

## Homework 19: Double integrals

This homework is due Wednesday, 10/25 resp Thursday 10/26.

- 1 a) (4 points) Write  $\log(x) = \ln(x)$  and compute  $\int_1^2 \int_2^4 \log(x)/(xy^2) dy dx$ .  
 b) (4 points) Now get

$$\int_1^2 \int_2^4 \log(x)/(xy^2) dx dy .$$

- c) (2 points) Why does this is not a counter example to Fubini's theorem?

### Solution:

- a) The first integral is  $\log(2)^2/8$ .  
 b) The second integral is  $3 \log(2)^2/4$ .  
 c) One would also have to change the bounds too.

- 2 For a function  $f(x)$  which is a probability density, meaning  $f \geq 0$ ,  $\int_{\mathbb{R}} f(x) dx = 1$ , the integral

$$E = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |x - y| f(x) f(y) dx dy$$

is called the Gini energy. The Gini index  $E/m$ , where  $m = \int_{-\infty}^{\infty} x f(x) dx$  is the **mean**, plays a role in computing wealth inequality.

- a) Find the Gini index for  $f(x) = 1$  on  $[0, 1]$  and  $f(x) = 0$  else.  
 b) Find the Gini energy for  $f(x) = \exp(-x^2)/\sqrt{\pi}$  (use technology)

**Solution:**

a)  $2/3$

b)  $\sqrt{2/\pi}$ .

- 3 We write  $dA$  if we don't specify the order of integration yet. Evaluate

$$\iint_R \log(1+x)\sqrt{x/y} dA$$

with  $\log(x) = \ln(x)$  over the region  $R = \{(x, y) \mid 0 \leq y \leq x, 0 \leq x \leq 1\}$ .

**Solution:**

Integrate first over  $y$ . Note that  $\log(1+x)$  does not depend on  $y$ . Then integrate over  $x$ . The answer is  $1/2$ .

- 4 a) Evaluate  $\int_0^2 \int_y^{2y} 6xy dx dy$  .

b) Compute this as a  $dydx$  integral:  $\int_0^2 \int_{\dots} 6xy dydx + \int_2^4 \int_{\dots} 6xy dydx$ .

**Solution:**

a)

$$\begin{aligned}\int_0^2 \int_y^{2y} 6xy \, dx \, dy &= \int_0^2 \frac{1}{2} 6x^2 y \Big|_{x=y}^{x=2y} \, dy \\ &= \frac{1}{2} \int_0^2 6y(4y^2 - y^2) \, dy = \frac{1}{2} \int_0^2 27y^3 \, dy \\ &= \frac{3}{2} \left( \frac{1}{4} 6y^4 \right) \Big|_0^2 = 6(4 - 0) = 36.\end{aligned}$$

b) Make a picture to get the region. Then compute

$$\int_0^2 \int_{x/2}^x xy \, dy \, dx + \int_2^4 \int_{x/2}^2 xy \, dy \, dx .$$

This is  $9 + 27 = 36$  again.**5** Evaluate  $\int_0^8 \int_{\sqrt[3]{y}}^2 4e^{x^4} \, dx \, dy$ .**Solution:**

$$\begin{aligned}4 \int_0^8 \int_{\sqrt[3]{y}}^2 e^{x^4} \, dx \, dy &= 4 \int_0^2 \int_0^{x^3} e^{x^4} \, dy \, dx \\ &= 4 \int_0^2 e^{x^4} [y]_{y=0}^{y=x^3} \, dx = 4 \int_0^2 x^3 e^{x^4} \, dx \\ &= e^{x^4} \Big|_0^2 = (e^{16} - 1)\end{aligned}$$

**Main definitions**

If  $R$  is a planar region and  $f(x, y)$  a function of two variables, the **double integral**  $\iint_R f(x, y) dA$  is the limit of the Riemann sum  $(1/n^2) \sum_{(i/n, j/n) \in R} f(i/n, j/n)$  for  $n \rightarrow \infty$ .

A **dydx-region** is of the form

$$R = \{(x, y) \mid a \leq x \leq b, c(x) \leq y \leq d(x)\} .$$

An integral over such a region is called a **dydx-integral**

$$\iint_R f dA = \int_a^b \int_{c(x)}^{d(x)} f(x, y) dy dx .$$

A **dx dy-region** is of the form

$$R = \{(x, y) \mid c \leq y \leq d, a(y) \leq x \leq b(y)\} .$$

An integral over such a region is called a **dx dy-integral**

$$\iint_R f dA = \int_c^d \int_{a(y)}^{b(y)} f(x, y) dx dy .$$

**Fubini's theorem** allows to switch the order of integration over a rectangle, if the function  $f$  is continuous:

$$\int_a^b \int_c^d f(x, y) dx dy = \int_c^d \int_a^b f(x, y) dy dx .$$