

Homework 21: Surface area

This homework is due Monday, 10/30 rsp Tuesday 10/31.

- 1 Find the surface area of the surface given by

$$z = \frac{2}{3}(x^{3/2} + y^{3/2}), \quad 0 \leq x \leq 3, \quad 0 \leq y \leq 3.$$

Solution:

We compute: $f_x = x^{1/2}$ and $f_y = y^{1/2}$. The surface area is

$$\int_0^3 \int_0^3 \sqrt{1 + x + y} \cdot dx dy = \frac{2}{3} \int_0^3 \{(2 + y)^{3/2} - (1 + y)^{3/2}\} dy$$

This evaluates to $28/15(7\sqrt{7} - 9)$.

- 2 Find the area of the surface given by the **helicoid**

$$\vec{r}(u, v) = \langle u \cos(v), u \sin(v), v \rangle.$$

with $0 \leq u \leq 1$, $0 \leq v \leq \pi$.

Solution:

We compute:

$$\vec{r}_u = \langle \cos v, \sin v, 0 \rangle \quad \vec{r}_v = \langle -u \sin v, u \cos v, 1 \rangle.$$

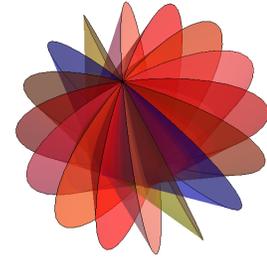
Therefore, $\vec{r}_u \times \vec{r}_v = \langle \sin v, -\cos v, u \rangle$. The area is given by

$$\int_0^\pi \int_0^1 \sqrt{1 + u^2} du dv = \frac{\pi}{2}(\sqrt{2} + \log(1 + \sqrt{2}))$$

A decorative paper lantern is made of 8 surfaces. Each is parametrized by

3 $\vec{r}(t, z) = \langle 10z \cos(t), 10z \sin(t), z \rangle$

with $0 \leq t \leq 2\pi$ and $0 \leq z \leq 1$ and then translated or rotated. Find the total surface area of the lantern.



Solution:

We compute

$$\vec{r}_t = \langle -10z \sin(t), 10z \cos(t), 0 \rangle$$

$$\vec{r}_z = \langle 10 \cos(t), 10 \sin(t), 1 \rangle$$

$$\vec{r}_t \times \vec{r}_z = \langle 10z \cos(t), 10z \sin(t), -100z \rangle$$

and the length is $|\vec{r}_t \times \vec{r}_z| = 10z\sqrt{101}$.

$$\int_0^{2\pi} \int_0^1 10z\sqrt{101} \, dz dt = 10\pi\sqrt{101} .$$

There are 8 pieces so that the final result is $\boxed{80\pi\sqrt{101}}$.

- 4 The figure shows the torus obtained by rotating about the z -axis the circle in the xz -plane with center $(5, 0, 0)$ and radius 1. Parametric equations for the torus are

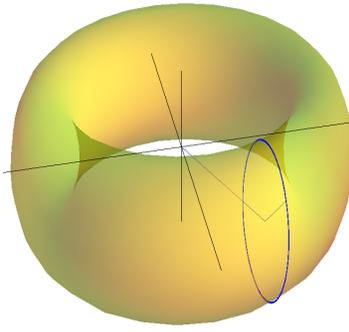
$$x = 5 \cos \theta + \cos \alpha \cos \theta$$

$$y = 5 \sin \theta + \cos \alpha \sin \theta$$

$$z = \sin \alpha ,$$

where θ and α are the angles shown in the figure. Find the surface

area of the torus.



Solution:

Let $\vec{r} = \langle x, y, z \rangle$. Then,

$$\vec{r}_\alpha = \langle -\sin \alpha \cos \theta, -\sin \alpha \sin \theta, \cos \alpha \rangle$$

and

$$\vec{r}_\theta = \langle -5 \sin \theta - \cos \alpha \sin \theta, 5 \cos \theta + \cos \alpha \cos \theta, 0 \rangle .$$

Thus

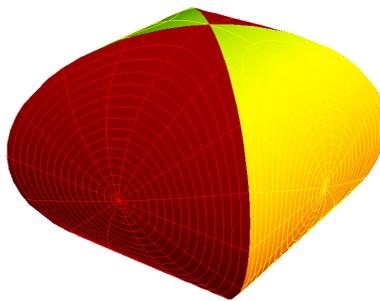
$$\begin{aligned} \vec{r}_\alpha \times \vec{r}_\theta &= \langle -5 \cos \alpha \cos \theta - \cos^2 \alpha \cos \theta, \\ &\quad -5 \cos \alpha \sin \theta - \cos^2 \alpha \sin \theta, -5 \sin \alpha - \cos \alpha \sin \alpha \rangle. \end{aligned}$$

Therefore $|\vec{r}_\alpha \times \vec{r}_\theta| = (5 + \cos \alpha)$ and so the surface area is

$$\int_0^{2\pi} \int_0^{2\pi} (5 + \cos \alpha) d\alpha d\theta = 2\pi \cdot 25\pi = 4\pi^2 5.$$

- 5 The volume and surface area of the solid obtained by intersecting the solid cylinder $y^2 + z^2 \leq 4$ with the solid cylinder $x^2 + z^2 \leq 4$ has been found by Archimedes. Find the surface area of the surface

S bounding this solid.



Solution:

In the picture above, the boundary of the solid is made up of two red and two yellow pieces. The yellow piece facing up lies on the cylinder $x^2 + z^2 = 1$ between $|x| \geq |y|$. The surface area of 1/8'th is

$$\int_0^2 \int_{-x}^x |r_x \times r_y| \, dy dx = \int_0^2 2x \frac{1}{\sqrt{1+x^2}} \, dx = 8 .$$

The answer is $8 \times 8 = 64$.

We can also use polar coordinates: the surface area of *one* piece is

$$\int_{-2}^2 \int_{-\arccos|y|}^{\arccos|y|} d\theta dy = \int_{-2}^2 2 \arccos |y| dy = 8 .$$

Again the surface area is $8 \cdot 8 = 64$.

Main definitions:

A surface $\vec{r}(u, v)$ parametrized on a parameter domain R has the **surface area**

$$\int \int_R |\vec{r}_u(u, v) \times \vec{r}_v(u, v)| \, du dv .$$

Examples:

$$\frac{\vec{r}(u, v)}{\langle \rho \cos(u) \sin(v), \rho \sin(u) \sin(v), \rho \cos(v) \rangle} \quad \frac{|\vec{r}_u \times \vec{r}_v|}{\rho^2 |\sin(v)|}$$
$$\langle u, v, f(u, v) \rangle \quad \sqrt{1 + f_u^2 + f_v^2}$$
$$\langle f(v) \cos(u), f(v) \sin(u), v \rangle \quad f(v) \sqrt{1 + f'(v)^2}$$