

Homework 18: Global extrema

This homework is due Monday, 10/23 resp Tuesday 10/24.

- 1 a) We suppose that the Cobb Douglas production formula $Q(L, K) = L^{1/4}K^{3/4} = 100$, which tells that the quantity Q is constant. What values of L and K minimizes the cost function $C(L, K) = 4L + 5K$ under the constraint $Q(L, K) = 100$?
- b) Is there a global maximum or minimum for $C(L, K)$ on the region $L \geq 0, K \geq 0$ without the constraint $Q = 100$? If yes, what is the maximum, or what is the minimum?

Solution:

$C(L, K) = 4L + 5K$, $g(L, K) = L^{1/4}K^{3/4} = 100 \Rightarrow \nabla C = \langle 4, 5 \rangle$, $\lambda \nabla g = \langle \lambda \frac{1}{4} b L^{-3/4} K^{3/4}, \lambda \frac{3}{4} L^{1/4} K^{-1/4} \rangle$. Then $\frac{4}{1/4} \left(\frac{L}{K}\right)^{3/4} = \frac{5}{3/4} \left(\frac{K}{L}\right)^{1/4}$ and $L^{1/4}K^{3/4} = 100 \Rightarrow \frac{5/4}{4(3/4)} = \left(\frac{L}{K}\right)^{3/4} \left(\frac{L}{K}\right)^{1/4} \Rightarrow L = \frac{K(5/4)}{4(3/4)}$. Plugging this into the production formula, we get $L = \left[\frac{K(5/4)}{4(3/4)}\right]^{1/4} K^{3/4} = 100$. Hence $K = \frac{100(4)^{1/4}(3/4)^{1/4}}{(5)^{1/4}(1/4)^{1/4}} = \frac{100}{1} \left(\frac{12}{5}\right)^{1/4}$ and $L = \frac{100(5)^{3/4}(1/4)^{3/4}}{4^{3/4}(3/4)^{3/4}} = \frac{100}{1} \left(\frac{5}{12}\right)^{3/4}$. There is a minimum 0 at $L = 0$ or $K = 0$. There is no global maximum. In the original problem, we had $L \geq 0, K \leq 0$ so that there was no minimum in that case.

- 2 Bolzano's theorem assures that on $D = \{x^2 + y^2 \leq 1\}$, a continuous function (and especially a differentiable function) has both a at least one maximum and at least one minimum on D . On an open disk essentially anything goes, as you can see here:

a) Find a differentiable $f(x, y)$ which has exactly one maximum and no minimum on $x^2 + y^2 < 1$. b) Engineer a differentiable function $f(x, y)$ which has exactly one maximum and exactly one minimum on $x^2 + y^2 < 1$ and no saddle point and no other critical point. c) Engineer a differentiable function $f(x, y)$ which has exactly two maxima and no other critical point on $x^2 + y^2 < 1$.

Solution:

a) $1 - x^2 - y^2$ does the job

b) $x^3 - x + y^3$ works c) here is an idea: take a two hill function so that the hills are at $x = a, -a$ for some $a < 1$ Then multiply with x so that the y axes has the same height. Then tilt the graph a bit by adding a multiple of y . $f(x, y) = x(x^4 - x^2 + 0.55y^2) + 0.5y$. The two saddles are outside the circle while the two maxima are inside. P.S. A neat example is $-(x^2 - 1)^2 - (x^2y - x - 1)^2$. It has overall in R^2 only two maxima and no other critical points. It is due to John Cook (<https://www.johndcook.com/blog/2017/10/04/no-critical-point-between-two-peaks>)

3 Find the absolute maximum and minimum values of

$$f(x, y) = e^{-x^2-y^2}(x^2 + 2y^2) \quad ,$$

on the disk $D = \{x^2 + y^2 \leq 4\}$.

Solution:

Inside D : $f_x = 2xe^{-x^2-y^2}(1 - x^2 - 2y^2) = 0$ implies $x = 0$ or $x^2 + 2y^2 = 1$. Then if $x = 0$, $f_y = 2ye^{-x^2-y^2}(2 - x^2 - 2y^2) = 0$ implies $y = 0$ or $2 - 2y^2 = 0$ giving the critical points $(0, 0)$, $(0, \pm 1)$. If $x^2 + 2y^2 = 1$, then $f_y = 0$ implies $y = 0$ giving the critical points $(\pm 1, 0)$. Now $f(0, 0) = 0$, $f(\pm 1, 0) = e^{-1}$ and $f(0, \pm 1) = 2e^{-1}$. On the boundary of D : $x^2 + y^2 = 4$, so $f(x, y) = e^{-4}(4 + y^2)$ and f is smallest when $y = 0$ and largest when $y^2 = 4$. But $f(\pm 2, 0) = 4e^{-4}$, $f(0, \pm 2) = 8e^{-4} = 0.146\dots$. Thus on D the global maximum of f is $f(0, \pm 1) = 2e^{-1} = 0.73\dots$ and the global minimum is $f(0, 0) = 0$.

