

Name:

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- Start by printing your name in the above box and please **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work.** Except for problems 1-3, we need to see **details** of your computation.
- All functions can be differentiated arbitrarily often unless otherwise specified.
- No notes, books, slide rules, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes to complete your work.

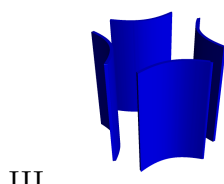
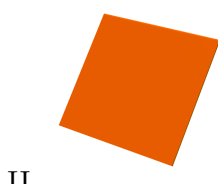
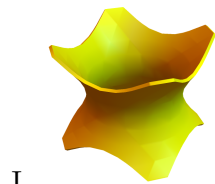
1		20
2		10
3		10
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6		10
7		10
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9		10
10		10
Total:		110

- 1) ☐ T ☐ F If $|\vec{r}'(t)| = 1$ for all t , then the curvature satisfies $\kappa(t) = |\vec{r}''(t)|$.
- 2) ☐ T ☐ F Let $\vec{r}(t)$ be a parametric curve. Suppose that at the point $\vec{r}(t)$ the unit tangent vector is $\langle 0, 1, 0 \rangle$ and the binormal vector is $\langle 0, 0, 1 \rangle$. Then the unit normal vector at the point is $\langle 1, 0, 0 \rangle$.
- 3) ☐ T ☐ F If $|\vec{v} + \vec{w}|^2 = 2\vec{v} \cdot \vec{w}$, then $\vec{v} = 2\vec{w}$.
- 4) ☐ T ☐ F The distance between two points P and Q is smaller than or equal to $|\vec{OP}| + |\vec{OQ}|$, where $O = (0, 0, 0)$ is the origin.
- 5) ☐ T ☐ F The line $\vec{r}(t) = \langle 5 + 2t, 2 + t, 3 + t \rangle$ is located on the plane $x - y - z = 0$.
- 6) ☐ T ☐ F The arc length of a circle with constant curvature 20 is $\pi/10$.
- 7) ☐ T ☐ F The surface $x^2 - y^2 + 4y = z^2 + 2z$ is an elliptic paraboloid.
- 8) ☐ T ☐ F For any two vectors, $|\vec{v} \cdot \vec{w}| \leq |\vec{v}| + |\vec{w}|$.
- 9) ☐ T ☐ F The acceleration of $\vec{r}(t) = \langle t, t, t \rangle$ is $\langle 0, 0, 0 \rangle$ everywhere.
- 10) ☐ T ☐ F If $\vec{v} = \vec{PQ} = \langle 2, 1, 1 \rangle$ then $|\vec{v}|$ is larger than the distance between P and Q .
- 11) ☐ T ☐ F If $\vec{v} \cdot \vec{w} > 0$, then the angle between $\vec{v}$ and $\vec{w}$ is larger than 90° .
- 12) ☐ T ☐ F If $\langle 5, 6, 4 \rangle \times \vec{x} = \vec{x}$ then $\vec{x} = \langle 0, 0, 0 \rangle$.
- 13) ☐ T ☐ F The curvature of $\vec{r}(t) = \langle t - 1, 1 - t, t \rangle$ is 0 at the point $t = 1$.
- 14) ☐ T ☐ F The line $\vec{r}(t) = t\langle 7, 7, 1 \rangle$ hits the plane $-x - y = 7z$ in a right angle.
- 15) ☐ T ☐ F The surface given in spherical coordinates as $\sin(\phi) = 1/\rho$ is a cylinder.
- 16) ☐ T ☐ F Given three vectors $\vec{u}, \vec{v}$ and $\vec{w}$, then $|(\vec{u} \cdot \vec{v})(\vec{w} \cdot \vec{u})| \leq |\vec{u}|^2 |\vec{v}| |\vec{w}|$.
- 17) ☐ T ☐ F The surface given in spherical coordinates as $\rho^2 = 1$ is a sphere.
- 18) ☐ T ☐ F The arc length of the curve $\langle 5 \sin(t), 1, 5 \cos(t) \rangle$ from $t = 0$ to $t = 2\pi$ is equal to 10π .
- 19) ☐ T ☐ F The surface parametrized by $\vec{r}(u, v) = \langle u^5 - v^5, u^5 + v^5, u^5 \rangle$ is a plane.
- 20) ☐ T ☐ F If the non-zero cross product of a vector $\vec{v}$ with a vector $\vec{w}$ is parallel to $\vec{v}$, then the dot product between $\vec{v}$ and $\vec{w}$ is zero.

