PARTIAL DIFFERENTIAL EQUATIONS , Math 21a

FUNCTIONS OF TWO VARIABLES. We consider functions $f(x,t)$ in two variables. Viewing the variable t as time, we can look at the function $x \mapsto f(x,t)$ of one variable evolving in time. The describing equation is a **partial differential equation** (PDE). It is a differential equation which involves the derivatives with respect to both space x and time t. The function $f(x,t)$ could denote the **temperature of a stick** or the **height of a water wave** at position x and time t.

PARTIAL DERIVATIVES. We write $f_x(x,t)$ and $f_t(x,t)$ for the **partial derivatives** with respect to x or t. The notation $f_{xx}(x,t)$ means that we differentiate twice with respect to x.

Example: for $f(x,t) = \cos(x+4t^2)$, we have
- $f_x(x,t) = -\sin(x+4t^2)$
- $f_t(x,t) = -8t\sin(x+4t^2)$.
- $f_{xx}(x,t) = -\cos(x+4t^2)$.

One also uses the notation $\frac{\partial f(x,y)}{\partial x}$ for the partial derivative with respect to x. Tired of all the "partial derivative signs", we always write $f_x(x,y)$ or $f_t(x,y)$ in this handout. This is an official abbreviation in the scientific literature.

PARTIAL DIFFERENTIAL EQUATIONS. A partial differential equation is an equation for an unknown function $f(x,t)$ in which at least two different partial derivatives occur.

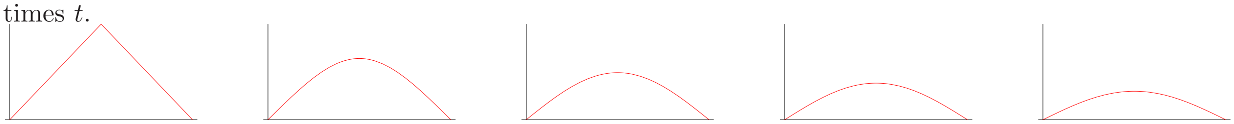
- $f_t(x,t) + f_x(x,t) = 0$ with $f(x,0) = \sin(x)$ has a solution $f(x,t) = \sin(x-t)$.
- $f_t(x,t) = f(x,t)$ has the solution $f(x,0)e^t$. The equation is **not** a PDE. Why not?
- $f_{tt}(x,t) - f_{xx}(x,t) = 0$ has a solution $f(x,t) = \sin(x-t) + \sin(x+t)$. Check it!

EXAMPLE: THE HEAT EQUATION. The temperature distribution $f(x,t)$ in a metal wire satisfies the **heat equation**

$$f_t(x,t) = \mu f_{xx}(x,t)$$

This PDE tells that the rate of change of the temperature at the point x is proportional to the second space derivative of $f(x,t)$ at x. A function $f(x) = f(x,0)$ defines an initial temperature distribution. The constant μ depends on the **heat conductivity** of the material. Metals for example conduct heat well and have a large μ.

VISUALIZATION. We can plot the graph of the function $f(x,t)$ or plot the temperature distribution for different times t.

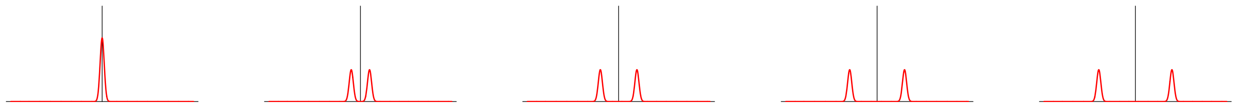


$f(x,0)$ $f(x,1)$ $f(x,2)$ $f(x,3)$ $f(x,4)$

EXAMPLE: THE WAVE EQUATION. The height of a wave $f(x,t)$ at time t and at position x satisfies the **wave equation**

$$f_{tt}(x,t) = c^2 f_{xx}(x,t),$$

VISUALIZATION. We can plot the wave height $f(x,t)$ as a function of x for different but fixed times t.



