Homework 13: Chain Rule

This homework is due Wednesday, 10/12 (Monday 10/10 is Columbus day and no class), rsp Tuesday 10/11.

1 a) Use the chain rule to find the derivative $df(\vec{r}(t))/dt$ at $t = 1$ for

$$f(x, y) = \sin(x^5 + y^2) ,$$

where $\vec{r}(t) = \langle x(t), y(t) \rangle = \langle t^4, 1/t \rangle$. To do so, compute $r'(1)$ and $\nabla f(\vec{r}(1))$ and then the dot product. b) Now compute the derivative directly by differentiating $\sin((t^4)^5 + 1/t^2)$ with respect to t. You should get the same thing.

Solution:
a) $\vec{r}\,'(t) = \langle 4t^3, -1/t^2 \rangle$ so that $\vec{r}(0) = \langle 4, -1 \rangle$. $\nabla f = \langle \cos(x^5 + y^2)5x^4, \cos(x^5 + y^2)2y \rangle$. At $(1, 1)$ this is $\langle 5\cos(2), 2\cos(2) \rangle$. The result is $18\cos(2)$. b) $f(\vec{r}(t)) = \sin(t^20 + 1/t^2)$. The derivative is $18\cos(2)$.

2 Find $dy/dx = y'(x)$ if x, y are related by

$$\sin(x) + \cos(y) = \sin(x)\cos(y) .$$

Solution:
Implicit differentiation with respect to x yields:

$$\cos(x) - \sin(y)y' = \cos(x)\cos(y) - \sin(x)\sin(y)y'$$

Therefore,

$$y' = \frac{\cos(x)\cos(y) - \cos(x)}{\sin(x)\sin(y) - \sin(y)}.$$

3 Find z_x and z_y for $yz = \log(x+z)$, where $\log = \ln$ is the natural log.

Solution:
Implicit differentiation with respect to x gives us:

$$yz_x = \frac{1+z_x}{x+z}$$

Hence $yz_x(x+z) = 1 + z_x$ or

$$z_x = \frac{1}{y(x+z)-1}.$$

Implicit differentiation with respect to y gives us:

$$z + yz_y = \frac{z_y}{x+z}$$

Hence $(x+z)z + (x+z)yz_y = z_y$ or

$$z_y = \frac{(x+z)z}{1-(x+z)y}.$$

4 The radius of a right circular cone is increasing at a rate of 1.8 while its height is decreasing at a rate of 2.5. At what rate is the

volume of the cone changing when the radius is 120 and the height is 140.

Solution:
Denote the radius and height of the cone by r and h respectively. Then, $\frac{\partial r}{\partial t} = 1.8$ and $\frac{\partial h}{\partial t} = -2.5$. The volume $V(r, h)$ is given by $\pi r^2 h/3$. By the chain rule,

$$\frac{\partial V}{\partial t} = \pi 2rh/3 \cdot 1.8 + \pi r^2/3 \cdot (-2.5).$$

Evaluating at $r = 120$ and $h = 140$,

$$\frac{\partial V}{\partial t}(120, 140) = \pi 2rh/3 \cdot 1.8 - \pi r^2/3 \cdot 2.5 = 8160\pi.$$

5 The Voltage V in a simple electrical circuit is slowly decreasing as the battery wears out. The resistance R is slowly increase as the resistor heats up. Use **Ohm's Law**, $V = IR$, to find how the current I is changing at the moment when $R = 400$, $I = 0.08$ $dV/dt = -0.01$ and $dR/dt = 0.03$.

Solution:
By the chain rule,

$$-0.01 = \frac{\partial V}{\partial t} = \frac{\partial I}{\partial t} R + I \frac{\partial R}{\partial t} = 400 \frac{\partial I}{\partial t} + 0.08 \cdot 0.03.$$

Thus, $\frac{\partial I}{\partial t} \approx -0.000031$.

Main definitions

Define the **gradient** $\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle$ or $\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle$.

The **multivariable chain rule** is

$$\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t) \; .$$

When written out in two dimensions, this is

$$\frac{d}{dt} f(x(t), y(t)) = f_x(x(t), y(t)) x'(t) + f_y(x(t), y(t)) y'(t)$$

Example: a bug walks on $\vec{r}(t) = \langle \cos(t), \sin(t)\rangle$ on a plane with temperature $f(x,y) = x^2 + 5y$. Find the temperature change $d/dt f(\vec{r}(t))$ at $(1,0)$. **Solution:** either compose $f(\vec{r}(t)) = \cos^2(t) + 5\sin(t)$ and differentiate at $t = 0$ to get $d/dt f(\vec{r}(t)) = 5\cos(0) = 5$. Or then find $\vec{r}\,'(0) = \langle 0, 1\rangle$ and the gradient $\nabla f(x,y) = \langle 2x, 5\rangle$ which is $\langle 0, 5\rangle$ at $(1,0)$. The chain rule assures that the dot product is the same.
We can use the chain rule for implicit differentiation

Implicit differentiation: If $f(x,y) = c$ is a curve, we can compute $y' = -f_x/f_y$.

In three dimensions, the **implicit differentiation formulas** derived from the chain rule are:

$$z_x(x,y) = -f_x(x,y,z)/f_z(x,y,z)$$

$$z_y(x,y) = -f_y(x,y,z)/f_z(x,y,z)$$