Total

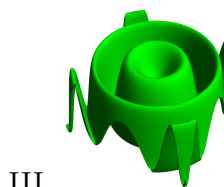
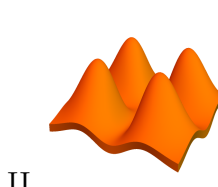
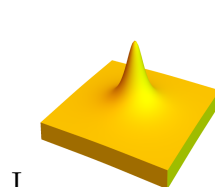
Problem 2) (10 points) No justifications are needed in this problem.

a) (2 points) Match the contour surfaces $g(x, y, z) = 1$. Enter O, if there is no match.



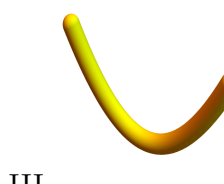
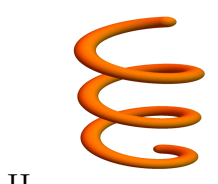
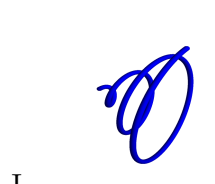
Function $g(x, y, z) = 1$	Enter O,I,II or III
$x^2 + y^2 - z^2$	
$x^2 y^2$	
$x - y$	
$x^2 + z^2$	

b) (2 points) Match the graphs of the functions $f(x, y)$. Enter O, if there is no match.



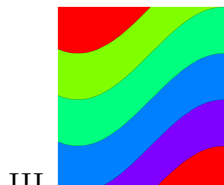
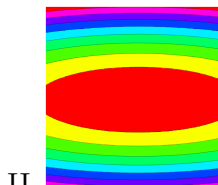
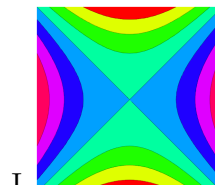
Function $f(x, y) =$	Enter O,I,II or III
$1/(1 + x^2 + y^2)$	
$x^2 y^2 e^{-x^2 - y^2}$	
$\sin(x^2 + y^2)$	
$\cos(y)$	

c) (2 points) Match the space curves with their parametrizations $\vec{r}(t)$. Enter O, if there is no match.



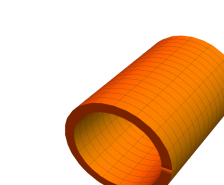
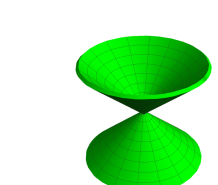
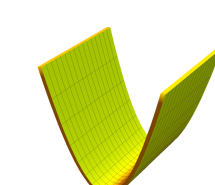
Parametrization $\vec{r}(t) =$	Enter O, I,II or III
$\langle t, t \sin(3t), t \cos(3t) \rangle$	
$\langle \cos(t), \sin(t), \cos(t) \rangle$	
$\langle \sin(t), \cos(t), t \rangle$	
$\langle t, t, t^2 \rangle$	

d) (2 points) Match functions g with contour plots in the xy-plane. Enter O, if there is no match.



Equation	Enter O, I,II or III
$\sin(x) - y$	
$x^2 - y^2$	
$10x^2 + y^2$	
$x^2 + 10y^2$	

e) (2 points) Match the quadrics. Enter O if no match.



Quadric	Enter O,I,II or III
$x^2 = z$	
$x^2 + y^2 = 1$	
$xy = 1$	
$y = 1$	
$x^2 + y^2 = z^2$	

a) (4 points) Mark what applies for any two vectors $\vec{v}$ and $\vec{w}$ in space.

