

Homework 28: Curl and Div

This homework is due Friday, 11/18 resp Tuesday 11/22.

- 1 Find a) the curl and b) the divergence of the vector field

$$\vec{F}(x, y, z) = \langle \sin(yz), \sin(zx), \sin(xy) \rangle .$$

c) Finally find $\text{div}(\text{curl}(\vec{F}))$.

- 2 Let f be a scalar field and $\vec{F}$ a vector field in space. Determine which expression is meaningful. If not, explain why. If so, state whether it is a scalar field or a vector field. In all problems, we deal with functions and vector fields in 3D space.

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| a) $\text{curl}(f)$ | b) $\text{grad}(f)$ |
| c) $\text{div}(\vec{F})$ | d) $\text{curl}(\text{grad}(f))$ |
| e) $\text{grad}(\vec{F})$ | f) $\text{grad}(\text{div}(\vec{F}))$ |
| g) $\text{div}(\text{grad}(f))$ | h) $\text{grad}(\text{div}(f))$ |
| i) $\text{curl}(\text{curl}(\vec{F}))$ | j) $\text{div}(\text{div}(\vec{F}))$ |
| k) $\text{grad}(f) \times \text{div}(\vec{F})$ | l) $\text{div}(\text{curl}(\text{grad } f))$ |

- 3 a) Is there a vector field $\vec{G}(x, y, z)$ such that

$$\text{curl}(\vec{G}) = \langle 5, 2, 3 \rangle$$

If yes, find one.

- b) Is there a vector field $\vec{G}(x, y, z)$ such that

$$\text{curl}(\vec{G}) = \langle xyz, -y^2z, yz^2 \rangle ?$$

If yes, find one.

- c) Assume $\vec{F}$ is a gradient field. Does this imply that there is a vector field $\vec{G}$ such that $\text{curl}(\vec{G}) = \vec{F}$? If yes, show it. If no, find a counter example.

- 4 a) Verify that any vector field of the form

$$\vec{F}(x, y, z) = \langle f(x), g(y), h(z) \rangle$$

is irrotational.

- b) Verify that any vector field of the form

$$\vec{F}(x, y, z) = \langle f(y, z), g(x, z), h(x, y) \rangle$$

is incompressible.

- c) Find a nonzero vector field $\vec{F}$ such that $\text{curl}(\vec{F}) = \langle 0, 0, 0 \rangle$.

- 5 a) Prove the identity $\text{div}(\nabla f \times \nabla g) = 0$.

- b) Show that the scalar function $f(x, y, z) = 3 \sin(z) + z^5 + y + x$ is the divergence of some vector field $\vec{F}$.

Main points

The **curl** of a vector field $\vec{F} = \langle P, Q, R \rangle$ is the vector field $\text{curl}(P, Q, R) = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$.

The curl measure rotation of a field. If $\vec{F}$ has zero curl everywhere it is **irrotational**. Remember, the curl of $\vec{F} = \langle P, Q \rangle$ is a scalar.

The **divergence** of a vector field $\vec{F}(x, y, z) = \langle P, Q, R \rangle$ is

$$\text{div}(\vec{F})(x, y, z) = P_x(x, y, z) + Q_y(x, y, z) + R_z(x, y, z).$$

$\text{div}(\vec{F})$ measures the expansion of a field. Zero divergence everywhere is called **incompressible**.

$\text{div}(\text{curl}(\vec{F})) = 0$ for all vector fields $\vec{F}$. $\text{curl}(\nabla f) = 0$ for all functions f .