11/1/2017 Second Hourly Practice 2 **Math 21a, Fall 2017**

Name:

MWF 9 Jameel Al-Aidroos
MWF 9 Dennis Tseng
MWF 10 Yu-Wei Fan
MWF 10 Koji Shimizu
MWF 11 Oliver Knill
MWF 11 Chenglong Yu
MWF 12 Stepan Paul
TTH 10 Matt Demers
TTH 10 Jun-Hou Fung
TTH 10 Peter Smillie
TTH 11:30 Aukosh Jagannath
TTH 11:30 Sebastian Vasey

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3,we need to see **details** of your computation.
- All functions can be differentiated arbitrarily often unless otherwise specified.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

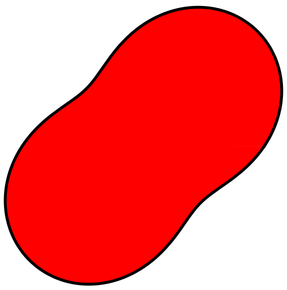
1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

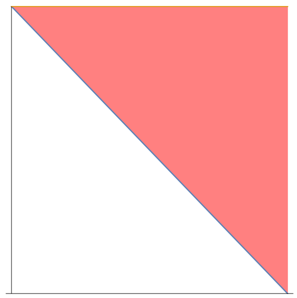
Problem 1) True/False questions (20 points), no justifications needed

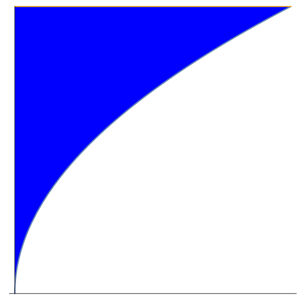
1)	T	F	The point $(-5, 3)$ is a critical point of $f(x, y) = 3x^2 + 5y^2$.
2)	T	F	If a function $f(x, y, z)$ has gradient satisfying $\lvert\nabla f\rvert = 1$ everywhere, then the level surface $f(x, y, z) = 1$ is a sphere.
3)	T	F	The chain rule assures that the vector $\nabla f(\vec{r}(t))$ and the velocity vector $\vec{r}\,'(t)$ for any curve $\vec{r}(t)$ on the level surface are perpendicular.
4)	T	F	The function $D(x, y) = f_{xx}f_{yy} - f_{xy}^2$ satisfies the formula $D_{\vec{d}}D = \nabla D \cdot \vec{d}$ for any unit vector $\vec{d}$.
5)	T	F	The function $f(x, y) = x^4 - 1$ has infinitely many critical points.
6)	T	F	The points $(0, 1)$ and $(0, -1)$ are maxima of $f(x, y) = y^2$ under the constraint $g(x, y) = x^2 + y^2 = 1$.
7)	T	F	The function $f(x, y) = y^2$ satisfies the partial differential equation $u_{yx}(x, y) = u_x(x, y)$.
8)	T	F	Let $f(x, y) = x^3y$. At every point (x, y) there is a direction $\vec{v}$ for which $D_{\vec{v}}f(x, y) = 0$.
9)	T	F	If $f_{xy} = f_{yx}$ then $f(x, y) = xy$.
10)	T	F	$g(x, y) = \int_0^x \int_0^y f(s, t) \;\; dtds$ satisfies the partial differential equation $g_{xy}(x, y) = f(x, y)$.
11)	T	F	If $f(x, y) = g(x, y) = x^2 + y^4$, then the Lagrange problem for maximizing f under the constraint $g(x, y) = 1$ has infinitely many solutions.
12)	T	F	The number $\lvert\nabla f(0, 0)\rvert$ is the maximal directional derivative $\lvert D_{\vec{v}}f(0, 0)\rvert$ among all unit vectors $\vec{v}$.
13)	T	F	Any continuous function $f(x, y)$ takes a global maximum as well as a global minimum on the region $0 \leq x^2 + y^2 \leq 1$.
14)	T	F	For any continuous function, $\int_0^1 \int_0^1 f(r, \theta)r \; drd\theta = \int_0^1 \int_0^1 f(x, y) \; dxdy$.
15)	T	F	If the Lagrange multiplier λ at a solution to a Lagrange problem is positive then this point is a minimum.
16)	T	F	The equation $f_{xy}f_{xx}f_{yy} = 1$ is an example of a partial differential equation.
17)	T	F	If the discriminant D appearing in the second derivative test of $f(x, y)$ is positive at $(0, 0)$ then $\lvert\nabla f(0, 0)\rvert > 0$.
18)	T	F	If $f(x, y)$ is a continuous function then $\int_7^9 \int_5^7 f(x, y) \;\; dxdy = \int_5^7 \int_7^9 f(x, y) \; dxdy$.
19)	T	F	If f has the critical point $(0, 0)$, then $f_y + f_x$ has the critical point $(0, 0)$.
20)	T	F	If $f(x, y)$ takes arbitrary large values, then $g(x, y) = \lvert\nabla f(x, y)\rvert$ takes arbitrary large values.

