11/1/2017 Second Hourly Practice 1 **Math 21a, Fall 2017**

Name:

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- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3,we need to see **details** of your computation.
- All functions can be differentiated arbitrarily often unless otherwise specified.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) True/False questions (20 points), no justifications needed

1) T **F** The function $f(x,y) = x^3y/(x^6+y^5)$ can be filled in at the origin with a value $f(0,0) = a$ so that f is continuous everywhere.

Solution:
Go to polar coordinates. The function diverges at $(0,0)$.

2) **T** F The chain rule assures that $\int_0^1 (\nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t))\, dt = f(\vec{r}(1)) - f(\vec{r}(0))$.

Solution:
Just integrate the chain rule and use the fundamental theorem of calculus.

3) **T** F The formula $\int_0^1 \int_0^1 f(x,y)\, dy\, dx = \int_0^1 \int_0^1 f(y,x)\, dy\, dx$ holds.

Solution:
Just change the variables x, y first on the left hand side to get $\int_0^1 \int_0^1 f(y,x)\, dx\, dy$, then use Fubini to get the right hand side.

4) **T** F If $u(x,t)$ solves the partial differential equation $u_t = u_x$, then so does the function u_x.

Solution:
We can use Clairaut to see that.

5) T **F** There is a surface S containing the curve $\vec{r}(t) = \langle t, t^2, t^3 \rangle$ for which the tangent plane to S at $(0,0,0)$ is $x + 2y + 3z = 0$.

Solution:
The velocity vector is perpendicular to the surface, not parallel to the surface.

6) **T** F For any two unit vectors $\vec{u}$ and $\vec{v}$, and any f, we have $D_{\vec{u}} D_{\vec{v}} f = D_{\vec{v}} D_{\vec{u}} f$.

Solution:
Write down the definitions, we have in both cases $f_{xx}u_1v_1 + f_{yy}u_2v_2 + f_{xy}u_1v_2 + f_{yx}u_2v_1$.

7) T **F** If the tangent plane to $z = f(x, y)$ at $(0, 0, f(0, 0))$ is $4 + 3x + 2y + z = 0$, then $L(x, y) = 4 + 3x + 2y$ is the linearization of $f(x, y)$ at $(0, 0)$.

Solution:
This is fast because $L(x, y) = -4 - 3x - 2y$

8) **T** F For $f(x, y) = x^3 e^{y^2 \cos y} - x^4 \cos y$ the function $f_{xyxyxyxyxy}$ is zero everywhere.

Solution:
Use Clairaut. We differentiate 5 times with respect to x.

9) **T** F The point $(0, 0)$ is a critical point of $f(x, y) = x^3 y^2$.

Solution:
There are many critical points but $(0, 0)$ belongs there.

10) T **F** The gradient of $f(x, y) = x^2 + y^2$ is a vector perpendicular to the surface $z = f(x, y)$.

Solution:
It is not a 3-vector.

11) T **F** If the function $f(x, y)$ attains an absolute maximum on the region $x^2+y^2 \leq 4$ at the point $(2, 0)$, then we must have $f_{xx}(2, 0) \leq 0$.

Solution:
The maximum does not need to be in the interior.

12) **T** F If $f(x, y) \leq 5$ for all values of (x, y), then $\int_0^{2\pi} \int_0^7 f(r \cos \theta, r \sin \theta) r \, dr \, d\theta \leq 5\pi(7^2)$.

Solution:
The integral is smaller or equal to $\int_0^{2\pi} \int_0^7 5r \, dr \, d\theta$ which is 5 times the area of the disk.

13) | T | F | (answer: T)

For any constant a, we have $\int_{-a}^{a} \int_0^{a} \left(e^{x^2} \sin y\right) \, dx \, dy = 0$.

Solution:
Use the symmetry

14) | T | F | (answer: F)

The linearization of the function $f(x, y) = e^{x^2+y}$ at the point $(0, 0)$ is the function $L(x, y) = 1 + 2x^2 e^{x^2+y} + y e^{x^2+y}$.

Solution:
The linearization is linear

15) | T | F | (answer: F)

Let $\vec{u}$ be the unit vector in the direction $\langle 1, 1 \rangle / \sqrt{2}$. Then $D_{\vec{u}} f = f_{xy}$.

Solution:
The directional derivative only invokes the first derivatives of f.

16) | T | F | (answer: F)

The integral of $f(x, y) = \sqrt{x^2 + y^2}$ over the unit disk is $\int_0^{2\pi} \int_0^1 r \, dr d\theta$.

