11/1/2017 Second Hourly Practice 4 **Math 21a, Fall 2017**

Name:

MWF 9 Jameel Al-Aidroos
MWF 9 Dennis Tseng
MWF 10 Yu-Wei Fan
MWF 10 Koji Shimizu
MWF 11 Oliver Knill
MWF 11 Chenglong Yu
MWF 12 Stepan Paul
TTH 10 Matt Demers
TTH 10 Jun-Hou Fung
TTH 10 Peter Smillie
TTH 11:30 Aukosh Jagannath
TTH 11:30 Sebastian Vasey

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3,8, we need to see **details** of your computation.
- All functions can be differentiated arbitrarily often unless otherwise specified.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

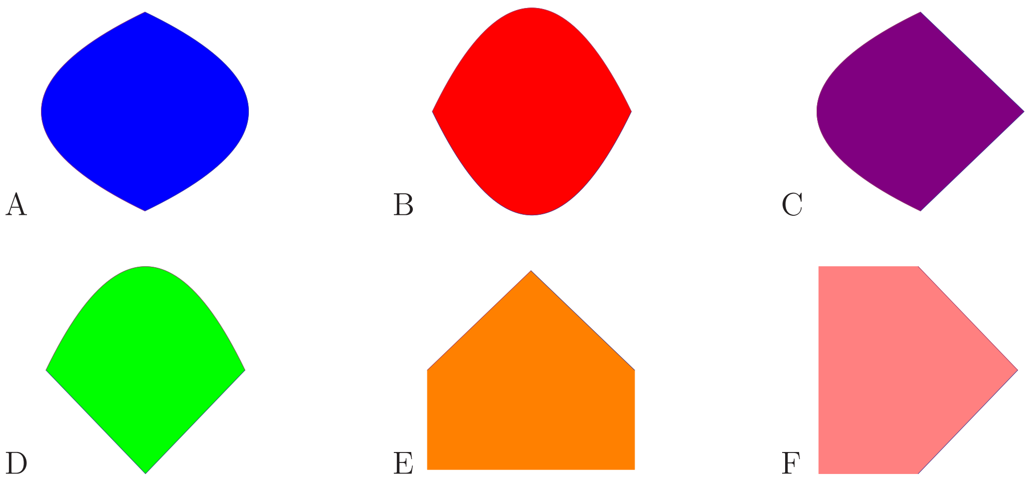
1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) True/False questions (20 points), no justifications needed

1)	T	F	For any continuous function $f(x,y)$, we have $\int_0^1 \int_1^2 f(x,y)\,dx\,dy = \int_1^2 \int_0^1 f(x,y)\,dx\,dy$.
2)	T	F	If $\vec{u}$ is a unit vector tangent to $f(x,y) = 1$ at $(0,0)$ and $f(0,0) = 1$, then $D_{\vec{u}}f(0,0)$ is zero.
3)	T	F	Assume f is zero on $x = y$ and $x = -y$, then $(0,0)$ is a critical point of f.
4)	T	F	If $(0,0)$ is the only local minimum of a function f and f has no local maxima, then $(0,0)$ is a global minimum.
5)	T	F	If $(0,0)$ is a critical point for f, and $f_{yy}(0,0) < 0$ then $(0,0)$ is not a local minimum.
6)	T	F	If $f(x,y)$ and $g(x,y)$ have the same non-constant linearization $L(x,y)$ at $(0,0)$ and $f(0,0) = g(0,0) = 0$, then the level sets $f = 0$ and $g = 0$ have the same tangent line at $(0,0)$.
7)	T	F	There are saddle points with positive discriminant $D > 0$.
8)	T	F	If R is the unit disc, then $\int\int_R x^2 - y^2 \; dxdy$ is zero.
9)	T	F	There is a nonzero function $f(x,y)$ for which the linearization $L(x,y)$ is equal to $2f(x,y)$.
10)	T	F	The directional derivative at a local minimum $(0,0)$ is positive in every direction.
11)	T	F	If $\vec{r}(t)$ is a curve on the surface $g(x,y,z) = 1$, then $\nabla g(\vec{r}(t)) \cdot \vec{r}\,'(t) = 0$.
12)	T	F	If $\|\nabla f(0,0)\| = 2$, there is a direction in which the directional derivative at $(0,0)$ is 2.
13)	T	F	If $D > 0$ at $(0,0)$ and $\nabla f(0,0) = 0$ and $f_{xx}(0,0) < 0$ then $f_{yy}(0,0) < 0$.
14)	T	F	$\int_0^1 \int_0^x f(x,y) \; dydx = \int_0^1 \int_y^1 f(x,y) \; dxdy$.
15)	T	F	The surface area of the sphere of radius L is $\int_0^\pi L^2 \sin(\phi) \; d\phi$.
16)	T	F	If $f(x,y) = g(x)$ is a function of x only, then $D = 0$ at every critical point.
17)	T	F	The gradient vector $\nabla f(x_0, y_0)$ is a vector which is perpendicular to the surface $z = f(x,y)$.
18)	T	F	If $\|\nabla f(0,0)\| = 2$, then there is a unit vector $\vec{v}$ such that $D_{\vec{v}}f(0,0) = 1$.
19)	T	F	The gradient of the function $f(x,y) = \int_x^y \sin(t) \; dt$ is $\langle -\sin(x), \sin(y) \rangle$.
20)	T	F	Assume $f(x,y) = x^2 + y^4$ and a curve $\vec{r}(t)$ satisfies $\vec{r}\,'(t) = \nabla f(\vec{r}(t))$, then $\frac{d}{dt} f(\vec{r}(t)) \geq 0$.