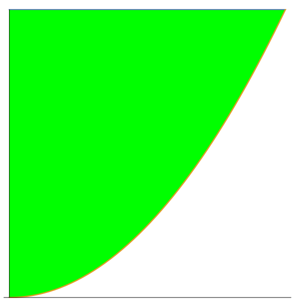
Problem 2) (10 points) No justifications needed

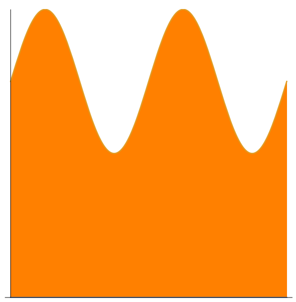
a) (6 points) Match the regions with the integrals. Each integral matches one region $A - F$.

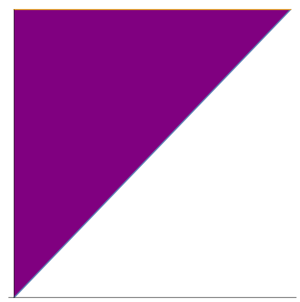

A


B


C


D


E


F

Enter A-F	Integral
	$\int_0^{2\pi} \int_0^{y} f(x,y)\ dxdy$
	$\int_0^{2\pi} \int_0^{\sqrt{y}} f(x,y)\ dxdy$
	$\int_0^{2\pi} \int_{2\pi-x}^{2\pi} f(x,y)\ dydx$
	$\int_0^{2\pi} \int_0^{y^2} f(x,y)\ dxdy$
	$\int_0^{2\pi} \int_0^{3+\sin(2x)} f(x,y)\ dydx$
	$\int_0^{2\pi} \int_0^{3+\sin(2t)} f(r,t)\ rdrdt$

b) (4 points) We define the **complexity** of a partial differential equation for $u(t,x)$ or $u(x,y)$ as the number of derivatives appearing in total. For example, the partial differential equation $u_{xxx} = u_{tx}$ has complexity 5 because 5 derivatives have been taken in total. As an expert in PDEs, you know a few of them. Write down the complexities of the partial differential equations. These are integers ≥ 2 in each case.

Complexity	Name
	Laplace for $u(x,y)$
	Wave for $u(t,x)$
	Transport for $u(t,x)$
	Heat for $u(t,x)$