Solution:
We have forgotten the factor r.

17) | T | F | (answer: F)

There is a function $f(x, y)$ for which $D_{\vec{v}} f(0, 0) = 1$ for all directions $\vec{v}$.

Solution:
Switching the direction gives a negative value.

18) | T | F | (answer: T)

Given $f(x, y(x)) = 0$, then $f_x + f_y \frac{dy}{dx} = 0$.

Solution:
This is implicit differentiation

19) | T | F | Any function on a closed and bounded region must have a critical point.

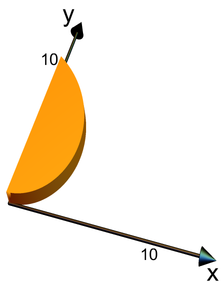
Solution:
Take a linear non-constant function like $x + y$.

20) | T | F | The integral $\iint_{x^2+y^2\leq 1} |f(x,y)|\, dxdy$ computes the surface area of the surface $z = f(x,y), x^2 + y^2 \leq 1$.

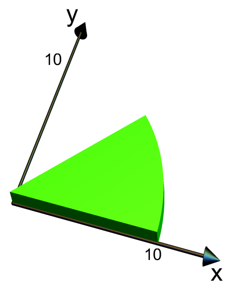
Solution:
The surface area is $\sqrt{1 + f_x^2 + f_y^2}$. In general the areas are different. Take $f = 2$ for example, then the integral under consideration is 2π while the surface area is π.

Problem 2) (10 points) No justifications needed

a) (6 points) Double integrals like $\iint_R 1\ dxdy$ or $\iint_R r\ drd\theta$ can be interpreted both as the **area** of the region R as well as the **volume** of the solid under the graph of the constant function $f(x,y) = 1$ or $f(\theta, r) = 1$. Match the regions with the integrals:



A



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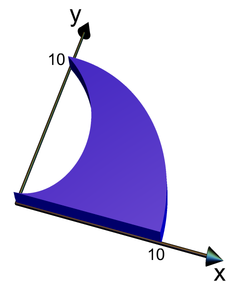

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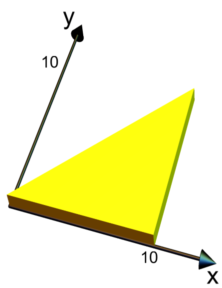

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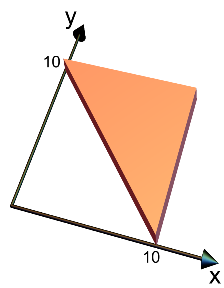

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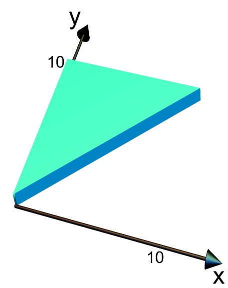
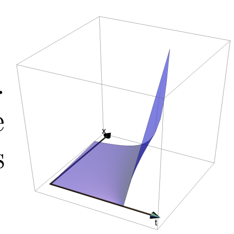


F

Enter A-F	Integral
	$\int_0^{10} \int_0^{x} 1\ dydx$
	$\int_0^{10} \int_0^{\pi/4} r\ d\theta dr$
	$\int_0^{\pi/2} \int_0^{20\theta/\pi} r\ drd\theta$
	$\int_0^{\pi/2} \int_{20\theta/\pi}^{10} r\ drd\theta$
	$\int_0^{10} \int_0^{y} 1\ dxdy$
	$\int_0^{10} \int_{10-x}^{10} 1\ dydx$

b) (4 points)

You know the **Transport**, **Wave**, **Heat**, or **Burgers** equation. Given in a possibly different order, these differential equations are $u_t = u_{xx}, u_t = u_x, u_{tt} = u_{xx}, u_t + uu_x = u_{xx}$. Check all the boxes where the given function solves the given PDE.



$f(x,t) = x/(1+t)$ solves	Name
	Burgers
	Transport

$f(x,t) = xt$ solves	Name
	Heat
	Wave

Solution:
a) The order is $DBACFE$.
b) $x/(1+t)$ solves the Burgers equation and xt solves the wave equation.

