

## Homework 6: Arc length and curvature

This homework is due Wednesday, 9/21 resp Thursday 9/22.

- 1 a) Find the arc length of the curve  $\vec{r}(t) = \langle \cos(t^3), \sin(t^3), t^3 \rangle$  where  $-1 \leq t \leq 1$ .  
b) Find the arc length of the curve  $\vec{r}(t) = \langle t^3, 24t, 6t^2 \rangle$ , where  $t \in [0, 10]$ .

- 2 Arc length can be defined in any dimensions. A curve in 4 dimension is parametrized as  $\vec{r}(t) = \langle x_1(t), x_2(t), x_3(t), x_4(t) \rangle$ . Find the arc length of the curve

$$\vec{r}(t) = \langle t, \log(t), 1/t, \log(t) \rangle ,$$

where  $\log(t) = \ln(t)$  is the natural log and  $1 \leq t \leq 4$ .

- 3 a) Use a calculator, Mathematica or Wolfram alpha to evaluate the arc length of the curve  $\vec{r}(t) = \langle t^{10}, t^5, t^{15} \rangle$  from  $t = 0$  to  $t = 2$ .  
b) Do the same with  $\vec{r}(t) = \langle t^2, t, t^3 \rangle$  from  $t = 0$  to  $t = 32$ . Compare with the result in a) and explain in a few words why you got the same result.

- 4 a) Use the formula

$$\kappa(t) = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}$$

to compute the curvature  $\kappa(t)$  of  $\vec{r}(t) = \langle t^3, t^2, t \rangle$  at  $t = 1$ .

- b) Find the curvature at  $t = 1$  for the curve  $\vec{r}(t) = 3\langle t^3, t^2, t \rangle$ .

**Hint.** There is an easy way to see the answer.

- 5 Find the parameter  $c$  such that the parabola  $y = cx^2$  has curvature 40 at the origin.

## Main definitions

If  $t \in [a, b] \mapsto \vec{r}(t)$  is a curve with velocity  $\vec{r}'(t)$  and speed  $|\vec{r}'(t)|$ , then

$$L = \int_a^b |\vec{r}'(t)| dt$$

is called the **arc length of the curve**. Written out in coordinates,  $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ , we have

$$L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} dt .$$

For curves in two dimensions, where  $\vec{r}(t) = \langle x(t), y(t) \rangle$  has two coordinates only, we have  $L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2} dt$ .

If  $\vec{r}(t)$  is a curve which has nonzero speed at  $t$ , then we can define  $\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$ , the **unit tangent vector**,  $\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}$ , the **normal vector** and  $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$  the **bi-normal vector**.

The **curvature** of a curve at the point  $\vec{r}(t)$  is defined as

$$\kappa(t) = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|} .$$

The curvature of a circle of radius  $r$  is equal to  $1/r$  at every point of the circle. The curvature is zero for a line.

A useful formula for curvature is

$$\kappa(t) = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3} .$$