Homework 26: Theorem of line integrals

This homework is due Monday, 11/14 rsp Tuesday 11/15.

1 a) Find a function f such that $\vec{F} = \nabla f$ if $\vec{F}(x,y) = \langle x^2+2xy^2+y, y^2+2x^2y+x\rangle$.
b) Use a) to evaluate $\int_C \vec{F} \cdot d\vec{r}$ along the parabola $y = 2x^2$ from $(-1,2)$ to $(2,8)$.

2 a) Find a function f such that $\vec{F} = \nabla f$ if $\vec{F}(x,y) = \langle 3y^2/(1+x^2), 6y\arctan(x)\rangle$.
b) Use a) to evaluate $\int_C \vec{F} \cdot d\vec{r}$ along the curve $\vec{r}(t) = \langle t^2, 2t\rangle$ with $0 \leq t \leq 1$.

3 On August 1, 2016, Lukas Irmler walked over a rope over the **Rheinfalls** in Switzerland.
There is a force field $\vec{F}$ present which consists part of the gravitational force and part by the wind forces: $\vec{F}(x,y,z) = \langle \sin(x), \cos(y), -10+z\rangle$. The path is given by $\vec{r}(t) = \langle 5t, t, 30 - \sin(t)/10\rangle$, where $0 \leq t \leq \pi$. Compute the work $\int_0^\pi \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$ done by Lukas during this stunt.

4 a) Verify that if $\vec{F} = \langle P, Q, R\rangle$ is conservative, then

$$P_y = Q_x, P_z = R_x, Q_z = R_y \, .$$

b) Is $\langle x^5 y, xy^2, zx\rangle$ conservative? If yes, find f such that $\vec{F} = \nabla f$, if not, give a reason.

5 a) Show that the line integral

$$\int_C \vec{F} \cdot d\vec{r}$$

with $F(x,y,z) = \langle y, x, xyz \rangle$ is not conservative by using problem 4).
b) Find two different curves from $(0,0,0)$ to $(1,1,0)$ for which the line integral is different.

Main points

This theorem is the first generalization of the fundamental theorem of calculus to higher dimensions. It tells that the work done along a path is the potential energy difference.

Fundamental theorem of line integrals: If $\vec{F} = \nabla f$, then

$$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt = f(\vec{r}(b)) - f(\vec{r}(a)) \, .$$

This theorem can be used to dramatically simplify the computation of a line integral. Just find the potential f and evaluate the difference of potential values.

Recall that a region R is called **simply connected** if every closed loop in R can be pulled together to a point within R.

The three concepts "gradient field", "closed loop property" and "conservative" are the same:
Gradient field $\leftrightarrow$ Conservative $\leftrightarrow$ Closed loop property
In simply connected open regions, these three properties are all equivalent to being irrotational $\mathrm{curl}(\vec{F}) = Q_x - P_y = 0$.