

Homework 9: Functions and Continuity

This homework is due Friday, 9/30 rsp Tuesday 10/4.

- 1 Determine which f is continuous in the plane. No reasoning is required. Each question one point. $\log = \ln$ denotes the natural log.
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|-------------------------------------|-----------------------------------|
| a) $f(x, y) = y(x^2 - 9)/(x - 3)$. | f) $f(x, y) = \log(\exp(x + y))$ |
| b) $f(x, y) = \log(xy)$ | g) $f(x, y) = y \exp(1/x)$ |
| c) $f(x, y) = 1/\log(\sin(xy)/2)$ | h) $f(x, y) = \sin(ye^{\cos(x)})$ |
| d) $f(x, y) = 1/\cos(x + y)$ | i) $f(x, y) = y \exp(x)$ |
| e) $f(x, y) = 1/\log(2 + x + y)$ | j) $f(x, y) = \exp(\log x + y)$ |

- 2 Investigate whether the following functions are continuous at $(0, 0)$. This means: investigate whether a function value at $(0, 0)$ which extends the definition so that the extended function is continuous? If the limit $(x, y) \rightarrow (0, 0)$ exists, tell what the limit is.

a) $f(x, y) = x^2/(x^3 + y^3)$. b) $f(x, y) = xy^2/(x^2 + y^2)$. It can be

helpful to use polar coordinates. You have to give some reasoning here.

- 3 Find the limit $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$ if it exists or show that the limit does not exist

a) $f(x, y) = \frac{6x^4y}{2x^5+y^5}$ b) $f(x, y) = \frac{x^6-y^6}{(x^2+y^2)^2}$

Also here is some reasoning required.

- 4 Determine the set of points where the following function is continuous $f(x, y) = \frac{e^x + e^y}{e^{xy} - 1}$.

- 5 Find the limit $(x, y) \rightarrow (0, 0)$ of the function $f(x, y) = \sin(x^2 + y^2) \log(x^2 + y^2)$, where $\log(x) = \ln(x)$.

Main definition

A function $f(x, y)$ with domain R is **continuous at** $(a, b) \in R$ if $f(x, y) \rightarrow f(a, b)$ for all choices $(x, y) \rightarrow (a, b)$. We even say is **continuous at a point** (a, b) **not in the domain** if there exists a finite value $f(a, b)$ such that $f(x, y) \rightarrow f(a, b)$ whenever $(x, y) \rightarrow (a, b)$ for $(x, y) \in R$. For example, the function $f(x, y) = y(x^2 - 1)/(x - 1)$ continuous everywhere even so $x = 1$ is not in the domain. We can fill in the value $f(1, y) = 2y$ as the function is equivalent to $y(x + 1)$, its analytic continuation. Also $f(x, y) = \sin(x^2 + y^2)/(x^2 + y^2)$ is continuous as l'Hopital for polar coordinates shows: with filling in the whole $f(0, 0) = 1$, it becomes continuous even at the point $(0, 0)$. In one dimensions, there are **jump discontinuities** like $f(x) = \text{sign}(x)$ or **poles** like $f(x) = 1/x$ or **oscillations** like $f(x) = \sin(1/x)$. These three prototypes can occur together in examples like $1/\sin(1/x)$ or $\arctan(1/\sin(1/x))$. Many questions about continuity in two dimensions are answered when writing the function in polar coordinates $x = r \cos(\theta)$, $y = r \sin(\theta)$ near the point in question. For $f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$ for example, the function becomes (just fill in $x = r \cos(\theta)$, $y = r \sin(\theta)$), in polar coordinates $f(r, \theta) = \cos(2\theta)$. The value of the function depends only on the angle. Arbitrarily close to $(0, 0)$, the function takes any values between -1 and 1 . The function is not continuous because no value can be found at $(0, 0)$ such that f can be continuously extended to it.