Homework 15: Directional Derivatives

This homework is due Monday, 10/16 rsp Tuesday 10/17.

1 a) Find the gradient of $f(x, y, z) = \sin(\pi(x + 5yz))$ at the point $P = (1, 3, 5)$. b) Use it to find the rate of change of f at P in the direction of the vector $\vec{u} = \langle 2/7, 3/7, 6/7 \rangle$.
c) Find a direction of the form $\vec{v} = \langle u, a, b \rangle$ in which the directional derivative of f at $(1, 3, 5)$ is $10\sqrt{2}\pi$.

Solution:
a) The gradient is

$$\nabla f(x, y, z) = \langle \pi \cos(\pi(x+5yz)), 5\pi z \cos(\pi(x+5yz)), 5\pi y \cos(\pi(x+5yz$$

and

$$\nabla f(1, 3, 5) = \langle \pi, 25\pi, 15\pi \rangle \ .$$

b) The directional derivative is $\pi \langle 1, 25, 15 \rangle \cdot \langle 2, 3, 6 \rangle / 7 = 167/7$.
c) To find the direction, take $\langle 0, a, b \rangle$ and solve $\langle \pi, 25\pi, 15\pi \rangle \cdot \langle 0, a, b \rangle = 10\sqrt{2}\pi$ for a with $b = \sqrt{1 - a^2}$. Its ok to solve the quadratic equation with a computer. The answer is about $(a, b) = (-0.375926, 0.92665)$.

2 a) (5 points) Find the directional derivative of the function $f(x, y) = \log(x^2 + y^2)$ at the point $P = (2, 1)$ in the direction of the vector $\vec{v} = \langle -1, 2 \rangle$. (We use the notation $\log = \ln$).
b) (5 points) Find the directional derivative of $f(x, y, z) = xy + yz + zx$ at the point $P = (1, -1, 3)$ in the direction from P to $Q = (2, 4, 5)$.

Solution:

(a) $\nabla f = \langle \frac{2x}{x^2+y^2}, \frac{2y}{x^2+y^2} \rangle$. At point $(2,1)$, it is $\langle 4/5, 2/5 \rangle$. The directional derivative is

$$\langle 4/5, 2/5 \rangle \cdot \frac{\langle -1, 2 \rangle}{|\langle -1, 2 \rangle|} = \frac{\langle 4/5, 2/5 \rangle \cdot \langle -1, 2 \rangle}{|\langle -1, 2 \rangle|} = 0.$$

(b) $\nabla f(x,y,z) = \langle y+z, x+z, y+x \rangle$, so $\nabla f(1,-1,3) = \langle 2,4,0 \rangle$. The unit vector in the direction of $\vec{PQ} = \langle 1,5,2 \rangle$ is $\vec{u} = \frac{1}{\sqrt{30}} \langle 1,5,2 \rangle$, so $D_u f(1,-1,3) = \nabla f(1,-1,3) \cdot \vec{u} = \langle 2,4,0 \rangle \cdot \frac{1}{\sqrt{30}} \langle 1,5,2 \rangle = \frac{22}{\sqrt{30}}$.

3 a) Find the direction of steepest descent for $f(x,y,z) = \frac{(x+y)}{z}$ at the point $P = (1,1,-1)$.
b) Find the value of the rate of change at $(1,1,-1)$ in that direction found in a).

Solution:

(a) The maximum rate of change occurs in the direction of the gradient. The gradient is

$$\nabla f = \langle \frac{1}{z}, \frac{1}{z}, -\frac{x+y}{z^2} \rangle, \qquad \nabla f(1,1,-1) = \langle -1,-1,-2 \rangle.$$

The direction of steepest decent is $\langle 1,1,2 \rangle$.

(b) The unit vector in the direction of the gradient is $\langle -\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}} \rangle$ and the rate of change is

$$\langle -\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}} \rangle \cdot \nabla f = \sqrt{6}.$$

It is not so clear whether one should define maximal rate of change as an absolute value. If one wants to emphasis the direction, then also $-\sqrt{6}$ should be a correct answer.

4 You know that a function $f(x,y,z)$ satisfies $f(0,0,0) = 33$ and $D_{\langle 1,1,1 \rangle/\sqrt{3}} f(0,0,0) = 4/\sqrt{3}$ and $D_{\langle 1,1,0 \rangle/\sqrt{2}} f(0,0,0) = 7/\sqrt{2}$ and $D_{\langle 1,2,2 \rangle/3} f(0,0,0) = 12$. Estimate $f(0.01, -0.001, 0.1)$ using linearization!

Solution:
Call the gradient $\langle a,b,c \rangle$. We know $a+b+c = 4$ and $a+b = 7$ and $a+2b+2c = 36$. The solution is $a = -28, b = 35, c = -3$. Now get $33 - 28 * 0.01 - 35 * 0.001 - 3 * 0.1 = 32.38$

5 On the course website there is a map of a neighborhood of Cambridge from 1891. The map features level curves of the height function $f(x,y)$ as well as points $A - K$ marked in red.

a) At which of the points A-K is the directional derivative $D_v f$ zero in every direction v and the second directional derivative $D_v(D_v f)$ negative in any direction. Think first of what this means.
b) Find two points for which all $D_v f$ are zero and where the second directional derivatives can take different signs.
c) Find a point where $f_x = 0$ and $f_y > 0$.
d) Find a point where $f_y = 0$ and $f_x > 0$.
e) Find a point where $f_y = 0$ and $f_x < 0$.