Taylor


Fourier

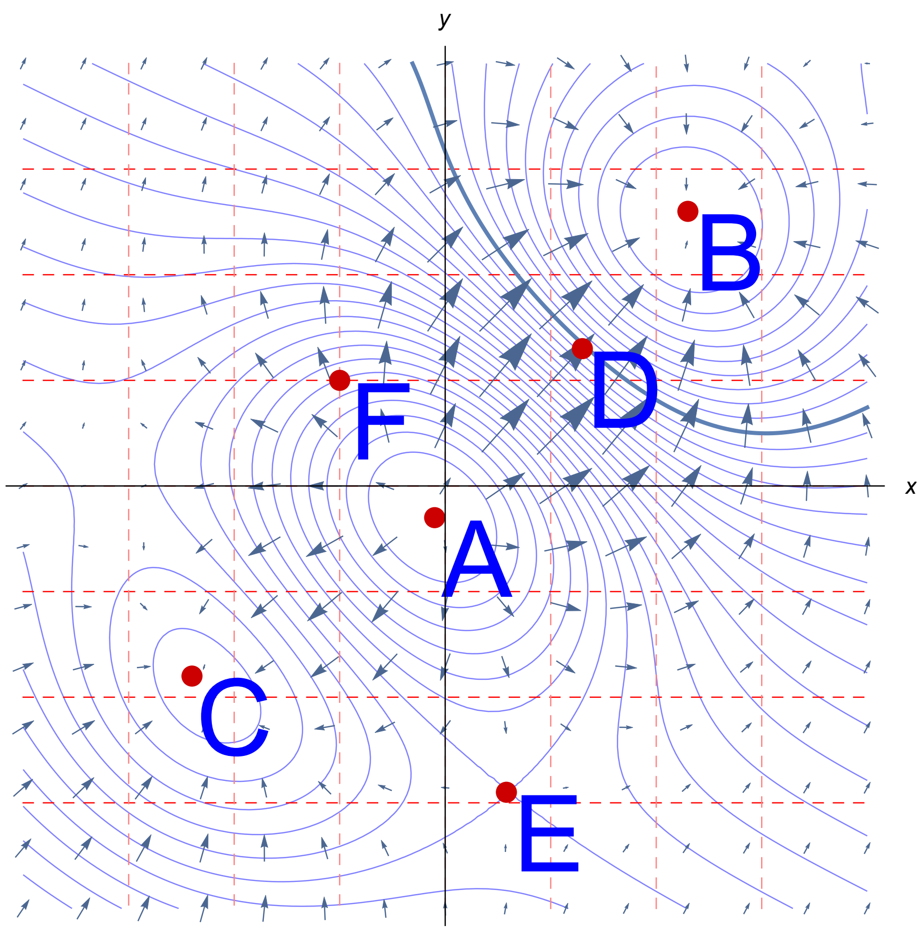

d'Alembert


Laplace

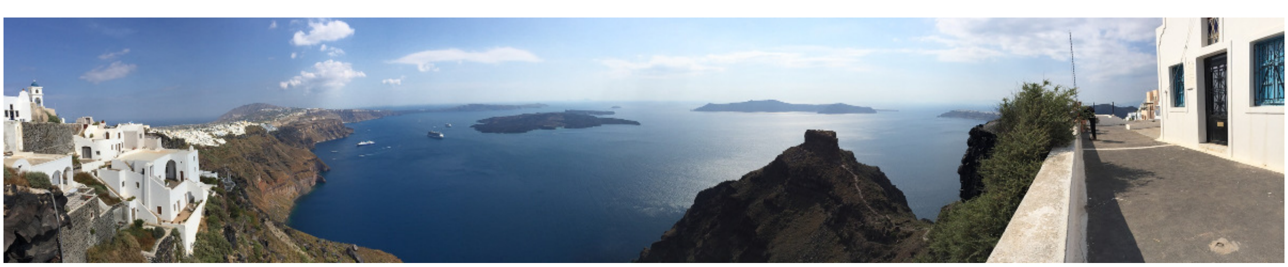
Problem 3) (10 points)

3a) (5 points) In the following contour plot of a height function $f(x,y)$, neighboring contours $f(x,y) = c$ have height distance 1. The arrows indicate the gradient of f. Every point A-F occurs at most once.

Which of the points is the global maximum on the visible region?	
Which of the points is a global minimum on the visible region?	
Which of the points is a global maximum for the function $\lvert\nabla f(x,y)\rvert^2$	
Which of the points is a saddle point?	
Which of the points has the property that $f_x f_y < 0$ at this point?	



