

Homework 1: Geometry and Distance

This homework is due Friday, 9/8 respectively Tuesday 9/12 at the beginning of class.

- 1 a) Find its center and radius of the sphere S given by $x^2 + y^2 + z^2 - 20x + 12y + 4z = 29$. b) Find the distance from the center of S defined in a) to $x^2 + y^2 + z^2 = 900$.
c) Find the minimal distance between the spheres. This is the minimal distance between two points where each is in one sphere.

Solution:

a) Complete the square:

$$\begin{aligned}(x^2 - 20x + 100) + (y^2 + 12y + 36) + (z^2 + 4z + 4) &= 100 + 36 + 4 + 29 \\(x - 10)^2 + (y + 6)^2 + (z - 1)^2 &= 13^2.\end{aligned}$$

Thus, the center of the sphere is at $(10, -6, -2)$ and the radius is 13.

b) The sphere $x^2 + y^2 + z^2 = 900$ has radius 30. It contains the center $(10, -6, -2)$ of the sphere S . The distance between the centers is $\sqrt{10^2 + 6^2 + 2^2} = \sqrt{140} = 2\sqrt{35}$. The center of the sphere S has distance $2\sqrt{35}$ from the center of the other sphere and $30 - 2\sqrt{35}$ is the distance between the center of S and the sphere of radius 30.

c) To get the distance between the two spheres, draw the line connecting the two centers and subtract the correct distances. We get $d = 30 - 2\sqrt{35} - 13$.

- 2 a) Find the distance a from $P = (-12, -4, -6)$ to $x = 0$.
b) Find the distance b from P to the x -axes.

- c) Find the distance c from P to the origin $O = (0, 0, 0)$.
d) Take a general point $P = (x, y, z)$ and repeat the computation for the values a, b, c as in a-c). What is $a^2 + b^2 - c^2$?

Solution:

a) The distance to the xy -plane is 12. b) The distance to the z axes is $\sqrt{4^2 + 6^2} = \sqrt{52}$. c) The distance to the origin is $\sqrt{12^2 + 4^2 + 6^2}$. d) We have $a = x$ and $b = \sqrt{y^2 + z^2}$ and $c = \sqrt{x^2 + y^2 + z^2}$. We see $a^2 + b^2 - c^2 = 0$.

- 3 a) Find an equation of the largest sphere with center $(4, 11, 9)$ that is contained in the first octant $\{x \geq 0, y \geq 0, z \geq 0\}$.
b) Find the equation for the sphere centered at $(6, 10, 8)$ which passes through the center $(4, 11, 9)$ of the sphere in a).

Solution:

a) The closest coordinate plane is the yz plane. It has distance 4, so the radius of the largest sphere must be 4. The equation is $(x - 4)^2 + (y - 11)^2 + (z - 9)^2 = 4^2$.
b) The distance between $(6, 10, 8)$ and $(4, 11, 9)$ is $\sqrt{6}$. Thus, the sphere's equation is $(x - 6)^2 + (y - 10)^2 + (z - 8)^2 = 6$.

- 4 a) Describe the surface $x^2 - (z^2 + 4z) = 12$ in \mathbb{R}^3 .
b) What is the surface $x^2 = z^2$ in three dimensional space \mathbb{R}^3 .
c) Draw the surfaces of a) and b) and their intersection.

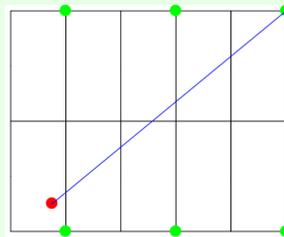
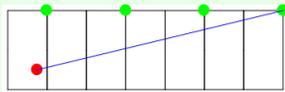
Solution:

a) This is a hyperbolic cylinder. b) This is a union of two planes $x = z$ and $x = -z$ containing the y -axis. c) The intersection consists of 2 lines parallel to the y -axes. The points through which the lines pass could be computed by solving a quadratic equation for x or z . It is $x = z = -3$.

- 5 You play billiard in the table $\{(x, y) \mid 0 \leq x \leq 4, 0 \leq y \leq 8\}$. a) Hit the ball at $(3, 2)$ to reach the hole $(4, 8)$ bouncing 3 times at the left wall and three times at the right wall and no other walls. Find the length of the shot.
- b) Hit from $(3, 2)$ to reach the hole $(4, 0)$ after hitting twice the left and twice the right wall as well as the top wall $y = 8$ once. What is the length of the trajectory?

Solution:

- a) Every time the ball hits a boundary, we can unfold the table. With Pythagoras, we have a distance $\sqrt{6^2 + (24 + 1)^2}$.
- b) Now, we also have to unfold the table vertically when hitting the upper wall. We have a distance $\sqrt{17^2 + 14^2}$.



Main definitions

Points in the **plane** or **space** are described using **coordinates** $P = (x, y)$ or $P = (x, y, z)$. Their signs define 4 **quadrants** or **octants** in space, regions which intersect at the **origin** $O = (0, 0)$ or $O = (0, 0, 0)$ and are separated by **coordinate planes** $\{x = 0\}$, $\{y = 0\}$, $\{z = 0\}$ intersecting in **coordinate axes** like the z -axes $\{y = 0, x = 0\}$.

The **Euclidean distance** between two points $P = (x, y, z)$ and $Q = (a, b, c)$ in space is defined as $d(P, Q) = \sqrt{(x - a)^2 + (y - b)^2 + (z - c)^2}$. The distance between a point P and a geometric object S is the minimal distance $d(P, Q)$ with Q located on S .

A **circle** of radius r centered at $P = (a, b)$ is the set of points in the plane which have distance r from P . A **sphere** of radius ρ centered at $P = (a, b, c)$ is the set of points in space which have distance ρ from P . The equation of a sphere is $(x - a)^2 + (y - b)^2 + (z - c)^2 = \rho^2$.

To **complete the square** of $x^2 + bx + c = 0$, add $(b/2)^2 - c$ on both sides to get $(x + b/2)^2 = (b/2)^2 - c$. Solving for x gives $x = -b/2 \pm \sqrt{(b/2)^2 - c}$. **Example:** $x^2 + 8x + y^2 = 9$. **Solution:** Add 16 on both sides to get $x^2 + 8x + 16 + y^2 = 25$ which is $(x + 4)^2 + y^2 = 25$, a circle of radius $r = 5$ centered at $(-4, 0)$.