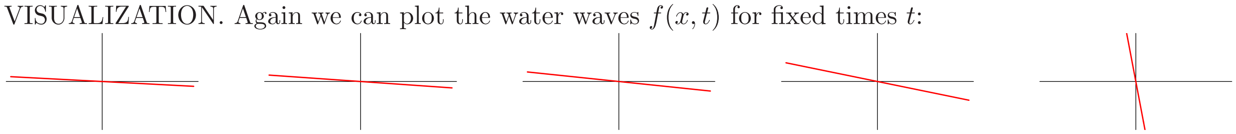
$f(x,0)$ $f(x,1)$ $f(x,2)$ $f(x,3)$ $f(x,4)$

EXAMPLE: THE BURGERS EQUATION. If waves approach the shore, the dynamics changes: low amplitude waves slow down and high altitude waves move faster. Additionally, waves start to dissipate and lose energy. A model is the **Burgers equation**

$$f_t(x,t) + f(x,t)f_x(x,t) = \mu f_{xx}(x,t),$$

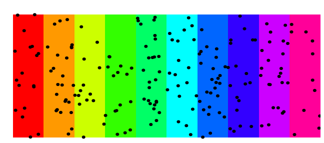
This partial differential equation can have **shocks**: the waves break. You see that at the beach. With positive μ, one can give explicit traveling waves $f(t,x) = (1 + e^{(2x-t)/(4\mu)})^{-1}$. Waves $f(t,x) = \frac{x}{t-1}\frac{\sqrt{\frac{1}{1-t}}e^{-x^2/(4\nu(t-1))}}{1+\sqrt{\frac{1}{1-t}}e^{-x^2/(4\nu(t-1))}}$ become discontinuous at $t = 1$.

VISUALIZATION. Again we can plot the water waves $f(x,t)$ for fixed times t:

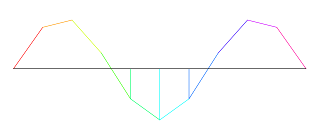


$f(x,0)$ $f(x,25)$ $f(x,5)$ $f(x,75)$ $f(x,0.99)$

TO THE DERIVATION OF THE HEAT EQUATION. The temperature $f(x,t)$ is proportional to the kinetic energy at the position x. Divide the stick into n adjacent cells and assume that in each time step, a fraction of the particles moves randomly to the right or to the left. If $f_k(t)$ is the **energy** of particles in cell k at time t, then the energy of particles at time $t+1$ is proportional the sum of $f_{k+1}(t) - f_k(t)$ and $f_{k-1}(t) - f_k(t)$ which is $(f_{k-1}(t) - 2f_k(t) + f_{k+1})(t))$. This is a discrete version of the second derivative because $dx^2 f_{xx}(x,t) \sim (f(x+dx,t) - 2f(x,t) + f(x-dx,t))$.



TO THE DERIVATION OF THE WAVE EQUATION. A wave can be modeled by n particles linked by springs. Assume that the water particles move up and down only. If $f_i(t)$ is the **height** of the particles, then the right particle pulls with a force $f_{i+1} - f_i$, the left particle with a force $f_{i-1} - f_i$. Again, $(f_{i-1}(t) - 2f_i(t) + f_{i+1})(t))$ is a discrete version of the second derivative f_{xx}. By Newtons law, the acceleration $f_{tt}(t,x)$ at position x is proportional to f_{xx}.



TO THE DERIVATION OF BURGERS EQUATION. Assume that $\mu = 0$ for a moment. If the wave f has height close to c, we see that $f_t(x,t) + cf_x(x,t) = 0$ which has the solution $f(x,t) = f(x-ct,0)$. The waves travel forward with a speed which depends on the height of the wave. Higher waves travel faster. The additional term μf_{xx} plays the same role as in the heat equation: the potential energy, which is proportional to the