Homework 25: Line integrals

This homework is due Friday, 11/11 rsp Tuesday 11/15.

1 a) Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$ if $\vec{F}(x,y,z) = \langle x+y, y-z, z^2+1\rangle$ and $\vec{r}(t) = \langle t^2, t^3, t^2\rangle$ with $0 \leq t \leq 2$.
b) Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$ if $\vec{F}(x,y,z) = \langle 5z, 5y, -5x\rangle$ and $\vec{r}(t) = \langle 2t, \sin(t), \cos(t)\rangle$, $0 \leq t \leq \pi$.

2 An electric current I produces a magnetic field $\vec{B}$ whose flow lines are circles circling the wire. Let $C : \langle r\cos(t), r\sin(t), 0\rangle$. Ampèrs law is $\int_C \vec{B} \cdot d\vec{r} = \mu_0 I$, where μ_0 is a constant called permeability. Show that the magnitude $B(r) = |\vec{B}|$ of the magnetic field at a distance r from the center of the wire is $B = \frac{\mu_0 I}{2\pi r}$. Note that B is a scalar function and $\vec{B}$ is a vector field. Use first the information provided to find the vector field $\vec{B}$.

3 Determine from each of the following cases, whether $\vec{F}$ is conservative or not. If it is, find a function f such that $\vec{F} = \nabla f$.

a) $\vec{F}(x,y) = \langle 7e^x \sin(y), 7e^x \cos(y)\rangle$
b) $\vec{F}(x,y) = \langle y + 4x^3 + y^6, -x + 6x^4y^5\rangle$
c) $\vec{F}(x,y,z) = \langle x+y, y+x, z^5 - \sin(z)\rangle$
d) $\vec{F}(x,y,z) = \langle x^5, z^5, y\rangle$

4 Evaluate the line integral $\int_C \langle 1 - ye^{-x}, e^{-x}\rangle \cdot d\vec{r}$, where C is the path $\vec{r(t)} = \langle t, 1 + t + \sin(\sin(t))\rangle$ and t is from 0 to π. You probably will have difficulty. A future "you" (who has a time machine) tells you that you can compute the integral also in a different way: find a function f which is a potential to the vector field, then evaluate $f(\vec{r}(\pi)) - f(\vec{r}(0))$. You can use this without justification for now. We will learn about this "warp" feature in the next lecture.

5 The topological notions appearing in this problem are cool but not very essential for the course. Determine whether or not the given set is (a) open, (b) connected, and (c) simply-connected.
a) $\{(x, y) \,|\, 0 < y < 3\}$, b) $\{(x, y) \,|\, 1 < |x| < 2\}$
c) $\{(x, y) \,|\, 1 \leq x^2 + y^2 \leq 4, y \geq 0\}$ d) $\{(x, y) \,|\, (x, y) \neq (1, 2)\}$
e) $\{(x, y, z) \,|\, (x, y, z) \neq (1, 2, 3)\}$

Main definitions

If $\vec{F}$ is a vector field and $C : t \mapsto \vec{r}(t)$ is a curve defined on the interval $[a, b]$ then $\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\; dt$ is called the **line integral** of $\vec{F}$ along the curve C.

A vector field is **conservative** in a region R if the line integral from A to B is path independent. It has the **closed loop property** if the line integral along any closed loop is zero. It is **irrotational** if $\text{curl}(F) = Q_x - P_y$ is zero everywhere in R.

A subset G of the plane is **open** if every point (x, y) in G is contained in a small disc centered at (x, y) which is also in G. (We want openness so that curves do not hit a boundary makeing us to worry about differentiability there). A subset is **connected**, if one can connect any two points in G with a curve inside G. A subset G is **simply connected** if it is connected and every closed curve in G can be pulled together to a point within G.

Clairaut test: Zero curl is necessary for a gradient field.