Object	always 0	can be $\neq 0$	always $\vec{0} = \langle 0, 0, 0 \rangle$	can be nonzero vector
$(\vec{v} \times \vec{w}) \times \vec{v}$				
$(\vec{v} \times \vec{w}) \cdot \vec{v}$				
$(\text{proj}_{\vec{v}} \vec{w}) \times \vec{w}$				
$(\text{proj}_{\vec{w}} \vec{v}) \cdot \vec{w}$				

b) (3 points) **Conic sections** (parabolas, ellipses or hyperbolas) can be seen as intersections of a two dimensional cone

$$x^2 + y^2 = z^2$$

with a 2D plane. Identify the quadrics in the following three cases:

Intersect with plane	Enter A-D
$z = 1$	
$z = \sqrt{2}x$	
$x = y + 1$	

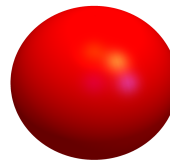


c) (3 points) Three dimensional cone is given by the equation

$$x^2 + y^2 + z^2 = w^2$$

in four dimensional space. If we intersect it with a three dimensional space, we get quadrics. We want you to identify a few quadrics. In the pictures the quadrics might be turned or scaled. You get a point for every right answer, meaning that you can miss one and still have full credit.

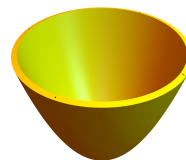
A



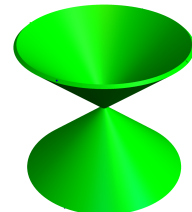
B



C



D



Intersect with the 3D plane	Enter A-D or O
$w = 0$	
$w = z + 1$	
$y = z$	
$w = 1$	

We are going into the **furniture design business**. Our first task is to construct a chair. Find the distance between the lines spanned by the parallel arm rests AB and CD , where

$$A = (1, 0, 2), B = (4, 1, 2)$$

and

$$C = (3, 3, 3), D = (6, 4, 3) .$$



Problem 5) (10 points)

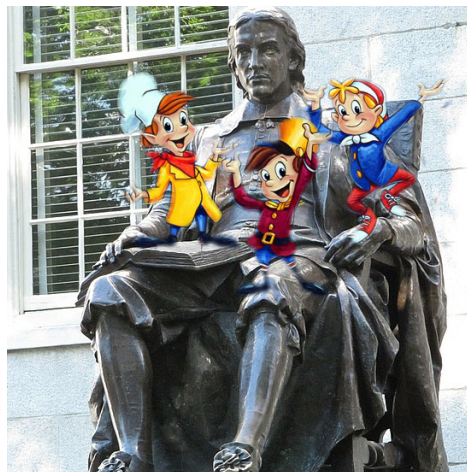
a) (6 points) The first derivative of parametrized curve is called “**velocity**”. You have also learned the terms “**acceleration**” and maybe “**jerk**” for the second and third derivative. Less well known are “**snap**”, “**crackle**”, “**pop**” for the fourth, fifth and sixth derivatives. Since we could not yet find the seventh derivative named, let’s call it the “**Harvard**”. Compute the “**Harvard**” of the curve

$$\vec{r}(t) = \langle \cos(2t) + t, \sin(2t), t^2 \rangle$$

at time $t = 0$.

b) (4 points) The parametrization $\vec{v}(t) = \vec{r}'(t)$ defines a new curve. It is located on a surface. Which of the following surface is it?

	Check one
cylinder	☐ A
cone	☐ B
plane	☐ C
ellipsoid	☐ D
paraboloid	☐ E



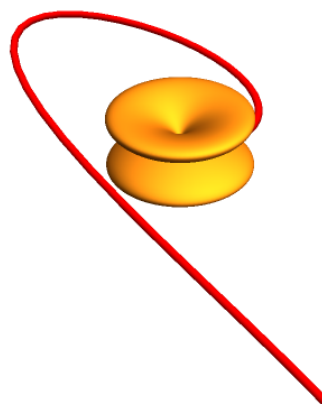
Problem 6) (10 points)

A kid plays with a **Yo-Yo**. It is accelerated periodically with $\vec{r}''(t) = \langle \sin(t), 0, \cos(t) - 10 \rangle$. Find the position of the Yo-Yo at time $t = 2\pi$ if the initial position is

$$\vec{r}(0) = \langle 5, 5, 0 \rangle$$

and the initial velocity is

$$\vec{r}'(0) = \langle 1, 1, 1 \rangle .$$



Problem 7) (10 points)

a) (5 points) Compute the arc length of the curve

$$\vec{r}(t) = \langle t, t^2, 2t^3/3 \rangle$$

if $0 \leq t \leq 4$.

b) (5 points) What is the curvature

$$\kappa(t) = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}$$

of this curve at $t = 0$?