- 4 a) Use Lagrange multipliers to find the maximum and minimum values of f subject to the given constraint

$$f(x, y) = \frac{1}{x} + \frac{1}{y}; \quad g(x, y) = \frac{1}{x^2} + \frac{1}{y^2} = 1.$$

- b) Is there a global maximum or global minimum of f on $g = 1$?
Is this a case for the Bolzano theorem?

Solution:

a) $f(x, y) = \frac{1}{x} + \frac{1}{y}$, $g(x, y) = \frac{1}{x^2} + \frac{1}{y^2} = 1 \Rightarrow \nabla f = \langle -x^{-2}, -y^{-2} \rangle = \lambda \nabla g = \langle -2\lambda x^{-3}, -2\lambda y^{-3} \rangle$. Then $-x^{-2} = -2\lambda x^{-3}$ or $x = 2\lambda$ and $-y^{-2} = -2\lambda y^{-3}$ or $y = 2\lambda$. Thus $x = y$, so $\frac{1}{x^2} + \frac{1}{y^2} = \frac{2}{x^2} = 1$ implies $x = \pm\sqrt{2}$ and the possible points are $(\pm\sqrt{2}, \pm\sqrt{2})$. The absolute maximum of f subject to $x^{-2} + y^{-2} = 1$ is then $f(\sqrt{2}, \sqrt{2}) = \sqrt{2}$ and the absolute minimum is $f(-\sqrt{2}, -\sqrt{2}) = -\sqrt{2}$.

b) There is a global minimum and maximum and you have found them in a). We could not apply Bolzano as the curve is not bounded. Why is there a global max and min? There is a global maximum on $(\sqrt{2}, \sqrt{2})$ and a global minimum on $(-\sqrt{2}, -\sqrt{2})$. There are four branches to the constraint $1/x^2 + 1/y^2 = 1$. In the first quadrant the function f is maximal at $(\sqrt{2}, \sqrt{2})$ and goes to 1 when either x goes to infinity or y goes to infinity. In the third quadrant, f is minimal at $(-\sqrt{2}, -\sqrt{2})$ and goes to -1 as x goes to infinity or minus infinity. On the second and third quadrant, the function is monotone going from -1 to 1.

- 5 A package in the shape of a rectangular box can be mailed by the **US Postal Service** if the sum of its length and girth (the perimeter of a cross-section perpendicular to the length) is at most 108. Find the dimensions of the package with largest volume $V(x, y, z) = xyz$ that can be mailed under the constraint $x + 2y + 2z \leq 108$.

Solution:

We want to extremize $V = xyz$ where $x + 2y + 2z \leq 108$, $x \geq 0$, $y \geq 0$, $z \geq 0$. First maximize V subject to $x + 2y + 2z = 108$ with x, y, z all positive. Then $\langle yz, xz, xy \rangle = \langle \lambda, 2\lambda, 2\lambda \rangle$ implies $2yz = xz$ or $x = 2y$ and $xz = xy$ or $z = y$. Thus $g(x, y, z) = 108$ implies $6y = 108$ or $y = 18 = z$, $x = 36$, so the volume is $V = 11,664$ cubic units. The maximal dimension is $(36, 18, 18)$.

At the boundary, where $x = 0$ or $y = 0$ or $z = 0$, we get 0 as a volume.

Main definitions:

Standard assumption is still that all functions have continuous first and second derivatives. Maximum means local maximum etc. A point (x_0, y_0) is an **absolute maximum** = **global maximum** on a domain R , if $f(x, y) \leq f(x_0, y_0)$ for all (x, y) in R .

To find a global maximum, we look at the local maxima and minima as well as the maxima and minima on the boundary. The latter is a Lagrange problem. If the domain is unbounded, we also have to look at the behavior of the function when $x, y \rightarrow \infty$.

Example.

$f(x, y) = x^2 + y^2 - x^4 - y^4$ has a local minimum at $(0, 0)$ but this is not a global minimum because $f(1000, 1000)$ for example is smaller than $f(0, 0) = 0$. If $f(x, y)$ is considered on the domain

$R = \{x^2 + y^2 \leq 1\}$ then the situation has changed and we need to look at extrema on the boundary too and $(0, 0)$, $(\pm 1, 0)$, $(0, \pm 1)$ are all global minima.

Bolzano theorem: A curve C or region R is bounded if there is r such that R is contained in a disc of radius r . A curve or region is closed if it contains all boundary points. A continuous function bounded and closed curve or bounded and closed region always has a global maximum and a global minimum on R . (You know this already in single variable with a continuous function on a closed interval).