Problem 3) (10 points)

3a) (7 points) All the parts of this problem refer to the labeled points and the differentiable function $f(x, y)$ whose level curves are shown in the following plot:

a) At the point $\boxed{}$, the gradient ∇f has maximal length

b) At the point $\boxed{}$, $f_x > 0$ and $f_y = 0$

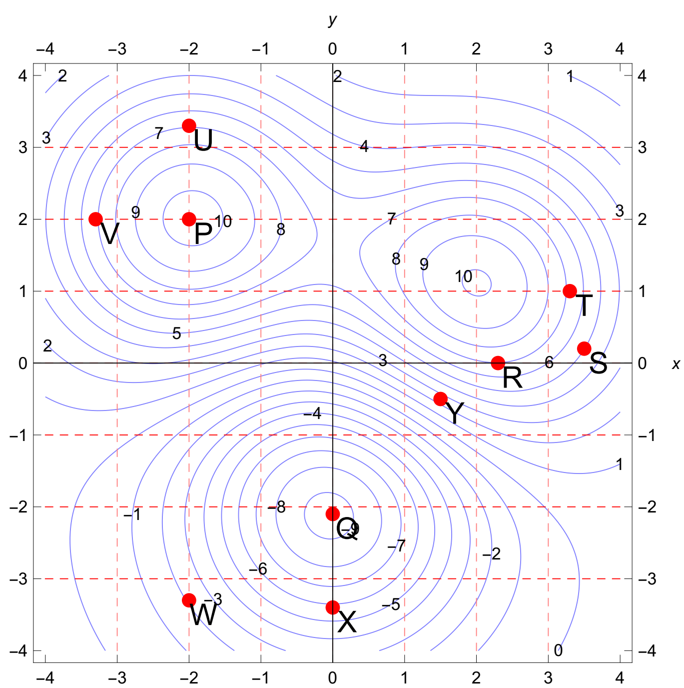
c) At the point $\boxed{}$, $f_x < 0$ and $f_y < 0$

d) At the point $\boxed{}$, $D_{\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle} f = 0$ and $f_x \neq 0$

e) At the point $\boxed{}$, f achieves a global min on $-4 \leq x \leq 4$ and $-4 \leq y \leq 4$

f) At the point $\boxed{}$, $\nabla f = \vec{0}$ and $f_{xx} < 0$

g) At the point $\boxed{}$, ∇f points straight toward the top of the page.



3b) (3 points) Check the cases where the maximum, minimum or saddle point of the function can be established **conclusively** using the second derivative test. Don't check the box if the test does not apply, (even if it might be a sort of minimum, maximum or saddle).

Critical Point	x^4+y^2	xy	x^2-y^4
Maximum			
Minimum			
Saddle point			

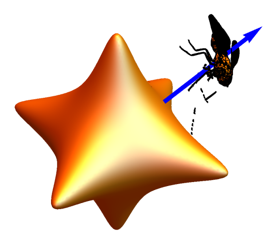
Solution:
a) The answer is $YVWSQPR$.
b) Only check "Saddle xy". The other cases are critical points but situations where $D=0$ and the second derivative test is inconclusive.

Problem 4) (10 points)

a) (4 points) A **math candy** of the form

$$f(x,y,z) = 3x^2y^2 + 3y^2z^2 + 3x^2z^2 + x^2 + y^2 + z^2 = 12$$

is leaning at $(1,1,1)$ at the plane tangent to it. Find that plane.



b) (3 points) Estimate $f(1.1, 1.01, 0.98)$ using linearization.

c) (3 points) A fruit fly just dipped some sugar from the candy at $(1,1,1)$ and moves along a path $\vec{r}(t)$ with constant speed 1 perpendicularly away from the candy. What is $\frac{d}{dt}f(\vec{r}(t))$ at the moment of take-off?

Solution:
a) $\nabla f(1,1,1)$ gives a normal vector to the tangent plane. As $\nabla f = \langle 6xy^2 + 6xz^2 + 2x, 6x^2y + 6yz^2 + 2y, 6y^2z + 6x^2z + 2z\rangle$, we have $\nabla f(1,1,1) = \langle 14, 14, 14\rangle$. The plane is $\boxed{14x + 14y + 14z = 42}$, where we got the constant on the right hand side by plugging in the point $(1,1,1)$.
b) The linearization $L(x,y,z)$ of $f(x,y,z)$ at $(1,1,1)$ is given by $L(x,y,z) = f(1,1,1) + \nabla f(1,1,1) \cdot \langle x-1, y-1, z-1\rangle = 12 + 14(x-1) + 14(y-1) + 14(z-1)$. So $f(1.1, 1.01, 0.98) \approx L(1.1, 1.01, 0.98) = 12 + 1.4 + 0.14 - 0.28 = 13.26$.
c) This is the length of the gradient as we know that this is the directional derivative in the direction of gradient. The length is $14\sqrt{3}$. If you should have forgotten the interpretation of the length of the gradient, here is the computation: by the chain rule, we have $\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$. Since the fly moves perpendicularly to the candy at the unit speed, $\vec{r}\,'(t)$ at the moment of take -off is the unit vector in the direction of $\nabla f(1,1,1) = \langle 14, 14, 14\rangle$, and thus it is $\frac{1}{\sqrt{3}}\langle 1,1,1\rangle$. Note that $\nabla f(1,1,1)$ points outward, so this is the direction of the fly's path.) Hence $\frac{d}{dt} f(\vec{r}(t))$ at the moment of take-off is $\nabla f(1,1,1) \cdot \frac{1}{\sqrt{3}}\langle 1,1,1\rangle = 14\sqrt{3}$.

