Homework 2: Vectors and Dot product

This homework is due Monday, 9/11 respectively Tuesday 9/12 at the beginning of class.

1 A **kite surfer** gets pulled with a force $\vec{F} = \langle 7, 1, 4 \rangle$. She moves with velocity $\vec{v} = \langle 4, -2, 1 \rangle$. The dot product of $\vec{F}$ with $\vec{v}$ is **power**. a) Find the angle between the $\vec{F}$ and $\vec{v}$. b) Find the **vector projection** of the $\vec{F}$ onto $\vec{v}$.

Solution:
(a) To find the angle between the force and velocity, we use the formula:

$$\cos\theta = \frac{F \cdot v}{|F||v|}.$$

The magnitude of the force is $\sqrt{7^2 + 1^2 + 4^2} = \sqrt{66}$. The magnitude of the velocity is $\sqrt{21}$. The dot product $F \cdot v = 30$. Thus,

$$\cos\theta = \frac{F \cdot v}{|F||v|} = \frac{30}{\sqrt{66} \cdot \sqrt{21}}$$

Hence, $\theta = \arccos(\frac{30}{\sqrt{1386}})$.
(b) The projection of $\vec{F}$ onto $\vec{v}$ is given by

$$\text{proj}_v F = \frac{F \cdot v}{|v|^2} v = \frac{30}{\sqrt{21}^2} \langle 4, -2, 1 \rangle = \frac{10}{7} \langle 4, -2, 1 \rangle.$$

2 Light shines long the vector $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and reflects at the three coordinate planes where the angle of incidence equals the

angle of reflection. Verify that the reflected ray is $-\vec{a}$. **Hint**. Reflect first at the xy-plane.

Solution:
If we reflect at the xy-plane, then the vector $\vec{a} = \langle a_1, a_2, a_3 \rangle$ gets changed to $\langle a_1, a_2, -a_3 \rangle$. You can see this by watching the reflection from above. Notice that the first two components stay the same. Do the same process with the other planes. If we next reflect $\langle a_1, a_2, -a_3 \rangle$ off the xz-plane, we get $\langle a_1, -a_2, -a_3 \rangle$. Finally, reflecting this vector off the yz-plane, we get the vector $\langle -a_1, -a_2, -a_3 \rangle$ as desired.

3 The **Harvard triple product** is defined in $\mathbb{R}^3$ as $\vec{u} \star \vec{v} \star \vec{w} = \vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{w} + \vec{w} \cdot \vec{u}$. a) Are there three unit vectors $\vec{u}, \vec{v}, \vec{w}$ for which the Harvard product is zero? If yes, find an example.
b) Are there three unit vectors $\vec{u}, \vec{v}, \vec{w}$ for which the Harvard product is 3? If yes, find an example.
c) Are there three unit vectors $\vec{u}, \vec{v}, \vec{w}$ for which the Harvard product is -3? If yes, find an example.
d) Are there three unit vectors $\vec{u}, \vec{v}, \vec{w}$ for which the Harvard product is 2? If yes, give a reason.

Solution:
a) Take i, j, k.
b) Take i, i, i.
c) Not possible. We must have $u = -v, u = -wandv = -w$.
d) Intermediate value theorem.

4 a) Find the angle between a space diagonal of a cube and the

diagonal in one of its faces.
b) The **hypercube** or **tesseract** has vertices $(\pm 1, \pm 1, \pm 1, \pm 1)$. Find the angle between the hyper diagonal connecting $(1,1,1,1)$ with $(-1,-1,-1,-1)$ and the space diagonal connecting $(1,1,1,1)$ with $(-1,-1,-1,1)$.

Solution:
(a) Consider the 'standard' cube with vertices whose coordinates are either 0 or 1. The line segment from $(0,0,0)$ to $(0,1,1)$ is a diagonal of one of its faces (the face on the yz-plane), while the line segment from $(0,0,0)$ to $(1,1,1)$ is a diagonal of the cube. These line segments can be expressed as the vectors $\langle 0,1,1\rangle$ and $\langle 1,1,1\rangle$, respectively. The cosine of the angle between them is then

$$\cos\theta = \frac{\langle 0,1,1\rangle \cdot \langle 1,1,1\rangle}{|\langle 0,1,1\rangle| \cdot |\langle 1,1,1\rangle|} = \frac{2}{\sqrt{2}\cdot\sqrt{3}} = \frac{2}{\sqrt{6}}.$$

Hence $\theta = \arccos(\frac{2}{\sqrt{6}})$.
(b) The vector connecting $(1,1,1,1)$ to $(-1,-1,-1,-1)$ is $\langle 2,2,2,2\rangle$. The vector connecting $(1,1,1,1)$ with $(-1,-1,-1,1)$ is $\langle 2,2,2,0\rangle$. Thus, the cosine of the angle between the hyper diagonal and the space diagonal is given by

$$\cos\theta = \frac{\langle 2,2,2,2\rangle \cdot \langle 2,2,2,0\rangle}{|\langle 2,2,2,2\rangle||\langle 2,2,2,0\rangle|} = \frac{12}{\sqrt{16}\cdot\sqrt{12}} = \frac{\sqrt{3}}{2}.$$

Finally, we calculate $\theta = \arccos(\frac{\sqrt{3}}{2})$.