Problem 2) (10 points) No justifications needed

a) (6 points) Match the regions with the integrals. Each integral matches one region $A-F$.



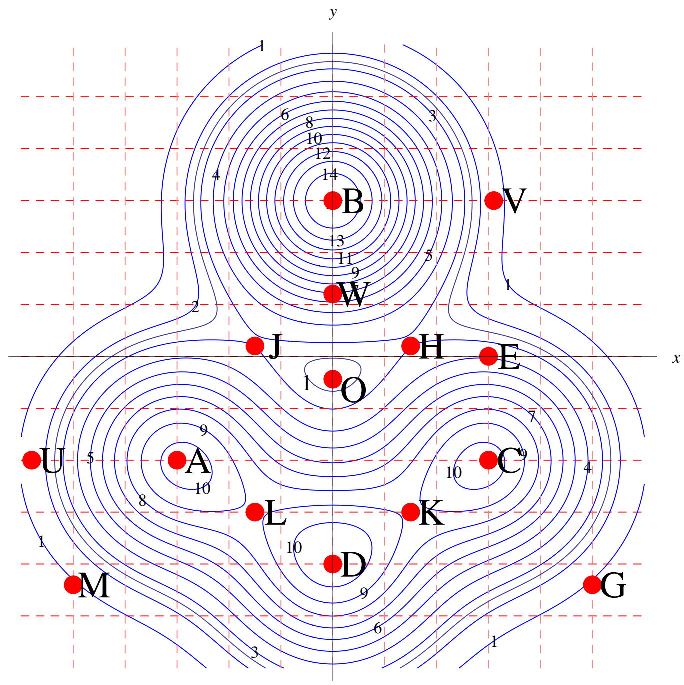
Enter A-F	Integral
	$\int_{-1}^{1}\int_{-1}^{2-\|y\|} f(x,y)\ dxdy$
	$\int_{-1}^{1}\int_{y^2}^{2-\|y\|} f(x,y)\ dxdy$
	$\int_{-1}^{1}\int_{x^2}^{2-x^2} f(x,y)\ dydx$
	$\int_{-1}^{1}\int_{\|x\|}^{2-x^2} f(x,y)\ dydx$
	$\int_{-1}^{1}\int_{y^2}^{2-y^2} f(x,y)\ dxdy$
	$\int_{-1}^{1}\int_{-1}^{2-\|x\|} f(x,y)\ dydx$

b) (4 points) Name the partial differential equations correctly. Each equation matches one name.

Fill in 1-4	Name
	Laplace
	Wave
	Transport
	Heat

Equation Number	PDE
1	$g_x - g_y = 0$
2	$g_{xx} - g_{yy} = 0$
3	$g_x - g_{yy} = 0$
4	$g_{xx} + g_{yy} = 0$

Problem 3) (10 points)



a) (6 points) Enter one label into each of the boxes.

	At which point is the length of the gradient maximal?
	At which point is the global maximum?
	At which point is $f_x > 0, f_y = 0$?
	At which point is $D_{\langle 1,1\rangle/\sqrt{2}} f = 0$, $D_{\langle 1,-1\rangle/\sqrt{2}} f < 0$?
	At which point is f maximal under the constraint $g(x,y) = y = 0$?
	At which point does f have a local minimum?

