2+z^2}\, e^{-(x^2+y^2+z^2)}\,dx\,dy\,dz = 2\pi \ .$$

Main definitions

The integration factor in cylindrical coordinates $(x, y, z) = (r\cos(\theta), r\sin(\theta), z)$ is r as in polar coordinates.

The integration factor in spherical coordinates $(x, y, z) = (\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi))$ is $\rho^2\sin(\phi)$. This was the surface area element $|\vec{r}_\phi \times \vec{r}_\theta|$

To evaluate an integral in spherical coordinates, we express the region in spherical coordinates, substitute $x = \rho\sin(\phi)\cos(\theta)$, $y = \rho\sin(\phi)\sin(\theta)$ and $z = \rho\cos(\phi))$ in the function and include the integration factor $\rho^2\sin(\phi)$.