5 a) Verify that if $\vec{a}, \vec{b}$ are nonzero, then $\vec{c} = |\vec{a}|\vec{b} + |\vec{b}|\vec{a}$ bisects the angle between $\vec{a}, \vec{b}$ if $\vec{c}$ is not zero.
b) Verify the parallelogram law $|\vec{a}+\vec{b}|^2 + |\vec{a}-\vec{b}|^2 = 2|\vec{a}|^2 + 2|\vec{b}|^2$.

Solution:

(a) The cosine of the angle between a and c is

$$\begin{aligned}\frac{\vec{a}\cdot\vec{c}}{|\vec{a}|\cdot|\vec{c}|} &= \frac{\vec{a}\cdot(|\vec{a}|\vec{b}+|\vec{b}|\vec{a})}{|\vec{a}|\cdot|\vec{c}|} \\ &= \frac{|\vec{a}|(\vec{a}\cdot\vec{b})+|\vec{a}|^2|\vec{b}|)}{|\vec{a}|\cdot|\vec{c}|} = \frac{(\vec{a}\cdot\vec{b})+|\vec{a}||\vec{b}|}{|\vec{c}|}.\end{aligned}$$

A similar calculation yields that the cosine of the angle between b and c is $\frac{(\vec{b}\cdot\vec{a}+|\vec{b}||\vec{a}|}{|\vec{c}|}$. Because these two expressions are exactly the same, this tells us that either c bisects the angle between a and b or that a and b are colinear. However, if a and b are collinear, c is also collinear with a and b and thus it still bisects the angle between them.

(b) We compute:

$$\begin{aligned}&|\vec{a}+\vec{b}|^2+|\vec{a}-\vec{b}|^2 = (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})+(\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b}) \\ =\ & (\vec{a}\cdot a+2\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{b})+(\vec{a}\cdot\vec{a}-2\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{b}) \\ =\ & 2(\vec{a}\cdot\vec{a}+\vec{b}\cdot\vec{b}) \\ =\ & 2(|\vec{a}|^2+|\vec{b}|^2) \\ =\ & 2|\vec{a}|^2+2|\vec{b}|^2.\end{aligned}$$

Main definitions

Two points $P = (a, b, c)$ and $Q = (x, y, z)$ define a **vector** $\vec{v} = \langle x - a, y - b, z - c \rangle$. We also write $\vec{v} = \vec{PQ}$. The numbers v_1, v_2, v_3 in $\vec{v} = \langle v_1, v_2, v_3 \rangle$ are the **components** of $\vec{v}$. The **length** $|\vec{v}|$ of a vector $\vec{v} = \vec{PQ}$ is defined as the distance $d(P, Q)$ from P to Q. A vector of length 1 is called a **unit vector**. The **addition** is $\vec{u} + \vec{v} = \langle u_1, u_2, u_3 \rangle + \langle v_1, v_2, v_3 \rangle = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$. The **scalar multiple** $\lambda\vec{u} = \lambda\langle u_1, u_2, u_3 \rangle = \langle \lambda u_1, \lambda u_2, \lambda u_3 \rangle$. The difference $\vec{u} - \vec{v}$ can be seen as $\vec{u} + (-\vec{v})$.

The **dot product** of two vectors $\vec{v} = \langle a, b, c \rangle$ and $\vec{w} = \langle p, q, r \rangle$ is defined as $\vec{v} \cdot \vec{w} = ap + bq + cr$. The **Cauchy-Schwarz inequality** tells $|\vec{v} \cdot \vec{w}| \leq |\vec{v}||\vec{w}|$.

The **angle** between two nonzero vectors is defined as the unique $\alpha \in [0, \pi]$ satisfying $\vec{v} \cdot \vec{w} = |\vec{v}|\ |\vec{w}| \cos(\alpha)$. Two vectors are called **orthogonal** or **perpendicular** if $\vec{v} \cdot \vec{w} = 0$. The zero vector $\vec{0}$ is orthogonal to any vector. For example, $\vec{v} = \langle 2, 3 \rangle$ is orthogonal to $\vec{w} = \langle -3, 2 \rangle$. The vector $\mathrm{P}(\vec{v}) = \frac{\vec{v} \cdot \vec{w}}{|\vec{w}|^2}\vec{w}$ is called the **projection** of $\vec{v}$ onto $\vec{w}$. The **scalar projection** $\frac{\vec{v} \cdot \vec{w}}{|\vec{w}|}$ is plus or minus the length of the projection of $\vec{v}$ onto $\vec{w}$. The vector $\vec{b} = \vec{v} - P(\vec{v})$ is a vector orthogonal to $\vec{w}$. **Pythagoras tells:** if $\vec{v}$ and $\vec{w}$ are orthogonal, then $|v - w|^2 = |v|^2 + |w|^2$.