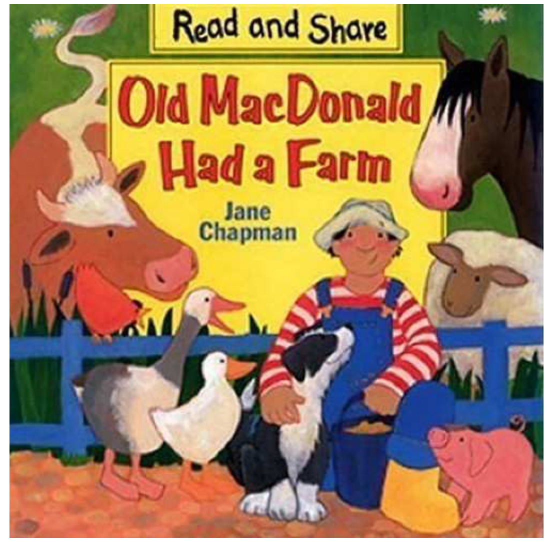
b) (4 points) Note that the zero vector is considered both parallel and perpendicular to any other vector.

	parallel	perp	
The gradient ∇f is always			to the surface $f = c$.
For a Lagrange minimum, ∇g is			to ∇f.
If $(0,0)$ is a min. of f then $\nabla f(0,0)$ is			to $\langle 1, 0\rangle$.
If $(0,0)$ is max. of f and $g = z - f(x,y)$ then ∇g is			to $\langle 0, 0, 1\rangle$.

Problem 4) (10 points)

A farm costs $f(x, y)$, where x is the number of cows and y is the number of ducks. There are 10 cows and 20 ducks and $f(10, 20) = 1000000$. We know that $f_x(x, y) = 2x$ and $f_y(x, y) = y^2$ for all x, y. Estimate $f(12, 19)$.

"Old MacDonald had a million dollar farm, E-I-E-I-O,
and on that farm he had $x = 10$ cows, E-I-E-I-O,
and on that farm he had $y = 20$ ducks, E-I-E-I-O,
with $f_x = 2x$ here and $f_y = y^2$ there,
and here two cows more, and there a duck less,
how much does the farm cost now, E-I-E-I-O?"



Problem 5) (10 points)

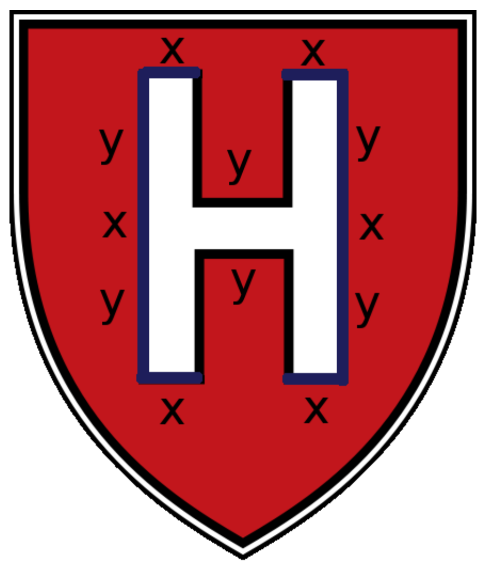
Find the Harvard H which has maximal area

$$f(x, y) = 5xy + 2x^2$$

with fixed exposed perimeter

$$6x + 4y = 88 \ .$$

Find the maximum using Lagrange.



Problem 6) (10 points)

a) (7 points) A minigolf on the cape has a hole at a local minimum of the function

$$f(x, y) = 3x^2 + 2x^3 + 2y^5 - 5y^2 \ .$$

Find all the critical points and classify them.

b) (3 points) A golfer hits tangent to the level curve $f(x, y) = 2$ through $(1, 1)$. Find this line.

About minigolf: the first standardized minigolf course appeared in 1916 in North Carolina. The world record on a round of minigolf is 18 strokes on 18 holes on eternite. No perfect round on concrete has been scored. The highest prizes reach 5000 dollars only so that nobody is known to make a living by competing in minigolf.

Problem 7) (10 points)

A circular track near Salem is a circle of radius 500 which is centered at the origin $(0, 0)$. A go-kart goes counter-clockwise around the track $\vec{r}(t)$. The cheering intensity is given by a function $f(x, y)$. The go-kart passes the point $(300, 400)$ at time $t = 0$ with velocity $\langle -4, 3 \rangle$. We know that $f_x(300, 400) = 2$ and $f_y(300, 400) = 10$. Find the rate of change

$$\frac{d}{dt} f(\vec{r}(t))$$

at $t = 0$.