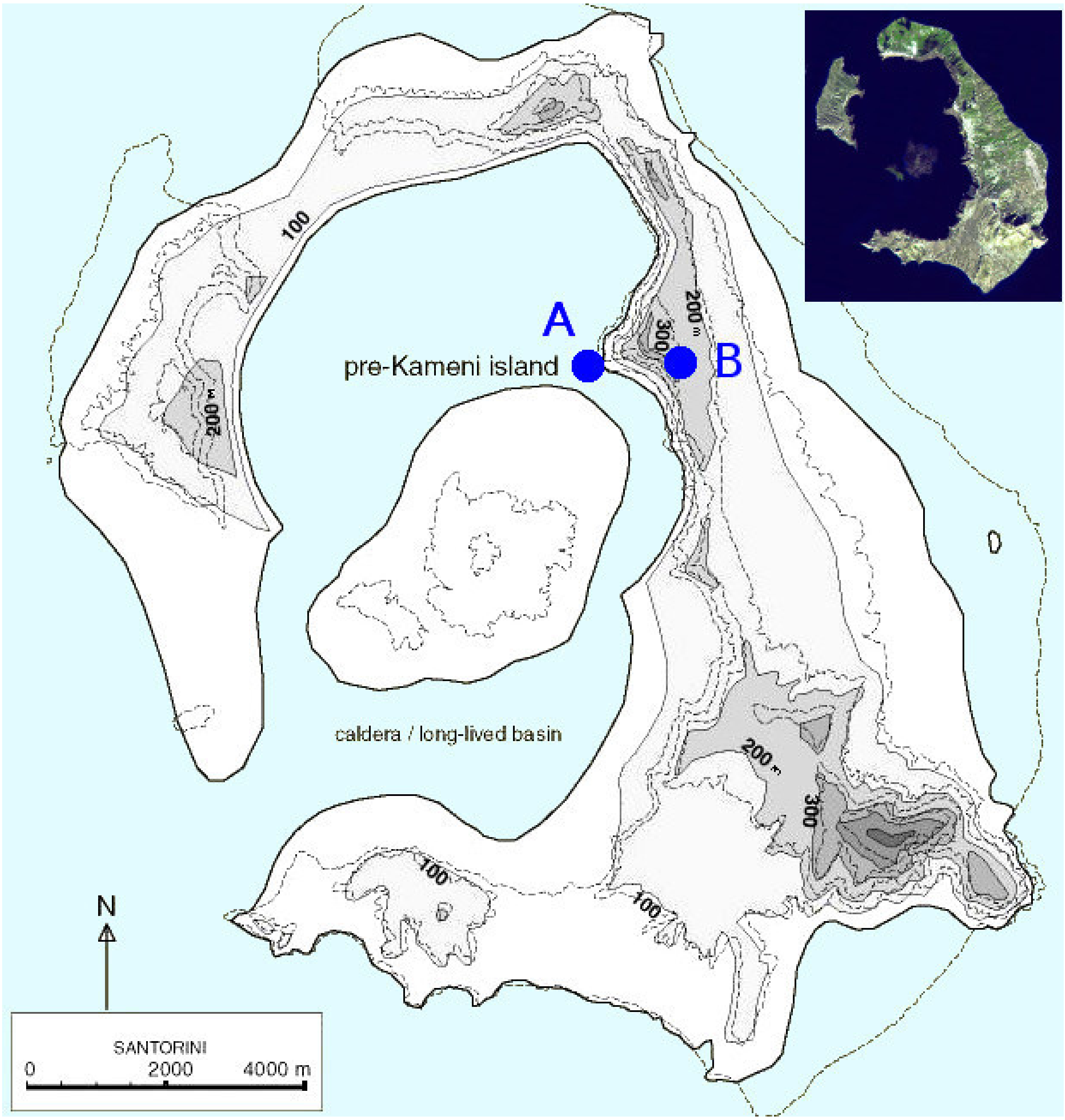
Part b) and c) of the problem are unrelated and on the new page.



Santorini panorama from Imerovigli with view onto Skaros rock, Caldera basin and volcanic island Nea Kameni. Photo: Oliver Knill, June 2015

3b) (2 points) You see a contour map of the Greek island of **Santorini**. Point A is on the water (0 elevation) Point B is **Skaros rock**, which used to be a fortification protecting merchants from pirates. Estimate the average directional derivative between A and B in the direction from A to B. Given elevation markers 100,200,300 are in meters.