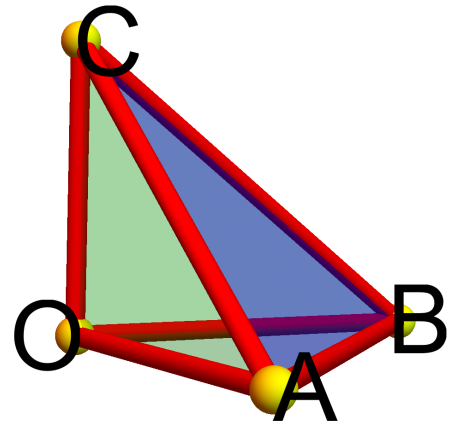
THE CURVATURE
OF A CURVE TELLS
HOW MUCH
THE CURVE
IS CURVED!

Problem 8) (10 points)

The triangle ABC is obtained by slicing a corner ABC off from a cube. One obtains a so called **tri-rectangular tetrahedron**.

a) (5 points) Find the square of the area of the triangle ABC , where $A = (3, 0, 0)$, $B = (0, 6, 0)$, $C = (0, 0, 8)$.

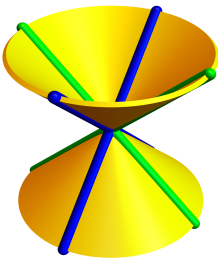
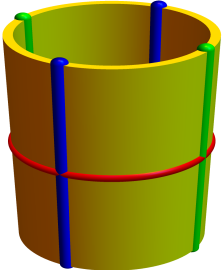
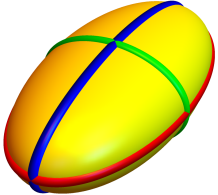
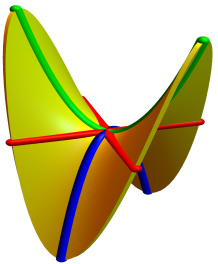
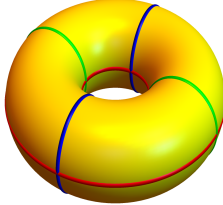
b) (5 points) Compute also the sum of the squares of the areas of the triangles OAB , OBC and OCA , where $O = (0, 0, 0)$ is the origin. The sum should get the same value you got in a).



P.S. the same computation can be repeated for arbitrary points $A = (a, 0, 0)$, $B = (0, b, 0)$, $C = (0, 0, c)$. It proves a not so well known theorem telling that the sum of the squares of the side wall areas is the square of the face area. It is a 3 dimensional version of Pythagoras and also goes by the name **de Gua theorem** or **Faulhaber theorem**.

Problem 9) (10 points)

The 3D printing venture "Math-Candy" (math-candy.com) asks you to do some product development. In each of the 5 following parametrizations, two entries are still missing, each entry being worth one candy (1 point).

a)		The surface $x^2 + y^2 = z^2$ is parametrized by $\vec{r}(\theta, z) = \langle \dots, \dots, z \rangle$
b)		The surface $x^2 + y^2 = 1$ is parametrized by $\vec{r}(\theta, z) = \langle \dots, \dots, z \rangle$
c)		The surface $2(x - 1)^2 + (y - 5)^2 + 4z^2 = 1$ is parametrized by $\vec{r}(\theta, \phi) = \langle \dots, \dots, \cos(\phi)/2 \rangle$
d)		The surface $x^2 - y^2 = z$ is parametrized by $\vec{r}(x, y) = \langle \dots, \dots, x^2 - y^2 \rangle$
e)		The surface $(\sqrt{x^2 + y^2} - 2)^2 + z^2 = 1$ is parametrized by $\vec{r}(\theta, \phi) = \langle (2 + \cos(\phi)) \cos(\theta), \dots, \dots \rangle$

A **pinball machine** is tilted in such a way that a ball in the xy -plane experiences a constant force $\vec{F} = \langle 0, -2 \rangle$. A ball of mass 1 is hit the left flipper at the point $\vec{r}(0) = \langle -1, 0 \rangle$ with velocity $\vec{r}'(0) = \langle 1/2, 5 \rangle$.

- (4 points) Find the velocity $\vec{r}'(t)$.
- (4 points) What trajectory $\vec{r}(t) = \langle x(t), y(t) \rangle$ does the ball follow?
- (2 points) As the ball hits the line $y = 0$, is it reachable by the player? In other words, does it hit $y = 0$ within the interval $x \in [1, 2]$?