Problem 8) (10 points)

a) (6 points) Find the integral

$$\int_0^1 \int_y^{y^{1/5}} \frac{e^x + x^7}{x - x^5} \, dx \, dy \ .$$

b) (4 points) Integrate

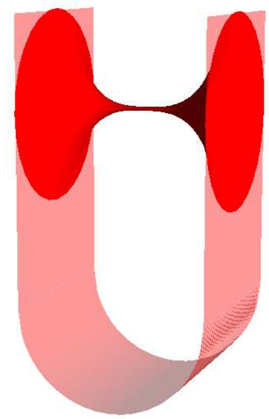
$$\int_{-1}^0 \int_0^{\sqrt{1-y^2}} \frac{e^{\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}} \, dxdy \ .$$

Problem 9) (10 points)

Find the surface area of the "wormhole"

$$\vec{r}(u,v) = \langle 3v^3, v^9 \cos(u), v^9 \sin(u) \rangle ,$$

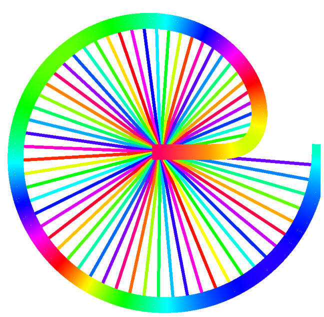
where $0 \leq u \leq 2\pi$ and $-1 \leq v \leq 1$.



Einstein-Rosen bridges *are hypothetical topological constructions which would allow shortcuts through space-time. Tunnels connecting different parts of the universe appear frequently in science fiction.*

Problem 10) (10 points)

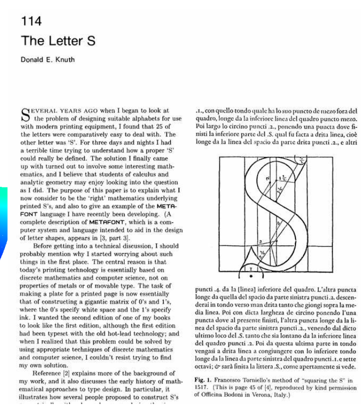
a) (5 points) We become typographer and design new mathematically defined **typeface** of the alphabet. The new letter "e" in this "21a" design is given by a polar region $r(t) \leq t^{1/7}$, with $0 \leq t \leq 2\pi$. Find the area of this region.



b) (5 points) Integrate

$$\int_0^1 \int_0^{\arccos(y)} \frac{1}{\cos(x)} \, dxdy .$$

Remark: Computer scientist **Donald Knuth** once wrote an entire article about "The Letter S".



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The Letter S

Donald E. Knuth

SEVERAL YEARS AGO when I began to look at the problem of designing suitable alphabets for use with modern printing equipment, I found that 25 of the letters were comparatively easy to deal with. The other letter was 'S'. For three days and nights I had a terrible time trying to understand how a proper 'S' could really be defined. The solution I finally came up with turned out to involve some interesting mathematics, and I believe that students of calculus and analytic geometry may enjoy looking into the question as I did. The purpose of this paper is to explain what I now consider to be the 'right' mathematics underlying printed S's, and also to give an example of the METAFONT language I have recently been developing. (A complete description of METAFONT, which is a computer system and language intended to aid in the design of letter shapes, appears in [3, part 3].

Before getting into a technical discussion, I should probably mention why I started worrying about such things in the first place. The central reason is that today's printing technology is essentially based on discrete mathematics and computer science, not on properties of metals or of movable type. The task of making a plate for a printed page is now essentially that of constructing a gigantic matrix of 0's and 1's, where the 0's specify white space and the 1's specify ink. I wanted the second edition of one of my books to look like the first edition, although the first edition had been typeset with the old hot-lead technology; and when I realized that this problem could be solved by using appropriate techniques of discrete mathematics and computer science, I couldn't resist trying to find my own solution.

Reference [2] explains more of the background of my work, and it also discusses the early history of mathematical approaches to type design. In particular, it illustrates how several people proposed to construct S's

.1., con quello tondo quale ha lo suo puncto de mezo fora del quadro, longe da la inferiore linea del quadro puncto mezo. Poi largo lo circino puncti .2., ponendo una puncta dove finisti la inferiore parte del .S. qual fu facta a drita linea, cioè longe da la linea del spacio da parte drita puncti .2., e altri

puncti .4. da la [linea] inferiore del quadro. L'altra puncta longe da quella del spacio da parte sinistra puncti .2. descenderai in tondo verso man drita tanto che giongi sopra la media linea. Poi con dicta largheza de circino ponendo l'una puncta dove al presente finisti, l'altra puncta longe da la linea del spacio da parte sinistra puncti .2., venendo dal dicto ultimo loco del .S. tanto che sia lontano da la inferiore linea del quadro puncti .2. Poi da questa ultima parte in tondo vengasi a drita linea a congiungere con lo inferiore tondo longe da la linea da parte sinistra del quadro puncti .1. e sette octavi; & sarà finita la littera .S., come apertamente si vede.

Fig. 1. Francesco Torniello's method of "squaring the S" in 1517. (This is page 45 of [4], reproduced by kind permission of Officina Bodoni in Verona, Italy.)