Problem 5) (10 points)

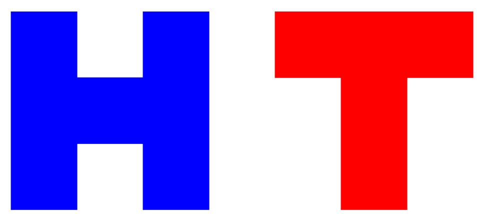
In order to figure out the **Egos** x and y of the US presidential candidates, we want to minimize the sum of the perimeter of the letters H and T written in units x and y if the total area is fixed. The letter H has area $7x^2$ and perimeter $16x$, the letter T has area $5y^2$ and perimeter $12y$. Minimize

$$f(x,y) = 16x + 12y \ .$$

under the constraint

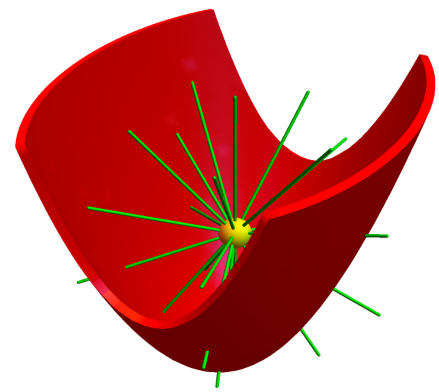
$$g(x,y) = 7x^2 + 5y^2 = 2016 \ .$$

We don't actually need to know x and y. **As political pundits, we are only interested in the ratio** y/x **at the minimum.** Find this ratio!



Solution:
Write down the Lagrange equations $16 = \lambda 14x, 12 = \lambda 10y$, then eliminate λ to get $y/x = 21/20$. There is slightly more "Ego" for T.

Problem 6) (10 points)



With $F(x,y,z) = 2x^2 + y^2 + z^2$ and the surface S parametrized by $\vec{r}(x,y) = \langle 2x, y, 2x^2 + y^2 - 1 \rangle$, the function $f(x,y) = F(\vec{r}(x,y))$ giving the value F on S is

$$f(x,y) = 4x^4 + 4x^2y^2 + 4x^2 + y^4 - y^2 + 1 \; .$$

a) (8 points) Find all the critical points of f and classify them with the second derivative test. Please organize your work carefully so that we can see your method and your conclusions easily.

b) (2 points) The minimum could be obtained by minimizing $F(x,y,z)$ on the surface $G(x,y,z) = x^2/2 + y^2 - 1 - z = 0$. We would then use a method found by some mathematician. Which one? Just check the name. No additional work is needed in b).

Fubini	Burgers	Laplace	Lagrange	Bolzano	Clairaut

Solution:
a) This is a standard problem for exrema without constraint. The gradient of f is $\langle 8x\,(2x^2 + y^2 - 1)\,, 4y\,(2x^2 + y^2 - 1)\rangle$. We have $f_{xx} = 8 + 48x^2 + 8y^2$. There are three critical points $(0,0)$, $(0, \sqrt{2}/2)$, $(0, -\sqrt{2}/2)$. The first is a saddle with $D = -16$, the other two are minima with $D = 48$ and $f_{xx} = 12$.

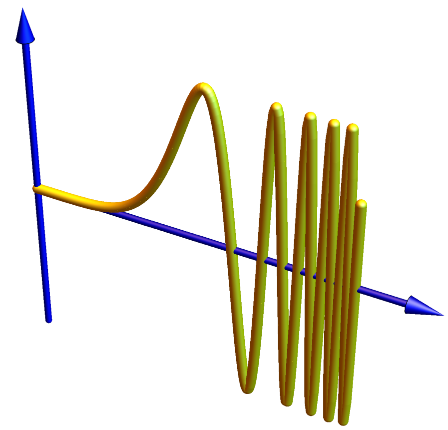
x	y	D	f_{xx}	Type
0	0	-16	8	saddle
0	$1/\sqrt{2}$	48	12	minimum
0	$-1/\sqrt{2}$	48	12	minimum

b) The problem has been reformulated as a Lagrange problem. Some were crossing Bolzano but since the surface is not bounded, one can not conclude the existence of a minimum from Bolzano.

