

Homework 22: Triple integrals

This homework is due Friday, 11/3 resp Tuesday 11/7.

- 1 a) Evaluate the iterated integral

$$\int_0^2 \int_0^z \int_0^{y^3} x^2 \, dx \, dy \, dz .$$

- b) Which of the three following scrambled versions make sense too? (you don't have to evaluate)

$$\int_0^2 \int_0^{y^3} \int_0^z x^2 \, dx \, dz \, dy .$$

$$\int_0^z \int_0^2 \int_0^{x^3} x^2 \, dx \, dz \, dx .$$

$$\int_0^2 \int_0^y \int_0^{z^3} x^2 \, dx \, dz \, dy .$$

Solution:

a) 1024/165.

b) Only the third makes sense.

- 2 Evaluate the triple integral

$$\int \int \int_E 2yz \cos(x^5) \, dV ,$$

where $E = \{(x, y, z) \mid 0 \leq x \leq \pi/2, 0 \leq y \leq x, x \leq z \leq 2x\}$.

Solution:

By the given description of E , we can write the triple integral as an iterated integral:

$$\iiint_E yz \cos(x^5) dV = \int_0^{\pi/2} \int_0^x \int_x^{2x} 2yz \cos(x^5) dz dy dx.$$

This we integrate in the usual way:

$$\begin{aligned} \int_0^{\pi/2} \int_0^x \int_x^{2x} 2yz \cos(x^5) dz dy dx &= \int_0^{\pi/2} \int_0^x y \cos(x^5) \cdot \frac{1}{2} 2z^2 \Big|_x^{2x} dy dx \\ &= \int_0^{\pi/2} \int_0^x 2y \cos(x^5) \cdot \frac{1}{2} \cdot 3x^2 dy dx \\ &= \frac{3}{2} \int_0^{\pi/2} \int_0^x x^2 2y \cos(x^5) dy dx \\ &= \frac{3}{2} \int_0^{\pi/2} x^2 \cos(x^5) \cdot y^2 \Big|_0^x dx \\ &= \frac{3}{2} \int_0^{\pi/2} x^2 \cos(x^5) \cdot x^2 dx \\ &= \frac{3}{4} \int_0^{\pi/2} 2x^4 \cos(x^5) dx \\ &= \frac{6}{20} \sin(x^5) \Big|_0^{\pi/2} \\ &= \frac{6}{20} \sin(\pi^5/32) \end{aligned}$$

The final answer is $\frac{3}{10} \sin(\pi^5/32)$.

3 Evaluate the triple integral

$$\int \int \int_E xy dV ,$$

where E is bounded by the parabolic cylinders $y = 3x^2$ and $x = 3y^2$ and the planes $z = 0$ and $z = x + y$.

Solution:

$$\iiint_E xy \, dV = \iint_R \int_0^{x+y} xy \, dz \, dA = \int_0^{1/3} \int_{3x^2}^{\sqrt{\frac{x}{3}}} \int_0^{x+y} xy \, dz \, dy \, dx$$

This is $\int_0^{1/3} \int_{3x^2}^{\sqrt{x}/3} xy(x+y) \, dz \, dy \, dx$. Evaluating the next integral gives

$$\int_0^{1/3} \frac{x^{5/2}}{9\sqrt{3}} - 9x^7 - \frac{9x^6}{2} + \frac{x^3}{6} \, dx = 1/2268 = 0.0004\dots$$

- 4 Use a triple integral to find the volume of the given solid enclosed by the paraboloid $x = y^2 + z^2$ and the plane $x = 25$.

Solution:

The paraboloid $x = y^2 + z^2$ intersects the plane $x = 25$ in the circle $y^2 + z^2 = 25$. Thus,

$$E = \{(x, y, z) \mid y^2 + z^2 \leq x \leq 25, y^2 + z^2 \leq 25\}$$

Let $D = \{(y, z) \mid y^2 + z^2 \leq 25\}$. Then using polar coordinates $y = r \cos \theta$ and $z = r \sin \theta$, we have

$$\begin{aligned} \iint_D \left(\int_{y^2+z^2}^{25} dx \right) dA &= \iint_D (25 - (y^2 + z^2)) \, dA \\ &= \int_0^{2\pi} \int_0^5 (25 - r^2) r \, dr \, d\theta \\ &= \int_0^{2\pi} d\theta \int_0^5 (25r - r^3) \, dr \\ &= 2\pi \left[\frac{25r^2}{2} - \frac{r^4}{4} \right]_0^5 \\ &= \frac{625\pi}{2} \end{aligned}$$

5 Find the moment of inertia

$$I = \int \int \int_E (x^2 + y^2) \, dz dx dy$$

about the z -axis of the solid $E : 0 \leq z \leq r^3 \leq 8$.

Solution:

$$\begin{aligned} I &= \iiint_E (x^2 + y^2) \, dV \\ &= \int_0^{2\pi} \int_0^2 \int_0^{r^3} r^3 \, dz r dr d\theta \\ &= 2\pi 2^7 / 7 = 256\pi / 7 \end{aligned}$$

For students who did the old version:

$$\begin{aligned} I_z &= \iiint_E (x^2 + y^2) \rho(x, y, z) \, dV \\ &= \iint_{x^2+y^2 \leq 10^2} \left[\int_{\sqrt{x^2+y^2}}^{10} (x^2 + y^2) \, dz \right] \, dA \\ &= \iint_{x^2+y^2 \leq 10^2} (x^2 + y^2) (10 - \sqrt{x^2 + y^2}) \, dA \\ &= \int_0^{2\pi} \int_0^{10} r^2 (10 - r) r \, dr \, d\theta \\ &= \int_0^{2\pi} d\theta \int_0^{10} (10r^3 - r^4) \, dr \\ &= (2\pi) \left[\frac{10}{4} r^4 - \frac{1}{5} r^5 \right]_0^{10} \\ &= 10000\pi \end{aligned}$$

Main definitions

If $f(x, y, z)$ is a function and E is a **solid**, then $\iiint_E f(x, y, z) dV$ is defined as the $n \rightarrow \infty$ limit of the Riemann sum

$$\frac{1}{n^3} \sum_{(\frac{i}{n}, \frac{j}{n}, \frac{k}{n}) \in E} f\left(\frac{i}{n}, \frac{j}{n}, \frac{k}{n}\right) .$$

Like $dA = dxdy$ was a symbol for a small area, $dV = dxdydz$ indicates a small volume. Triple integrals are solved as a nested list of single integrals.

If $f(x, y, z) = 1$ then $\iiint_E 1 dxdydz$ is the volume of the solid

A common situation is where the triple integral is reduced to a double integral

$$\int \int_R \left[\int_{g(x,y)}^{h(x,y)} f(x, y, z) dz \right] dxdy .$$

This is by far the most common case. For example, if $g(x, y) = 0$ and $f(x, y, z) = 1$, then

$$\int \int_R \left[\int_0^{h(x,y)} 1 dz \right] dxdy = \int \int_R h(x, y) dxdy$$

is the signed volume of the solid under the graph of h . In variable calculus, where you were sometimes able to compute triple integrals by reducing to a single integral. You would have written $\int_a^b A(z) dz$ to compute the volume of a solid sandwiched between $z = a$ and $z = b$ for which the area of the cross section at height z is $A(z)$. In multi variable calculus we are much more flexible as we can now also reduce to a double integral.