Derivative	Check one
2	
0.2	
0.02	



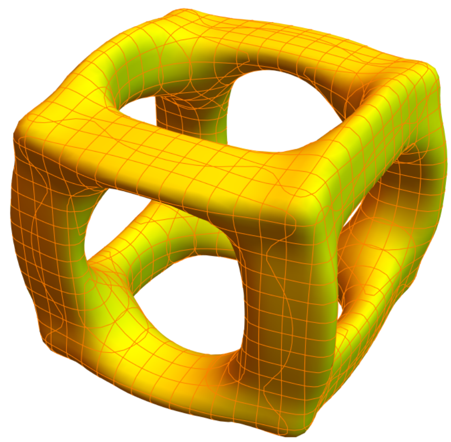
Source: http://www.decadevolcano.net, the picture shows a reconstruction of pre-Minoan Thera done by Druitt and Francaviglia from 1991. The island of today is shown in dottet curves. A satelite picture of the Santorini Caldera with the Nea Kameni volcano in the center is seen in the upper right corner.

3c) (3 points) Which statements about a critical point with discriminant $D \neq 0$ always hold for a smooth function $f(x, y)$?

Critical Point	$f_{xx} > 0$	$f_{yy} < 0$	$f_x > 0$	$f_y < 0$
Maximum				
Minimum				
Saddle point				

Problem 4) (10 points)

a) (6 points) Let $g(x,y) = (6y^2-5)^2(x^2+y^2-1)^2$. Find the gradient of g at the points $(1,-1)$, $(-1,1)$ and $(1,1)$.



b) (4 points) A student from the **Harvard graduate school of design** contemplates the surface

$$f(x,y,z) = g(x,y) + g(y,z) + g(z,x) = 3$$

shown in the picture. She first discovers the formula

$$\begin{aligned} \nabla f(1,-1,1) &= \langle g_x(1,-1) + g_y(1,1) \ , \\ & \quad g_x(-1,1) + g_y(1,-1) \ , \\ & \quad g_x(1,1) + g_y(-1,1) \rangle \ . \end{aligned}$$

Without verifying this, find the tangent plane at $(1,-1,1)$.

Problem 5) (10 points)

Octagons are used in architecture designs, in symbolism, for rugs or in traffic signs. Use the Lagrange method to find the octagon with maximal area

$$f(x,y) = (x+2y)^2 - 2y^2$$

if the circumference

$$g(x,y) = 4x + 4y\sqrt{2} = 8 \ .$$

is fixed.

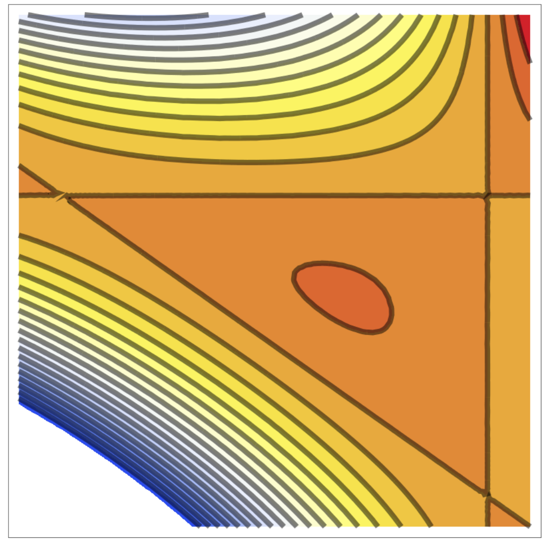
Problem 6) (10 points)

a) (8 points) Find and classify the four critical points of the **"triangle function"**

$$f(x,y) = x^2y + y^2x - y^2 - y$$

using the second derivative test. There is no need to find the values of f.

b) (2 points) State whether any of the four points is a global maximum or minimum on the entire plane.