Solution:
a) These are hills. A,K,D,E
b) B and G c) H d) J e) C

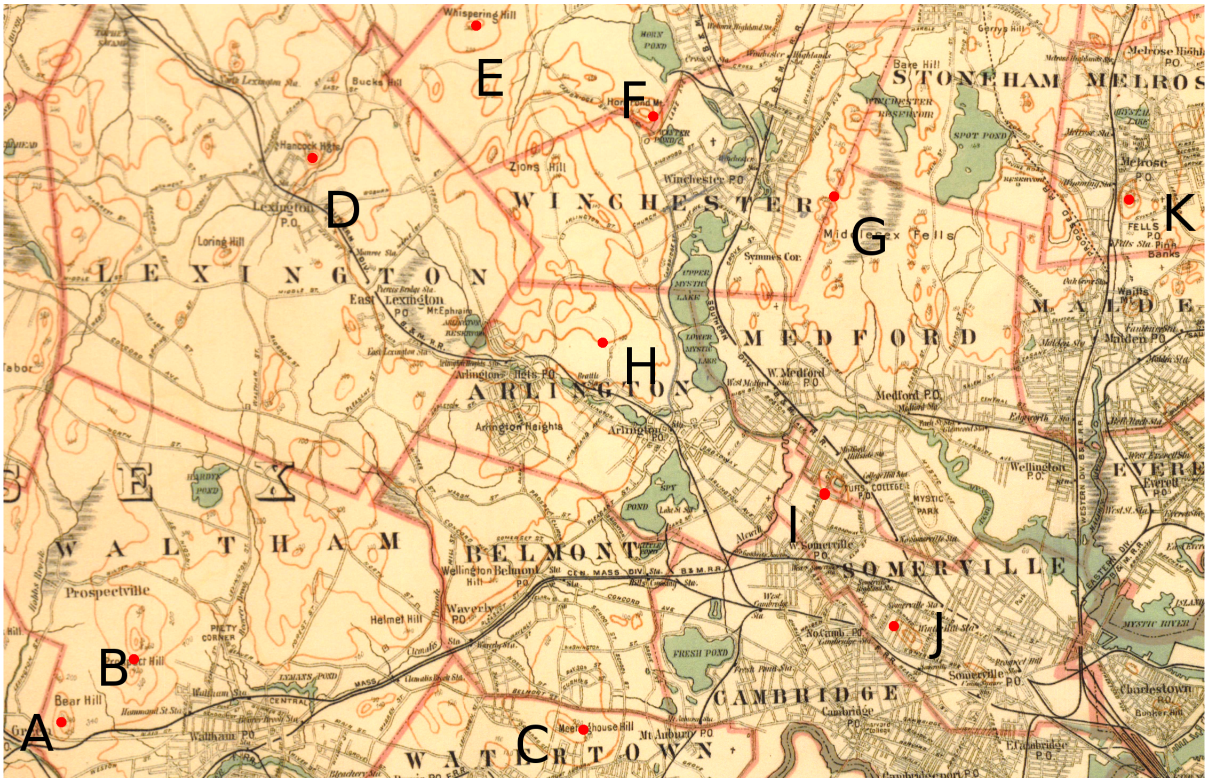
Main definition:

If f is a function of several variables and $\vec{v}$ is a unit vector then $D_{\vec{v}} f = \nabla f \cdot \vec{v}$ is the **directional derivative** of f in the direction $\vec{v}$.

For $\vec{v} = \nabla f / |\nabla f|$, the directional derivative is

$$D_v f = \nabla f \cdot \nabla f / |\nabla f| = |\nabla f| \ ,$$

so that f **increases** in the direction of the gradient. The value $|\nabla f|$ is the maximal slope.



A
B
C
D
E
F
G
H
I
J
K
Whispering Hill
Hancock Hill
Bucks Hill
Lexington P.O.
Loring Hill
L E X I N G T O N
East Lexington P.O.
Zions Hill
W I N C H E S T E R
Winchester P.O.
Symmes Cor.
UPPER MYSTIC LAKE
LOWER MYSTIC LAKE
S T O N E H A M
M E L R O S
Melrose P.O.
SPOT POND
Middlesex Fells
FELLS P.O.
M E D F O R D
W. Medford P.O.
Medford P.O.
M A L D E
Malden P.O.
A R L I N G T O N
Arlington Heights
TUFTS COLLEGE
MYSTIC PARK
Wellington P.O.
E V E R E
Everett P.O.
W A L T H A M
Prospectville
PIETY CORNER
Helmet Hill
Bear Hill
Waltham P.O.
B E L M O N T
Wellington Hill
Waverly P.O.
FRESH POND
S O M E R V I L L E
Somerville P.O.
C A M B R I D G E
Cambridge P.O.
Mt Auburn P.O.
W A T E R T O W N
Meetinghouse Hill
Charlestown P.O.
MYSTIC RIVER
E. Cambridge P.O.
Tabor