## Homework 27: Greens theorem

This homework is due Wednesday, 11/16 rsp Thursday 11/17.

**1** Evaluate the line integral

$$\int_C \vec{F} \cdot \vec{dr} \ ,$$

where $\vec{F}(x,y) = \langle 16y + 3x^{15} - 1, \cos(9y)y^{300} \rangle$ and $C$ consists of the line segments from $(0,1)$ to $(0,0)$ and from (0,0) to (1,0) and the curve $y = \sqrt{1 - x^2}$ from $(1,0)$ to $(0,1)$.

**2** a) Find the line integral $\int_C \vec{F} \cdot \vec{dr}$ with

$$\vec{F}(x,y) = \langle y^2 \cos(x) + \sin(\sin(x)), 15x + x^2 + 2y\sin(x) + \sin(\sin(y)) \rangle$$

where $C$ is the triangular path from (0,0) to (2,6) to (2,0) to (0,0). Watch the orientation of the curve!
b) Evaluate $\int_C \vec{F} \cdot \vec{dr}$

$$\vec{F}(x,y) = \langle 8y - \ln(x^2 + y^2), 2\arctan(y/x) \rangle$$

$C$ is the circle $(x-6)^2 + (y-7)^2 = 25$ oriented counterclockwise.

**3** A classical problem asks to compute the area of the region bounded by the **hypocycloid**

$$\vec{r}(t) = \langle 4\cos^3(t), 4\sin^3(t) \rangle, 0 \leq t \leq 2\pi \ .$$

We can not do that directly. Guess which theorem to use, then use it!

**4** Calculate $\int_C \vec{F} \cdot \vec{dr}$, where $\vec{F}(x,y) = \langle x^2 + y, 3x - y^2 \rangle$ and $C$ is the ellipse $x^2/100 + y^2/16 = 1$ oriented clockwise.

**5** Use Green's Theorem to evaluate

$$\int_C \langle \sin(\sqrt{1+x^3}), 7x \rangle \, d\vec{r} \,,$$

where $C$ is the boundary of the region $K(4)$. You see in the picture $K(0), K(1), K(2), K(3), K(4)$. The first $K(0)$ is an equilateral triangle of length 1. The second $K(1)$ is $K(0)$ with 3 equilateral triangles of length $1/3$ added. $K(2)$ is $K(1)$ with $3 * 4^1$ equilateral triangles of length $1/9$ added. $K(3)$ is $K(2)$ with $3 * 4^2$ of length $1/27$ added and $K(4)$ is $K(3)$ with $3*4^3$ triangles of length $1/81$ added. Remark. We could now find the line integral in the limit $K = K(\infty)$, a **fractal** called the **Koch snowflake** It has dimension $\log(4)/\log(3) = 1.26...$ which is between 1 and 2.



## Main points

The **curl** of a vector field $\vec{F}(x, y) = \langle P(x,y), Q(x,y) \rangle$ is the scalar field

$$\text{curl}(F)(x, y) = Q_x(x, y) - P_y(x, y) \,.$$

**Green's theorem**: If $\vec{F}(x, y) = \langle P(x,y), Q(x,y) \rangle$ is a vector field and $G$ is a region for which the boundary $C$ is parametrized so that $R$ is "to the left", then

$$\int_C \vec{F} \cdot d\vec{r} = \iint_G \text{curl}(F) \, dxdy \,.$$