Problem 7) (10 points)

The region R defined by

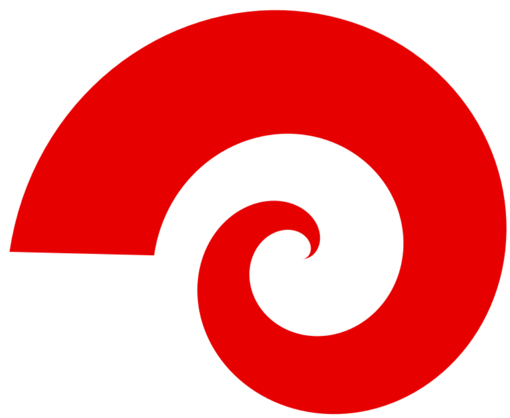
$$\theta \leq r(\theta) \leq 2\theta$$

with

$$0 \leq \theta \leq 3\pi$$

is shown in the picture. Compute its **moment of inertia**

$$\iint_R x^2 + y^2 \; dA \; .$$

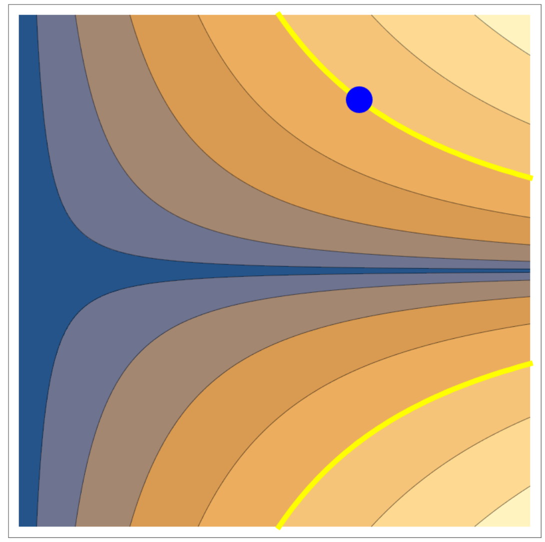


Problem 8) (10 points)

a) (5 points) Find a vector perpendicular to the **tangent line** of the curve

$$f(x,y) = 5(x^3y^2)^{1/5} = 100$$

at $(20, 20)$. The picture shows a contour map of f.



b) (5 points) Use the same function in a) to estimate

$$f(21,19) = 5(21^3 \cdot 19^2)^{1/5}$$

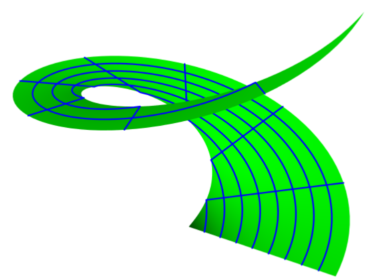
by **linearizing** f near $(20, 20)$.

Problem 9) (10 points)

We compute the **surface area** of the surface

$$\vec{r}(u,v) = \langle v\cos(u), v\sin(u), u \rangle$$

over the region $R : 0 \leq u \leq 2\pi, u \leq v \leq 2\pi$.



a) (5 points) First verify that the integral is of the form

$$\iint_R \sqrt{1+v^2}\; dudv \; .$$

b) (5 points) Now compute the surface area integral.

Problem 9) (10 points)

The **Ramanujan constant** $e^{\pi\sqrt{163}} = 262537412640768743.99999999999925...$ is close to an integer. There is an elaborate story about why this is so. Here, we just want to estimate the logarithm of this constant roughly.

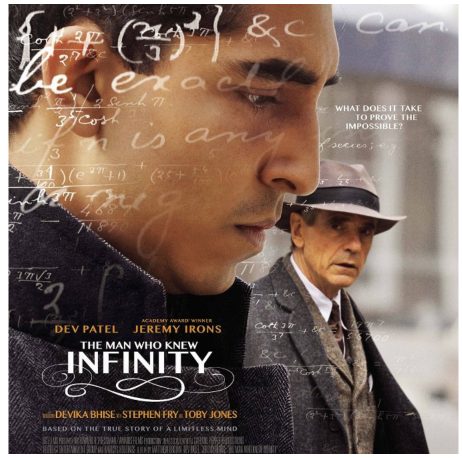
Let

$$f(x,y) = x\sqrt{y}\ .$$

Estimate

$$f(3.141, 163) = 3.141\sqrt{163}$$

near $(x_0, y_0) = (3, 169)$ using linear approximation.



Ramanujan is featured in the movie: "The Man who knew infinity", 2015