Homework 29: Flux integral, Stokes I

This homework is due Monday, 11/21 rsp Tuesday 11/22 just before Thanksgiving. If no orientation is given for a surface, the orientation is assumed to be "outwards". There is one problem which gives a first exposure to this important theorem. The Monday/Tuesday lecture before thanksgiving will cover the theorem again.

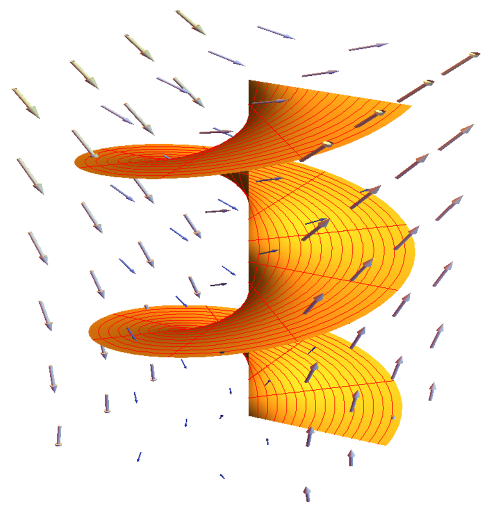
1 Without using Stokes, evaluate the flux integral $\int\int_S \vec{F} \cdot d\vec{S}$ if

$$\vec{F}(x, y, z) = \langle 3z, 3y, 3x \rangle \ ,$$

and S is the helicoid

$$\vec{r}(u, v) = \langle u \cos v, u \sin v, v \rangle, 0 \leq u \leq 1, 0 \leq v \leq 4\pi$$

which has an upward orientation.



2 Evaluate the flux integral $\int\int_S \vec{F} \cdot d\vec{S}$ for the vector field

$$\vec{F}(x, y, z) = \langle x, y, 5 \rangle \ ,$$

where S is the boundary of the solid region enclosed by the cylinder $x^2 + z^2 = 1$ and the planes $y = 0$ and $y = 7$ (there are three surfaces). We will learn later how to compute this faster

3 The temperature $f(x, y, z)$ at a point (x, y, z) is equal to the distance from the origin $(0, 0, 0)$. Find the flux of the heat flow field $\vec{F} = -\nabla f$ across a sphere S of radius 2 centered at $(0, 0, 0)$.

4 Let $\vec{F}(x,y,z)$ be an inverse square field, that is

$$\vec{F}(x,y,z) = c\langle x,y,z\rangle/\rho^3$$

with $\rho = \sqrt{x^2+y^2+z^2}$. Show that the flux of $\vec{F}$ across a sphere S with center at the origin and radius R is independent of the radius of the sphere.

5 Use Stokes theorem to evaluate the flux integral $\int\int_S \text{curl}(\vec{F}) \cdot d\vec{S}$ for the vector field

$$\vec{F}(x,y,z) = \langle (x^5+y^5)z, x, y\rangle \ ,$$

where S is the surface $x^2 + y^2/4 + z^8 = 25, z \geq 0$, oriented upwards. Stokes theorem expresses this as a line integral along the boundary curve $\vec{r}(t) = \langle 5\cos(t), 10\sin(t), 0\rangle$, $0 \leq t \leq 2\pi$.

Main points

If a surface S is parametrized as $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$ over a domain G in the uv-plane and $\vec{F}$ is a vector field, then the **flux integral** of $\vec{F}$ through S is

$$\int\int_G \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\ dudv \ .$$

If $d\vec{S} = (\vec{r}_u \times \vec{r}_v)\ dudv$ represents an infinitesimal normal vector to the surface, this can be written as $\int\int_S \vec{F} \cdot d\vec{S}$. The interpretation is that if $\vec{F} =$ fluid velocity field, then $\int\int_S \vec{F} \cdot d\vec{S}$ is the amount of fluid passing through S in unit time.

Stokes theorem tells that if S be a surface bounded by a curve C and $\vec{F}$ be a vector field, then

$$\int\int_S \text{curl}(\vec{F})\ \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r} \ .$$