Problem 7) (10 points)

Integrate

$$\int_0^1 \int_{(1-y)^{1/4}}^1 \sin(x^5)\, dxdy \;.$$



The figure just shows a fancy plot of the function $\sin(x^5)$.

Solution:
Change the order of integration to get

$$\int_0^1 \int_{1-x^4}^1 \sin(x^5)\, dy\, dx \;.$$

This simplifies to

$$\int_0^1 x^4 \sin(x^5)\, dx = (1-\cos(1))/5 \;.$$

Problem 8) (10 points)

Integrate the double integral

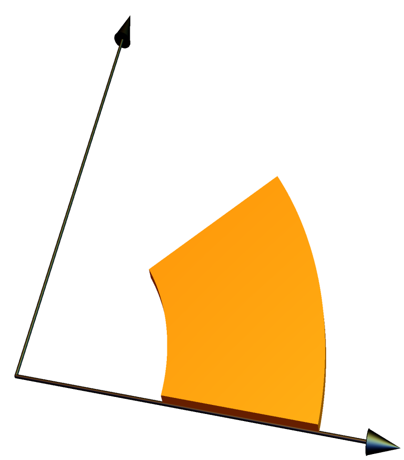
$$\iint_R x^2 \, dxdy \;,$$

where R is the region

$$1 \leq x^2 + y^2 \leq 4$$

and

$$x \geq 0, y \geq 0, y \leq x \;.$$



Solution:
Use polar coordinates. We integrate $\int_0^{\pi/4} (15/4)\cos^2(\theta)\ d\theta$. Now use the double angle formula. We end up with $\boxed{(15\pi + 30)/32}$.

Problem 9) (10 points)

a) (7 points) Compute $A = |\vec{r}_\theta \times \vec{r}_\phi|$ for the half cylinder parametrized by

$$\vec{r}(\theta, \phi) = \langle \cos(\theta), \sin(\theta), \cos(\phi) \rangle \ .$$

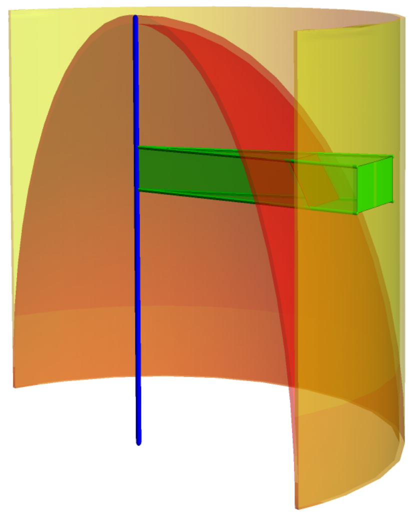
with $0 \leq \phi \leq \pi/2$ and $0 \leq \theta \leq \pi$ and use this to find the surface area of the half cylinder

b) (3 points) Compute $B = |\vec{r}_\theta \times \vec{r}_\phi|$ for the quarter sphere parametrized by

$$\vec{r}(\theta, \phi) = \langle \sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi) \rangle$$

with $0 \leq \phi \leq \pi/2$ and $0 \leq \theta \leq \pi$ to show that (remarkably!) it is the same factor than in part a).

Remark: The fact that the surface area elements A and B are the same has been realized by Archimedes already. It allowed him to compute the surface area of the sphere in terms of the surface area of the cylinder.



Solution:
a) The integration factor is $A = \sin(\phi)$. The integral is $\int_0^\pi \int_0^{\pi/2} \sin(\phi) = \pi$.
b) The integration factor is again $\sin(\phi)$. We did not ask for the integral as the integral is the same.
P.S. It is remarkable that one can project the sphere onto the cylinder to see that the areas are the same. You can do that also in a way how Archimedes might have derived it: compare a thin slice of size dz on each surface. The surface area of the cylinder is $2\pi dz$ which integrates to 2π, the area of the full cylinder of height 1 and radius 1. At height $z = \cos(\phi)$ the radius is $r\sin(\phi)$ and since the surface is slanted, the length of the cross section of the piece is $dz/\sin(\phi)$ Now the surface area piece is is $2\pi r \sin(\phi) dz/\sin(\phi) = 2\pi dz$ again.